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On Soft Decoding of Coded QAM using Integer Codes

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Abstract

A soft decoding algorithm of coded quadrature amplitude modulation (QAM) using single error correcting integer codes over a finite ring of integers modulo m is presented. Although the integer codes under consideration are designed to correct only single errors in a QAM constellation, soft decoding is shown to improve the error correcting capabilities of these codes. Computer simulation results show that for AWGN channels the soft decoding algorithm can achieve between 4 dB and 6 dB coding gain relative to uncoded QAM.

1. INTRODUCTION

Coded modulation is an efficiently combined scheme of coding and modulation techniques. It has been investigated extensively by Ungerboeck [1, 2], Imai and Hirakawa [3] and others. In 1982, Ungerboeck constructed a trellis code that maps the input sequence into signal points of a fixed signal constellation by a method referred to as *set partitioning*. This method is called trellis coded modulation (TCM). An alternative which allows us to deal with a variety of constellations is block coded modulation [4, 5]. In block coded modulation, each point of the signal constellation corresponds to a symbol of a finite ring of integers modulo m denoted by $\mathbb{Z}_m$. An information sequence is mapped into a sequence of symbols in $\mathbb{Z}_m$ and coded by a code over $\mathbb{Z}_m$.

In this article, we consider soft decoding of a block-coded modulation scheme using integer codes over $\mathbb{Z}_m^n$ that is capable of correcting single errors in a two-dimensional lattice $\mathbb{Z}_t^2$, where $m \geq t^2$. A class of integer codes can be useful in coded modulation, since each point in a signal constellation can be represented by an integer (see Fig. 1). The most common errors will be those which change a point to its nearest neighbour in the grid, i.e., a grid point to the left, right, top or bottom of the point. Neither the Hamming distance nor the Lee distance are appropriate for handling these errors in a QAM signal. In [6], Huber proposed codes over Gaussian integers with a two-dimensional modular distance called Mannheim distance to improve the situation. Although Huber's codes require fixed parameters, our construction of integer codes are more flexible and intended for multi-dimensional modulation.

Due to the simple structure of coded QAM using integer codes, the exact expressions for the bit error probability (BEP) over additive white Gaussian noise (AWGN) channels has been derived [7], where an integer code of rate 3/4 is constructed over $\mathbb{Z}_{17}^4$ and it shown that the code gains 4 dB in a range of BEP $10^{-5} \sim 10^{-8}$. Moreover, the integer code performs 1 dB better than TCM at the same rate. By applying soft decoding to coded QAM using integer codes, we will show that it is possible to achieve an additional gain of 0.5 dB$\sim$4 dB.

The organization of this article is as follows. In Section 2, we describe three methods for constructing integer codes. Next, we present the framework of coded QAM using integer codes in Section 3. The soft decoding algorithm for integer codes is proposed in Section 4. Section 5 presents the results of our simulation experiments.

2. CONSTRUCTION OF INTEGER CODES

An integer code $\mathcal{C}^{(n)}(d, \boldsymbol{w}) \subset \mathbb{Z}_m^n$ of length n is defined by

$$\mathcal{C}^{(n)}(d, \boldsymbol{w}) = \left\{ \boldsymbol{c} \in \mathbb{Z}_m^n | \sum_{i=1}^{n} c_i w_i = d \pmod{m} \right\} \quad (1)$$

where $d \in \mathbb{Z}_m$, $\boldsymbol{c} = (c_1, c_2, \ldots, c_n) \in \mathbb{Z}_m^n$ is a codeword vector and $\boldsymbol{w} = (w_1, w_2, \ldots, w_n) \in \mathbb{Z}_m^n$ is a fixed weight coefficient vector [5]. The code $\mathcal{C}^{(n)}(d, \boldsymbol{w})$ is said to be a single $(\pm 1, \pm t)$-error correctable if it corrects a single error vector of the set

$$\mathcal{E} = \{(e_1, 0, \ldots, 0), \ldots, (0, \ldots, 0, e_n)\}$$

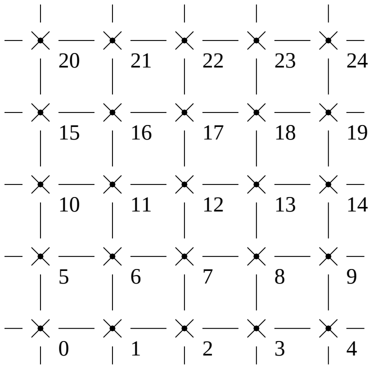

Figure 1: Two-dimensional lattice $\mathbb{Z}_5^2$ indexed by elements of $\mathbb{Z}_{25}$

where $e_i \in \{\pm 1, \pm t\}$ and $1 \le i \le n$. For example, the weight vector $\boldsymbol{w} = (1, 2, 3, 6)$ gives a single $(\pm 1, \pm 4)$-error correctable integer code of length 4 over $\mathbb{Z}_{17}$.

To identify each error vector in $\mathcal{E}$, it is necessary to choose the values of $\boldsymbol{w}$ and m in such a way that the syndrome values $e_i w_i \pmod{m}$ $(1 \le i \le n)$ are unique for vectors in $\mathcal{E}$. Hence, if $\mathcal{C}^{(n)}(d, \boldsymbol{w})$ is single $(\pm 1, \pm t)$-error correctable, the following inequality always holds

$$m \ge 4n + 1. \tag{2}$$

An integer code is called *perfect* if $m = 4n + 1$ and *quasi-perfect* if $m > 4n + 1$ is the smallest value of m for which an integer code exists.

We briefly present three $(\pm 1, \pm t)$-correctable integer code constructions that were presented in [8]. The first two give perfect codes while the last gives quasi-perfect codes.

Construction A.
Consider a weight vector $\boldsymbol{w}^{(a)}$, where $w_i^{(a)} = t^{2i-2} \bmod m$. It has been shown in [8] that $\mathcal{C}^{(n)}(d, \boldsymbol{w}^{(a)}) \subseteq \mathbb{Z}_m^n$, where $d \in \mathbb{Z}_m$, is a perfect single $(\pm 1, \pm t)$-error correcting integer code $\mathcal{C}^{(n)}(d, \boldsymbol{w}^{(a)}) \subseteq \mathbb{Z}_m^n$ for any prime $m = 4n + 1$ and any t that generates $\mathbb{Z}_m$.

Construction B.
Consider the weight vector $\boldsymbol{w}^{(b)}$, where $w_i^{(b)}$ are distinct elements from the integer set $\mathcal{A}^{(t)} \subset \mathbb{Z}_m$, defined by

$$\mathcal{A}^{(t)} = \big\{(p-1)t + q \,|\, 1 \le p \le \lfloor t/2 \rfloor, p \le q \le t - p\big\}.$$

It follows that

$$n = |\mathcal{A}^{(t)}| = \begin{cases} t^2/4 & \text{if } t \text{ is even,} \\ (t^2-1)/4 & \text{if } t \text{ is odd.} \end{cases} \tag{3}$$

It has been proved in [8] that for any $m = t^2 + 1$, $\mathcal{C}^{(n)}(d, \boldsymbol{w}^{(b)})$ is $(\pm 1, \pm t)$-correctable for arbitrary values of d where n is given in (3). It follows that the code $\mathcal{C}^{(n)}(d, \boldsymbol{w}^{(b)})$, $d \in \mathbb{Z}_m$ is perfect for even values of t. If t is odd, $\mathcal{C}^{(n)}(d, \boldsymbol{w}^{(b)})$ gives $m = 4n + 2$ where $m = t^2 + 1$ and $n = (t^2 - 1)/4$. It implies that the integer code obtained is quasi-perfect, because it does not achieve the bound $4n + 1$.

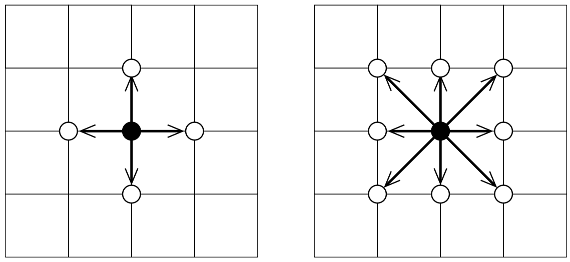

Figure 2: Examples of errors of type ‘cross’ (left) and type ‘square’ (right) on a two-dimensional constellation.

It should be noted that Construction B has been extended to construct integer codes suitable for correcting single errors on k-dimensional lattices [9].

Construction C.
Consider the weight sequence $\boldsymbol{w}^{(c)}$ with weights $w_i^{(c)} = 2i - 1$, $1 \le i \le n$. It has been shown in [8] that $\mathcal{C}^{(n)}(d, \boldsymbol{w}^{(c)}) \subseteq \mathbb{Z}_m^n$ is a quasi-perfect integer code for even values of t, provided that t and $2n+1$ are mutually prime. The codes $\mathcal{C}^{(n)}(d, \boldsymbol{w}^{(c)})$ are called *odd-weight* codes.

All the codes presented above are supposed to correct single errors of the type ‘cross’ in a two-dimensional lattice $\mathbb{Z}_t^2$. Another interesting type of errors to be investigated are errors of the type ‘square’ that consist of $\pm 1, \pm t, \pm(t-1)$, and $\pm(t+1)$. For example, the weight vector $\boldsymbol{w} = (1, 2)$ gives a code of length 2 over $\mathbb{Z}_{17}$ which corrects single errors of type ‘square’ in the two-dimensional lattice $\mathbb{Z}_4^2$ (see Fig. 2). It is possible to find integer codes capable of correcting other error patterns as well as errors of type ‘square’ using exhaustive search methods. However, it is not yet known whether it is possible to systematically construct such integer codes.

3. HARD DECODING ALGORITHM

Suppose that $\mathcal{C}^{(n)}(d, \boldsymbol{w})$ is a single $(\pm 1, \pm t)$-error correctable code. If $m \ge t^2$, we may apply the subset $\mathcal{C}_{t^2}^{(n)}(d, \boldsymbol{w}) \triangleq \mathcal{C}^{(n)}(d, \boldsymbol{w}) \cap \mathbb{Z}_{t^2}^n$ to coded t^2-QAM, where each point (a, b) of the signal constellation is represented by an integer $at + b$ in $\mathbb{Z}_{t^2} \subset \mathbb{Z}_m$.

Then, a point (a, b) in $\mathbb{Z}_t^2$ has four nearest neighbours at distance 1 in $\mathbb{Z}_t^2$, i.e., $(a, b-1)$, $(a, b+1)$, $(a-1, b)$ and $(a+1, b)$. Using the mapping $c = a \cdot t + b \pmod m$, this corresponds to the errors ± 1, $\pm t$ in $\mathbb{Z}_m$. For example, the nearest neighbours of point 12 in the $\mathbb{Z}_5^2$ constellation depicted in Fig. 1 are $7, 11, 13$, and 17, i.e., 12 ± 1, 12 ± 5. each of the error values $t, 1, -1$, and $-t$ automatically corresponds to one of the four vectors $(0, 1), (1, 0), (-1, 0)$, and $(0, -1)$, respectively.

In coded QAM, each codeword $\boldsymbol{c} \in \mathcal{C}_{t^2}^{(n)}(d, \boldsymbol{w})$ is modulated by a t^2 quadrature amplitude modulator. The modulated codeword is represented by $(\boldsymbol{s}_1, \boldsymbol{s}_2, \ldots, \boldsymbol{s}_n)$ where $\boldsymbol{s}_i = (a_i, b_i)$ is a signal point in the constellation $\mathbb{Z}_t^2$ for $i = 1, \ldots, n$.

Each component of a modulated codeword is transmitted over an AWGN channel. The sampled channel output sequence is given by $(\boldsymbol{z}_1, \boldsymbol{z}_2, \ldots, \boldsymbol{z}_n)$, where a pair of output components $\boldsymbol{z}_i = (x_i, y_i) \in \mathbb{R}^2$ is given by $x_i = a_i + n_{2i-1}$ and $y_i = b_i + n_{2i}$ with AWGN samples n_{2i-1} and n_{2i}, $i = 1, \ldots, n$.

In coded QAM, the decoder makes *hard decisions* on each channel output pair (x_i, y_i) to estimate the transmitted code symbol and selects the nearest signal point $(\bar{a}_i, \bar{b}_i)$ to (x_i, y_i) in the constellation. The associated number $\bar{a}_i + \bar{b}_i t$ of $(\bar{a}_i, \bar{b}_i)$ can be treated as the received value of code symbol transmitted, denoted by $r_i \in \mathbb{Z}_m$ $(1 \le i \le n)$.

4. SOFT DECODING ALGORITHM

In soft decoding we utilize the analog received samples (x_i, y_i), $1 \le i \le n$, to find the most probable codeword to be transmitted in the sense of the maximum likelihood estimation.

We propose the following algorithm for coded t^2-QAM using $\mathcal{C}_{t^2}^{(n)}(d, \boldsymbol{w})$ over $\mathbb{Z}_m$ where $m \ge t^2$.

In: The channel output $\boldsymbol{z}$ and the received sequence $\boldsymbol{r} = (r_1, r_2, \ldots, r_n)$.

Out: The decoded codeword $\tilde{\boldsymbol{c}} = (\tilde{c}_1, \tilde{c}_2, \ldots, \tilde{c}_n)$.

Step 1. Calculate the squared distance $\Delta^2[i, \varepsilon]$ between (x_i, y_i) and each of the signal points associated with $r_i + \epsilon \pmod{t^2}$ where $\epsilon \in \{-t, -1, 0, 1, t\}$.

Step 2. Compute the syndrome value $s = \sum_{i=1}^n w_i r_i \pmod m$.

Step 3. Let $\mathcal{E}[s]$ be the set of all the vectors $\boldsymbol{e} = (e_1, e_2, \ldots, e_n) \in \{-t, -1, 0, 1, t\}^n$ such that

$$\sum_{i=1}^{n} w_i e_i = s \pmod m.$$

Then find the vector $\boldsymbol{e}^* = (e_1^*, e_2^*, \ldots, e_n^*) \in \mathcal{E}[s]$ that minimizes $\sum_{i=1}^n \Delta^2[i, e_i^*]$.

Step 4. Output $\tilde{\boldsymbol{c}} = \boldsymbol{r} - \boldsymbol{e}^*$ and stop.

The above algorithm accomplishes maximum likelihood decoding for an AWGN channel. In Step 3 an exhaustive search is performed to find $\boldsymbol{e}^*$ among $\mathcal{E}[s]$, the cardinality of which is roughly estimated by $5^n/4n$. It is reasonable if n is relatively small, say $n = 4$. For a large value of n, we can utilize a trellis with $n+1$ layers, in each of which there are m states. Each state in the ith layer for $i = 0, 1, \ldots, n$ is indexed by $k = 0, 1, \ldots, m-1$. A pair of numbers $(d_k^{(i)}, e_k^{(i)})$ is attached to the kth state in the ith layer. Here $d_0^{(0)} = 0$ and $d_k^{(0)} = \infty$ for $k \in \mathbb{Z}_m \backslash \{0\}$ and $d_k^{(i)}$ is given by

$$d_k^{(i)} = \min_{\epsilon \in \{0, \pm 1, \pm t\}} \left\{ d_{k+\epsilon w_i}^{(i-1)} + \Delta^2[i, \epsilon] \right\} \qquad (4)$$
$$\text{for } i = 1, 2, \ldots, n \text{ and } k = 0, 1, \ldots, m-1.$$

Moreover, $e_k^{(i)} = \epsilon^*$ where ϵ^* is an element in $\{0, \pm 1, \pm t\}$ that achieves the minimum value of (4). The Viterbi algorithm can sequentially calculate $\{(d_k^{(i)}, e_k^{(i)}), k = 0, 1, \ldots, m-1\}$ for $i = 1, 2, \ldots, n$ in a similar way discussed in [10, 11]. After the calculation is completed, e_n^* is given by the value of $e_s^{(n)}$ associated to the s-th state in the n-th layer where s is the syndrome value obtained in Step 2. The other $e_{n-1}^*, \ldots, e_1^*$ are computed in descending order by using the following recursive equations on k_i, $i = n-1, \ldots, 1$ with the initial value $k_n = s$:

$$k_i = k_{i+1} + e_{i+1}^* w_{i+1}, \qquad (5)$$
$$e_i^* = e_{k_i}^{(i)}. \qquad (6)$$

Although we consider only errors of type 'cross' in the algorithm presented, it is easy to make the algorithm handle more general error patterns such as the error type 'square'. In case of the error type 'square', the set of error values is $\{-t-1, -t, -t+1, -1, 0, 1, t-1, t, t+1\}$ in steps 1 and 3.

5. SIMULATION RESULTS

To give an impression of the performance of the soft decoding algorithm, we determine the bit error probability of soft decoding for coded t^2-QAM ($t = 4$) using integer codes. The simulation results will be presented for two types of errors, that is, 'cross' and 'square'.

5.1. Errors of Type 'Cross'

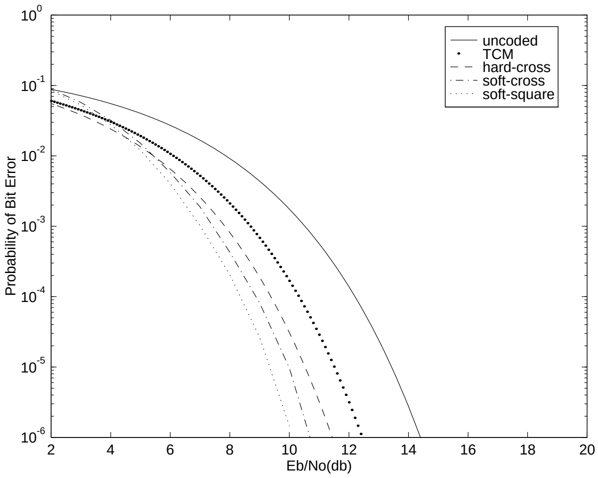

Figure 3: A comparison of symbol error probability versus signal-to-noise ratio between uncoded and coded (type 'cross' using 'hard' and 'soft' decoding algorithms) 16-QAM over AWGN channels. The code used is $\mathcal{C}_{16}^{(4)}(0, \boldsymbol{w}_c)$ with $\boldsymbol{w}_c = (1, 2, 3, 6)$.

We use a single $(\pm 1, \pm 4)$-error correcting integer code $\mathcal{C}^{(4)}(0, \boldsymbol{w}_c)$ over $\mathbb{Z}_{17}$, where the weight vector $\boldsymbol{w}_c$ is given by $\boldsymbol{w}_c = (1, 2, 3, 6)$. This integer code can be obtained from Construction B in Section 2. Moreover, we consider the subset $\mathcal{C}_{16}^{(4)}(0, \boldsymbol{w}_c) = \mathcal{C}^{(4)}(0, \boldsymbol{w}_c) \cap \mathbb{Z}_{16}^4$ to allocate codewords on the 16-QAM constellation.

According to the method shown in [8], we can systematically encode a binary source string of length 11 into a codeword $\boldsymbol{c}$ of $\mathcal{C}_{16}^{(4)}(0, \boldsymbol{w}_c)$ where $\boldsymbol{c}$ can be represented by a binary string of length 16. Hence the rate of this code is 11/16.

Figure 3 shows the performance of the soft decoding algorithm for $\mathcal{C}_{16}^{(4)}(0, \boldsymbol{w}_c)$ over an AWGN channel. The horizontal and vertical axes stand for E_b/N_0 in dB and bit error rate (BER), respectively where E_b is average signal energy per bit and $N_0/2$ is the variance of the two-side power spectral density of AWGN. In Figure 3, three curves for uncoded 16-QAM (solid), TCM (thick dotted), hard decision (dashed) are obtained from the exact formula derived in [7]. Other two curves for soft decoding (one-point dashed and thin dotted) have been obtained by computer simulation where 1,000,000 binary source block of length 11 were generated equiprobably and encoded into codewords of $\mathcal{C}_{16}^{(4)}(0, \boldsymbol{w}_c)$. The bit error probability has been estimated by

$$\frac{\#\text{ erroneous symbols in soft decoding}}{tN \times R}$$

where $t = 4, N = 1,000,000$, and R is the rate of the code ($R = 11/16$).

From this figure, we can see that the integer code gains about 3dB compared with uncoded 16-QAM in a range of BEP $10^{-4} \sim 10^{-6}$. And, we find that integer coded modulation with hard decoding (dashed curve) can gain 1dB more than TCM (thick dotted curve) in 16-QAM. Moreover, by using soft decoding we can obtain 0.5 dB more coding gain compared with the hard decoding. It means applying soft decoding to integer coded modulation enlarges decodability of the code. In fact, most of double errors have been corrected in a high SNR region. In a low SNR region less than 6 dB, soft decoding results in worse performance than hard decoding. The larger error values become, the more frequently the soft decoding tends to decode the erroneous codeword since the erroneous one might be much nearer to the received signal than the correct one in a low SNR environment.

Moreover, by replacing errors of type 'cross' by errors of type 'square' in the proposed algorithm, we can correct even errors of type 'square' using the same code. As a result, we can obtain an additional gain of 0.5 dB (the thin dotted curve) compared with the results of soft decoding for errors of type 'cross'. Of course, the computational complexity of soft decoding increases when handling errors of type 'cross' in the proposed algorithm. Though a straightforward search of the maximum likelihood error vector may increase the complexity by $(9/5)^n$, the Viterbi algorithm suppressed it by double in our experiments where $n = 4$.

5.2. Error of Type 'Square'

The effect of soft decoding is more significant for the error type 'square' as shown in Figure 4 where $N = 1,000,000$. In this experiment, we used the aforementioned integer code $\mathcal{C}^{(2)}(0, \boldsymbol{w}_s)$ over $\mathbb{Z}_{17}$ where $\boldsymbol{w}_s = (1.2)$. The rate of this code is $R = 1/2$ as described above. Similar to Figure 3, three curves for uncoded 16 QAM (solid), TCM (dots), hard decision (dashed) in Fig. 4 are obtained from the exact formula. Note that the shapes of the curves are slightly different from those in Figure 3 because we calculated bit error probability by normalizing symbol error probability by the rate of the code. The coding gain using soft decoding is about 3dB at BEP 10^{-6}. The performance is surprisingly good given the simplicity of the code used in the experiments.

6. CONCLUSION

We proposed soft decoding for coded QAM using integer codes to improve the error correcting capability of the codes. The computer simulation results show

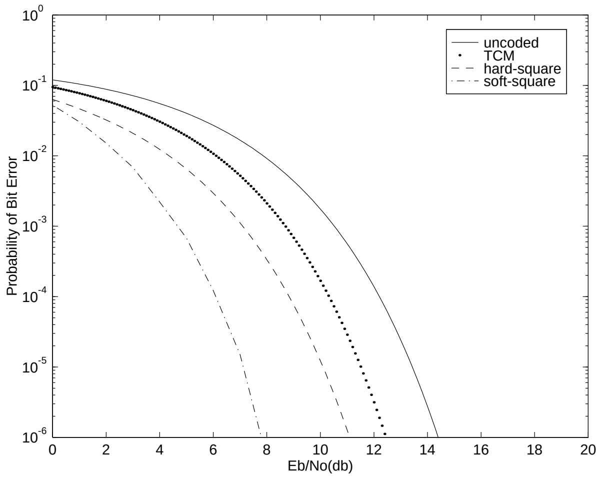

Figure 4: A comparison of symbol error probability versus signal-to-noise ratio between uncoded and coded (type 'square' using 'hard' and 'soft' decoding algorithms) 16-QAM over AWGN channels. The code used is $\mathcal{C}^{(2)}(0, \boldsymbol{w}_s)$ with $\boldsymbol{w}_s = (1.2)$.

that combining soft decoding with integer coded modulation gives a 0.5 dB $\sim$ 3 dB coding gain relative to hard decoding of integer coded modulation. It is significant and remarkable that in case of correcting errors of type 'square', the proposed code allows simple soft decoding without the Viterbi algorithm.

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