

## REVIEW

by Prof. D.Sc. Virginia Kiryakova,

Institute of Mathematics and Informatics (IMI) – Bulgarian Academy of Sciences (BAS)

on the documents and publications, presented by

Asso. Prof. Dr. Emilia Grigorova Bazhlekova

for defense of Dissertation Thesis entitled:

„Subordination principle for generalized fractional evolution equations“  
for awarding of the scientific degree „Doctor of Science”

in the Higher education area 4. Natural sciences, mathematics and informatics, professional field 4.5. Mathematics, scientific specialty “Mathematical analysis”

I am a member of the Academic Board on this procedure according to Order № 216 / 20.07.2022 of Director of IMI-BAS. At the first meeting of this Academic Board I was appointed as a Reviewer (Minutes № 1 / 21.07.2022). Let us mention that the Institute of Mathematics and Informatics (IMI)-BAS possesses a valid accreditation by the National Accreditation Agency for the Doctoral program “Mathematical Analysis”.

I have been acquainted with the required documents and presented thesis and publications by the candidate Emilia Bazhlekova. Checking them, I am convinced that the presented stuff and the candidate satisfy all the requirements of Ch. 2 of the Law Act for Development of Academic Staff in R. Bulgaria (abbrev. further as “Law”). She defended a PhD thesis in 2001 in the Eindhoven University of Technology – the Netherlands and possesses a legalized Diploma № 000001/16.12.2011.

All specific science-metric indicators from the Regulations of IMI-BAS (item 3, (1),1, updated 2022) related to application of the Law are satisfied and essentially exceeded.

### 1. Short CV data for applicant

Mrs. E. Bazhlekova graduated from the Mathematical high school “Geo Milev” in the town of Pleven; then had her higher education in Mathematics with qualification „Ms.Sc.“ in the Faculty of Mathematics and Informatics – Sofia University “St. Kl. Ohridski” (1986) and next 1 year specialization (1986-1987) there, and has been also an Assist. Prof. there on Mathematical Analysis course (1991-1992). In the period 1989-1992 she was a regular time PhD student in IMI-BAS, Dept. Complex Analysis with a scientific advisor Academy Corr. member Ivan Dimovski, on the topics related to nonlocal operational calculi. She has been employed in IMI-BAS since 1995, initially as a mathematician, then as Assist. Prof. and since 2014 – habilitated as Asso. Prof. She has been (and continue) a long years scientific secretary of the Dept. “Analysis, Geometry and Topology”.

As a result of her scientific specialization in the Netherlands (Applied Analysis Group, Eindhoven University of Technology) in the period 1997-2001, she prepared and defended there (1 Oct. 2001) a PhD thesis on the subject „*Fractional evolution equations in Banach spaces*” with an international academic board (jury).

**The research interests and contributions of Dr. Bazhlekova are in the field of Mathematical Analysis** – fractional calculus and fractional order differential equations; operator-theoretical approach to abstract problems in Banach spaces; spectral series for the solutions of

boundary value problems, estimations and asymptotic; subordination principle; special functions of Mittag-Leffler type; integral transforms; convolutional calculus and its applications to Duhamel representations of solutions; numerical algorithms for nonlocal boundary value problems; applications of fractional calculus and differential equation to mathematical modeling, to problems of mechanics, etc.

Educational activities – as Assist. Prof. on Mathematical Analysis in FMI – Sofia University, lecturer in high school on Mathematics and English, consultations and preparation of students and PhD students, etc.

## 2. Description of the presented materials:

- **Dissertation thesis (the Thesis) for awarding of the scientific degree “Doctor of Science” on the subject „Subordination principle for generalized fractional evolution equations“** (In English) – consisting of 200 pages: Foreword, Introduction, 8 chapters, Annotation of the scientific contributions, Bibliography (list of 110 items), Index. The thesis is on the scientific specialty Mathematical Analysis, and especially – on Fractional Calculus, Evolution Differential Equations, Operator Methods, Special Functions, and applications.
- **Summary (Autoreferat) of the Thesis** – 33 pages, in Bulgarian and with English translation.
- **Author’s list of scientific contributions in the Thesis and related publications** (in Bulgarian and English).
- **List of citations for the publications on the Thesis.**
- **Scientific publications related to the results in the Thesis – 11 scientific articles.** All of them are already published, in English, 6 of them are single-authored.

**The candidate has not applied with these publications in the previous promotion cases** (for PhD. – in 2001, or for Asso. Prof. – in 2014): all these works are from the next period 2015-2021.

**The presented 11 publications are as follows:**

- **Scientific articles in reviewed and indexed journals – 10 items**
- **Scientific articles in indexed proceedings of international conferences – 1 item**
  - **8 of these articles are in journals with impact factor** (Web of Science, Clarivate Analytics), in Q1 – 5 items; Q2 – 1 item; Q3 – 2 items; with total IF = 16.393 (published in the journals: *Integral Transforms and Special Functions*; *Mathematical Methods in Applied Sciences*; *Fractional Calculus and Applied Analysis*; *J. of Computational and Applied Mathematics*, *Mathematics*, *Fractal and Fractional*)
  - **1 article – in a journal indexed in WoS and Scopus** (*Mathematics*, in 2015, but afterwards it has IF in Q1)
  - **1 article – in a journal indexed in Scopus with SJR=0.268**, in Q3 (*Intern. J. Appl. Math.*)
  - **1 article – in proceedings of international conference with impact rang (indexed in Scopus) with SRJ = 0.163** but not assigned to any quartal (in: *American Institute of Physics Conf. Proc.*)

The data on the criteria “Г” and “Д” in the table below, are presented accurately with the corresponding scores (points), according to the minimal specific requirements from the IMI-BAS

regulations (that are higher than the corresponding national ones). These data are also reflected in the Academy system Sonix.

<b>Groups of indicators</b>	<b>Contents</b>	<b>Minimal requirement - DrSc</b>	<b>Applicant's values</b>
<b>Г</b>	<b>Sum of Indicators from 5 to 10</b> Publications on the Thesis (11 items): 5 publs in Q1 ( <b>250 p.</b> ) 1 publ. in Q2 ( <b>40 p.</b> ) 2 publ. in Q3 ( <b>60 p.</b> ) <hr/> 2 publs with SJR ( <b>40 p.</b> ) 1 publ. in Scopus without SJR ( <b>12 p.</b> )	<b>100 points</b>	<b>402 points</b>
	<b>Of the publications on the Thesis, these are:</b>		
	Publs. in editions with IF or SJR	<b>7 items</b>	<b>10 items</b>
	Publs. in editions with IF (8 items are with total IF: 16.339)	<b>4 items</b>	<b>8 items</b>
<b>Д</b>	<b>Indicator 11</b> Citations (without selfcit.) of the publications on the Thesis which are in the data basis of Scopus: <b>90 items (x 6 p.)</b> (but most of these are also in the data basis of WoS)	<b>100 points</b>	<b>540 points</b>

**This table confirms that all required science metric indicators are satisfied, and even essentially exceeded.**

### **3. Analysis of the scientific results and contributions in the presented Thesis and related publications**

The subject of the Thesis is in the field of Fractional Calculus (FC, as Calculus where the operators of integration and differentiation can be of arbitrary, incl. fractional order), especially – there is a deep analysis on the fractional evolution differential equations based on the principle of subordination – as a main contribution of Thesis' author. With her studies and results, Bazhlekova has opened a new path for this trend in FC, whose beginning was introduced by Prüss (1993) for Volterra integral equations, but now – for a wide class of abstract differential equations of fractional order – with numerous applications in mathematical models for phenomena of natural and social aspect. Some of her own studies with the help of this method were presented in Chapter 3 (of 13 pages only) in her previous (PhD) thesis of 2001. Then, in next 20 years, but mainly in the period 2015-2021, Bazhlekova successfully extended this theory and is already recognized as its main initiator among the researchers in the field of fractional differential equations and their applications.

The tools used in the Thesis and related publications include not only these of Fractional Calculus and Special Functions, but also of Functional Analysis, Semigroup Theory, Resolvent Operators, Abstract Differential Equations, Bernstein Functions and classes of related functions.

### **The results in the Thesis can be characterized in the following main trends:**

In Introduction and Chapter 1 there are presented in a condensed form the basic notions, notations, operators of FC, special functions and their properties.

In Chapter 2, the Bernstein functions and Volterra integral equations are introduced; and the two basic theorems on the subordination principle are proven: Th. 2.1 (reducing the subordination to a problem for the Bernstein functions) and Th. 2.2.

Chapter 3 (also [B5],[B9]) is devoted to the subordination principle for abstract fractional (in time and in space) evolution equations (involving Caputo derivative and an operator  $(-A)^\alpha$ , where  $A$  generates a  $C_0$ -semigroup in Banach space) – Th.3.1-Th.3.5. As applications, integral representations for the fundamental solution are given and explicit representations in terms of special functions (of Mittag-Leffler, Mainardi, and confluent hypergeometric functions).

In the rest of the Thesis, the studies are for generalized evolution equations with fractional derivatives w.r.t. time.

Chapter 4 (also [B10]) is devoted to an analogue of fractional order (with a Riemann-Liouville derivative) of the Jeffrey heat conduction equation, satisfying two different subordination principles, and so – different types of behavior – of diffusion regime and of wave propagation (Th. 4.1.-Th.4.4). The integral representation of the solution is suitable for numerical calculation and visualization.

Chapter 5 ([B1],[B2],[B3]) contains „subordination“ results for subdiffusion equation of distributed order, and also with more general memory kernels. A Cauchy problem is considered with the generalized convolution derivative introduced by Kochubei, and with an operator  $A$  generating a bounded  $C_0$ -semigroup. The scalar case is investigated in details, when the solutions are relaxation functions. In the particular case with Caputo derivatives, the relaxation functions are represented by the Mittag-Leffler functions. The estimates in Th. 5.3 are useful for investigation of boundary value problems with expansions in eigenfunctions.

In Chapter 6 (and [B6]) the studies of evolution equations with relaxation with different orders derivatives (multi-term) w.r.t. time are based on the use and on the author's results (Th. 6.2) for the multinomial Mittag-Leffler function (by Luchko et al.) and on its extension to a multinomial Prabhakar analogue (introduced by the author), Th. 6.3-Th.6.4 (conditions when this function is completely monotone).

In Chapter 7 (and [B4],[B7]) it is considered a partial answer (Asserts. 7.1, 7.2) of an open problem (stated by Gorenflo-Luchko-Stojanovic, see §7.1) for possible interpretation of the fundamental solution of the diffusion-wave equations of distributed order as probability density. The subordination relations here are presented in Th.7.1, Th.7.2.

In Chapter 8 (and [B4],[B8],[B11]) the subordination principle is illustrated by applications to mathematical models describing waves in viscoelastic media, as fractional distributed order generalizations of the Maxwell, Jeffrey and Zener laws. It is proven that the relaxation modulus is a completely monotone function. Again, the tools of the special functions are used, for example the binomial Mittag-Leffler function.

Let me comment shortly on the basic idea behind the subordination principle, on the development and applications of which in fractional calculus is devoted this Thesis: Let us have two given Cauchy problems – (P) и (P\*). For the problem (P) it is said that it is subordinate to the problem (P\*), if it is solvable whenever (P\*) is solvable, and the solution  $u(x,t)$  of (P) possesses an

integral representation via the solution  $u^*(x,t)$  of (P\*) with a kernel which is a probability density. This principle allows to obtain / represent the solutions of complicated equations / problems by means of the (known) solutions of simpler / classical problems, and to derive their properties. In this sense, according to reviewer's view, there is certain analogy with the transmutation method (that arose also in studies in the field of differential equations, by Delsarte and Lions).

My personal research interests in this domain allow me to highly evaluate the use of the tools of the so-called special functions of fractional calculus, and the results of Bazhlekova that are parts of the Thesis and her related publications – for the functions of Mittag-Leffler, Mainardi, multinomial Mittag-Leffler function and its Prabhakar-type extension introduced by the author.

**The Thesis' Abstract (and Author's List of Contributions) reflects adequately the scientific contributions of Emilia Bazhlekova in her Thesis. To avoid unnecessary repetitions, I am avoiding their description here.**

#### **4. General description of the research and organizational activities of the candidate:**

Asso. Prof. Dr. Bazhlekova is author of more than 50 scientific publications, 18 of which are in editions with IF (with total IF=31), 17 are in editions with SJR, 3 are in other editions indexed in WoS and Scopus, 11 – are referred to in other world data bases.

Bazhlekova has taken active part and continues it, in numerous research projects on the topics of the Thesis: with National Science Fund, on series of European science programs, with Russian academy and bilateral agreements between Bulgarian and Serbian academies, on budget projects of the Institute, etc. She was a member of organizing committees of international conferences in Bulgaria, as: TMSF'1996, TMSF'1999, TMSF'2011, CAA'2013, TMSF'2017, etc. Long years scientific secretary of the Dept. "Analysis, Geometry and Topology" in IMI-BAS.

She is actively working as a member of one of basic research groups ("Applied Mathematical Analysis") at the same department, which is closely related to the subject of the Thesis, and annually reports a series of research publications in prestigious journals, and their so numerous citations.

#### **5. Impact of candidate's results in works of other authors:**

The publications of E. Bazhlekova (over 50) have totally more than 1500 citations (without selfcitations). The data for her personal Hirsh index are, respectively:

**h = 9** (Scopus, without selfcits.), **h = 10** (Web of Science), **h = 15** (Google Scholar & Harzing's Publish or Perish). These mentioned citations are mainly in prestigious international editions, and by foreign authors. This is confirmed by the lists presented by the author, the data inserted in the academy system Sonix, and by her scientific profiles in the World data bases:

- Google Scholar: <https://scholar.google.com/citations?user=S1V5wscAAAAJ&hl=en>  
50 publications; over 1580 citations; h=15
- Scopus (Bazhlekova, Emilia; ID 6507427736):  
<https://www.scopus.com/authid/detail.uri?authorId=6507427736>  
37 publications, of which 18 are with Impact Factor; over 360 citations;
- Web of Science: <https://www.webofscience.com/wos/woscc/citation-report/ef1e6d5b-70de-40bb-ab0a-ca0097fd7950-53e76e0b> -> 36 publications, 365 citations (10.14 average per article), h=10
- Research Gate:  
50 publications; over 1100 citations; h=13 (without selfcits. h=12)  
<https://www.researchgate.net/profile/Emilia-Bazhlekova>

**For the publications related with the Thesis** (11 items not used for applications to previous author's procedures), Bazhlekova has presented a detailed list of the observed **citations: 90 citations, of which 77 items are in journals with IF, and others – with SJR**. Let me mention that according to Bazhlekova's GoogleScholar profile, her 2<sup>nd</sup> most cited work (after the PhD thesis of 2001) is the article on the topic (and with same title as) of the Thesis: „Supordination principle for fractional evolution equations” – with more than 111 citations. If we add also the found other 87 citations of publications on the Thesis, that however have increased substantially in the recent months (after preparation and presentation of the Thesis), I would suppose that **the citations on the „subordination principle“ of Bazhlekova are already more that 200, or so**. A Google search (V.K., 7 Oct. 2022) for "Subordination"+"Bazhlekova" returned 1980 results.

## **6. Approbation of the results:**

The results of Asso. Prof. Bazhlekova on the papers related to the Thesis are published in prestigious international editions, indexed in the world data bases and are cited indeed many times. They have been reported at a series of international scientific forums (more than 10), both in Bulgaria and abroad (Serbia, Greece, etc.), also at national seminars and annual sessions. These results are also essential part of the working programs of a series of research projects (with National Science Fund, IMI-BAS, international and European program, bilateral agreements).

## **7. Evaluation of the personal contributions of the candidate:**

As a rule, a thesis for the scientific degree “Doctor of Science” is prepared under self-studies and should consists of personal contributions by the author.

Of the 11 articles of Bazhlekova related to the Thesis, 6 are single-authored, and 5 are with 1 co-author, in these she definitely has a leading and / or equal role. In the Thesis there are described only results obtained by the author; while the co-authors in [B7]-[B11] have their contributions in fixing the mathematical models and their applications, and in the numerical evaluations. The whole ideology and the basic analytical results on the development and applications of the theory are personal contribution of E. Bazhlekova, resulting from long years professional development as an internationally recognized leading experts in her domain.

There is no detected plagiarism.

## **8. Critical notes: None.**

I have a personal recommendation to Asso. Prof. Bazhlekova to continue her research work and to speed up her carrier to a procedure for Full Professor. With her results and published works, she has covered long ago the criteria for this academic post.

## **7. Personal impressions:**

I know personally Emilia since 1989, when she entered in our department “Complex Analysis” – at IMI as a PhD student and later, as a mathematician. Then, and also after her return to Bulgaria and Institute, she always made impression with her modesty, diligence, good mathematical culture, and taste for precise investigation of significant problems of Analysis. I have the pleasure to attend in 2001 the defense of her PhD thesis in the Eindhoven University of Technology, as a member of the international academic board (jury) on this procedure, and to witness the excellent presentation that led to unanimously taken decision to award the PhD diploma.

Most of the results and publications of Bazhlekova are devoted to operational methods for solving differential equations of fractional order and with essential results in applications of fractional calculus. She is a member of the international editorial board of the leading journal in this domain, "Fractional Calculus and Applied Analysis", and part of its success is due also to her professional expert activities. I have been personally convinced in the highly positive opinions for her results and personality by the world leading figures in our field.

My personal opinion – based on our joint work in IMI-BAS team on the topic, on the presented publications in leading specialized journals and on her prestige among the colleagues in fractional calculus – is that Asso. Prof. Dr. Emilia Bazhlekova is a well established and world recognized scholar in the field.

## **7. Conclusion:**

The review of the presented Thesis, its Summary, related publications and documents show that Asso. Prof. Dr. Bazhlekova is an internationally established expert on Mathematical Analysis, and especially in the rapidly developing area of Fractional Calculus and its applications. Her personal contributions on the topic of the Thesis are undoubted.

I can evaluate the contributions in this Thesis as theoretical generalizations and developing a theory in a difficult but rather hot area of Analysis. The presented results have not only theoretical but also scientific-applicable nature, and they conform to and support the contemporary development of the fractional calculus and fractional differential equations.

According to the Regulations of IMI-BAS for application of the Law, all quantitative indicators for awarding the scientific degree "Doctor of Science" in IMI-BAS are fulfilled in the case of Bazhlekova's thesis, and these are as follows:

1. of the presented publications for awarding the scientific degree "Doctor of Science" the candidate should have at least 7 publications in editions with IF or SJR – such are 10 of all presented 11 items;

2. of these, at least 4 should be in journals with IF – such are 8 items;

3. the presented works on the Thesis should have citations for at least 100 points – the candidate presented a list of more than 90 citations in Scopus (although most of the are also in WoS) for 540 points. While the number of citations of all candidate's publications is more than 1500.

I confirm that in the presented thesis and related publications, which are close to my personal scientific specialty, **there is no observed plagiarism.**

It is evident that all required indicators are essentially exceeded.

**Having all above said in mind, I conclude that the Thesis presented by Asso. Prof. Dr. Emilia Bazhlekova satisfies all requirements of the Law (Law Act for Development of Academic Staff in R. Bulgaria) and of the Regulations of BAS and IMI-BAS for such procedure. So, I propose to the Academic Board to award to the candidate the scientific degree "Doctor of Science" in the Higher education area 4. Natural sciences, mathematics and informatics, professional field 4.5. Mathematics, scientific specialty "Mathematical analysis".**

**I give my positive conclusion for awarding of the scientific degree „Doctor of Science” to Emilia G. Bazhlekova.**

October 10, 2022

Reviewer and member of Academic board:  
(Prof. DSc. V. Kiryakova)