Review

on the **Thesis** "**Subordination principle for generalized fractional evolution equations**" submitted by **Assoc. Prof. Dr. Emilia Grigorova Bazhlekova** for awarding the **Scientific degree** "Doctor of Science" in the the field of Higher education 4. Natural sciences, mathematics and informatics, Professional direction 4.5 "Mathematics Scientific specialty Mathematical analysis,

from **Prof. DSc. Tsviatko Vassilev Rangelov, IMI, BAS**, member of the Scientific jury, order N:216/20.07.2022 of the Director of IMI, BAS.

1) Assoc. Prof. E. Bazhlekova graduated from the Faculty of Mathematics and Informatics, Sofia University, majoring in Mathematics in 1986. In 2001, after postgraduate studies, she defended her PhD thesis "Fractional evolution equations in Banach spaces"in Technical University of Eindhoven, the Netherlands. From 1989 to 1993 she was PhD student (1989 - 1993) and a mathematician (1995 - 2004) in the Department of Complex Analysis, a mathematician and Ass. Prof. in the Department of Analysis, Geometry and Topology (2011 - 2014) and Assoc. Prof. since 2014.

The scientific activity of Assoc. Prof. E. Bazhlekova is in the field of differential equations with fractional derivatives, special functions, convolutional analysis and applications. The thesis is devoted to the subordination principle for generalized fractional evolution equations and applications to sub-diffusion equations, diffusion-wave equations etc.

2) Problems with fractional derivatives is an actual domain of analysis and differential equations. In the last years they are a subject of investigation not only by mathematicians, but also by scientists in the field of mechanics and physics, due to their applications in continuum media, viscoelasticity, system stability etc. Although the operators with fractional derivatives of order α with positive integer α are differential operators, the processes being modeled with them as well as the methods for solution of Couchy problems and of boundary-value problems are completely different, mainly because of nonlocal character of the equations with fractional derivatives. This necessitates the development of new methods, approaches and principles in their research. The present dissertation is devoted to the study of such principle - principle of subordination. Principle of subordination is originally formulated for stochastic processes, associated with a diffusion equation by S. Bochner (1949). Along with the copious amount of follow-up papers, monographs related to the topics, discussed by Assoc. Prof E. Bazhlekova should be noted. Cited and used in the dissertation are e.g. J. Prüs (1993); I. Podlubni (1999); R. Gorenflo, H. Kibas, F. Mainardi, S. Rogosin (2014);F. Mainardi (2010), R. Schilling, R. Song, Z. Vondracek (2010); W. Arendt, C. Batty, M. Hieber, F. Neubrander (2011); J. Paneva-Konovska (2016) and others.

Because in the next text below, the principle of subordination is mentioned often, we will indicate the general definition, given in the dissertation also suitable for fractional differential equations:

Given two Cauchy problems (P) and (P_*), the problem (P) is called a subordinate of the problem (P_*) if it is solvable (it has a unique solution, continuously dependent on the initial data) and the solution $u(x,t)$ of (P) is present via the solution $u_*(x,t)$ of (P_*) through the integral dependence $u(x,t) = \int_0^\infty \varphi(t,\tau)u_*(x,\tau)d\tau$. Here, the kernel $\varphi(t,\tau)$ is a probability density with respect to $\tau \geq 0$ while $t > 0$ is considered as a parameter, i.e., $\varphi(t,\tau) \geq 0$ и $\int_0^\infty \varphi(t,\tau)d\tau = 1$.

3) The aim of the dissertation is to study the principle of subordination for general fractional evolution equations and to develop a methodology enabling the establishment of subordination dependence between two equations. Given the variety and active development of equations with fractional derivatives and their applications in modeling and studying evolutionary processes in mechanics, physics, biology, etc. there is a wide field of application of the presented methodology.

The proposed dissertation is written in English, contains 200 pages, consists of an introduction, 8 chapters, a list of 110 titles of the literature used and an index. The first two chapters (1, 2) are introductory and contain the main used definitions, integral transformations and special functions, as well as two general subordination theorems (Theorems 2.4 and 2.5), which summarize results published by the author in 2000 and are proved by using ideas from J. Prüs (1993). Chapter 3 studies the subordination principle for evolution equations with fractional derivatives in space and time. In the remaining chapters (4, 5, 6) and in (7,8) a number of new results are obtained, by applying the subordination principle to evolution equations such as Jeffrey's fractional equation for heat conduction, sub-diffusion equations, diffusion-wave equations, equations describing wave propagation in viscoelastic media.

4) I will dwell in more detail on the problems solved in the present dissertation and scientific contributions. New results are presented in each chapter (3 - 8) and the fact that they are cited 90 times shows their relevance. All new results and contributions are sufficiently well described in the thesis, in the abstract, and in the contribution report. Therefore, I will indicate only some of them, which, in my opinion, represent a basis for future use and application of the principle of subordination.

- One of the main contributions is in chapter 3 in Theorem 3.1, in which, applying the principle of subordination, new results are obtained for the abstract Cauchy problem with fractional derivatives in time and space. Let ${}^{C}D_t^{\beta}u(t)$ be the fractional Caputo derivative with time, A be the generator of a bounded C_0 - semigroup in the Banach space X, $0 < \alpha, \beta \leq 1$ and $S_{\alpha,\beta}(t)$ denotes the resolution operator for the problem

$$
{}^{C}D_t^{\beta}u(t) = -(-A)^{\alpha}u(t), \quad u(0) = v \in X.
$$

Then, applying the subordination principle in Theorem 3.1 is proved the representation $S_{\alpha,\beta}(t) = \int_0^{\infty} \psi_{\alpha,\beta}(t,\tau)S_{1,1}(\tau)d\tau$, $\tau > 0$, where $\psi_{\alpha,\tau}$ is the subordination kernel. This relationship between the resolving operator $S_{\alpha,\beta}(t)$ of the problem with fractional derivatives and the resolving operator $S_{1,1}(t)$ for the parabolic problem justifies the principle of subordination and its applications.

Contribution (Ch. 3) is also the integral representation of the kernel of subordination $\psi_{\alpha,\beta}(t)$ in Theorem 3.5, the analyticity result of the resolving operator in Theorem 3.6, as well as an integral representation of the resolving operator through a fundamental solution.

- On the example of a parabolic equation with fractional derivatives

$$
\left(1 + aD_t^{\alpha}\right)u'(t) = \left(1 + bD_t^{\alpha}\right)Au(t), \quad \alpha \in (0,1], \quad 0 \leq a, b,
$$

of Jeffrey's type a general property is demonstrated after applying the principle of subordination. Depending on the numbers a, b, the following 2 types are established: for $a < b$ the equation models diffusion, while for $a > b$ it models propagation of waves. This distinction helps the correct classification of the evolutionary fractional differential equations. A number of results in the dissertation are obtained developing this new idea of diffuse or wave behavior of solutions, (e.g. Theorems 4.1, 4.2) and the corresponding representations of the fundamental solution depending on a and b.

- The principle of subordination is applied and a representation of the solutions of a generalized sub-diffuse problem in Theorem 5.5 is obtained, involving evolutionary distributed order equations of the form

$$\int_0^1 \mu(\beta)^C D_t^\beta u(t) d\beta = Au(t) \quad \text{и} \quad u'(t) = \int_0^1 \mu(\beta) D_t^\beta Au(t) d\beta,$$

with initial conditions $u(0) = a \in X$ and ${}^C D_t^\beta, D_t^\beta$ are fractional Caputo derivatives and Riemann-Louisville, respectively, $\mu(\beta)$ is a discrete or continuous distribution. In the scalar case, the generalized relaxation equation is studied in detail (Theorem 5.7), which is applied to the inverse problem with a source.

- Evolutionary equations with several time derivatives of a different order are investigated

$$
\begin{aligned}
{}^C D_t^\alpha u(t) + \sum_{j=1}^m b_j^C D_t^{\alpha_j} u(t) &= Au(t) + f(t) \quad \text{и} \\
u'(t) = D_t^{1-\alpha} Au(t) + \sum_{j=1}^m b_j D_t^{1-\alpha_j} Au(t) &+ f(t), \quad t > 0
\end{aligned}
$$

where $1 \geq \alpha > \alpha_1 > \ldots > \alpha_m > 0$, $b_j > 0$, $j = 1, \ldots, m$, and A is an operator generating a C_0 semigroup. With the principle of subordination, a representation of their solutions (Theorem 6.1) with a Mittag-Leffler multinomial function, of the type of Prabhakar is derived. New estimates of the relaxation functions are also obtained (Theorem 6.7).

- For diffuse-wave equations with several time derivatives of different order

$$c^C D_t^\alpha u(t) + \sum_{j=1}^m c_j^C D_t^{\alpha_j} u(t) = Au(t) \quad u(0) = a \in X, \quad u'(0) = 0,$$

where $\alpha \in [1, 2]$, $\alpha > \alpha_1 > \ldots > \alpha_m > 0$, $\alpha - \alpha_m \leq 1$, $c, c_j > 0$, and A generates strongly continuous cosine operator function, is obtained integral representation of the subordination kernel (Theorem 7.4).

- For distributed order equations, under additional conditions, it is proved that the fundamental solution is a probability density. This makes it possible to apply the principle of subordination (Theorem 7.8).

- It is shown that the relaxation moduli for a number of generalized viscoelastic models (Theorems 8.2 - 8.4) are completely monotonic functions.

5) The abstract correctly reflects the content and contributions of the dissertation work.

6) The results obtained in the dissertation by Acoss. Prof. E. Bazhlekova are new, they are the subject of 11 articles. They have been published after 2015 in renowned journals in mathematics, such as: Fract. Calc. Appl. Anal. - 2; Integr. Transf. Spec. Funct. - 2; Mathematics - 2; J. Comput. Appl. Math. - 1; Int. J. Appl. Math. - 1; Math. Met. Appl. Sci. - 1;AIP Conf. Proc. - 1; Fractal Fract. - 1. With impact factor are 8 publications [10, 12 - 15, 18, 20, 22], 2 are with SJR and 1 is indexed in Scopus, but without IF/SJR (numbering is according to the bibliography in the dissertation). Five of the articles,were co-authored by I. Bazhlekov and S. Pchenichkov and 6 are independent. Considering the theoretical results obtained in the papers and included in the dissertation, I accept that the contribution of Acoss. Prof. E. Bazhlekova in joint publications is essential. Also the results of none of the above 11 publications are not included in the PhD thesis of Assoc. Prof. E. Bajlekova and as well are not included in the materials for her competition for associate professor in 2014. In relation to the Rules of the BAS for the implementation of the RSARB and in connection with Art. 2 of the IMI Rules for the "minimum required score by set of indicators" for Assoc. Prof. E. Bajlekova is obtained the following: A - 50 points; Б - 100 points; B - 200 points; Г - 402 points; Д - 540 points; E - 40 points, which means that this requirement is fulfilled.

I will emphasize that for points according to indicators Г and Д, only the publications included in the dissertation are counted and their citations, which are 90 for the period after 2015.

7) I have the following remarks:

a) Three different numberings for the cited literature are used in the abstract and in the dissertation. In my opinion, only the numbering in the thesis should be used. Also the numbering of sections as well as the theorems in the abstract and in the dissertation is different.

b) The terms fundamental solution and Green's function are used for the same functions. A fundamental solution is a solution in sense of distributions of a differential equation with a right-hand side δ function of Dirac. A Green's function is a solution of a boundary value problem, with a right-hand side δ function or of an initial-boundary value problem, with an initial condition δ function. I think that the definitions of a fundamental solution and of a Green's function should be given in Ch. 1 in the dissertation.

These notes do not refer to the scientific contributions received in the dissertation.

c) I believe that the dissertation could be proposed for publication as a monograph.

8) Conclusion: The dissertation of Assoc. Prof. E. Bazhlekova is in an up-to-date and intensively developing field of mathematical analysis. It is prepared on high scientific level and fully satisfies the requirements of ZRASRB and the Rules of BAS and IMI-BAS guidelines for its implementation. Also there is no plagiarism in the dissertation and in the papers.

I recommend to the scientific jury to award Assoc. Prof. Dr. Emilia Grigorova Bazhlekova the scientific degree "Doctor of Sciences"in the field of Higher education 4. Natural sciences, mathematics and informatics, Professional direction 4.5 ”Mathematics”, Scientific specialty Mathematical analysis.

October 04, 2022 Signature:

Ts. Rangelov