PAPER

Derivation on Bit Error Probability of Coded QAM Using Integer Codes

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SUMMARY In this paper we present the exact expressions for the bit error probability over a Gaussian noise channel of coded QAM using single error correcting integer codes. It is shown that the proposed integer codes have a better performance with respect to the lower on the bit error probability for trellis coded modulation.

key words: *integer codes, finite rings, QAM, BEP, AWGN*

1. Introduction

Coded modulation refers to the process of combined and jointly optimized channel coding and modulation schemes. It has been studied extensively by Ungerboeck [1], [2], Imai and Hirakawa [3] and others. In 1982, Ungerboeck constructed a trellis code that maps the input sequence into signal points of a fixed signal constellation by a method referred to as *set partitioning*. This technique is referred to as trellis coded modulation (TCM). An alternative technique that allows the coding theorists to deal with larger and more complicated constellations, is block coded modulation [4]–[6].

The concept of block coded modulation is the following. Each signal point of the signal constellation that is under consideration is matched by a symbol of a finite ring, for instance by symbols of the ring $\mathbb{Z}_A$ of integers modulo A. The information sequence is mapped into a sequence of symbols of $\mathbb{Z}_A$ and coded by a code over the same ring. Hence, based on the correspondence between elements of the ring and signal points of the constellation, the encoder transforms the input information sequence into a sequence of signals.

Here we restrict ourselves to the basic definitions and refer the reader to the above mentioned papers for more details.

Any linear code C can be represented by a generator matrix or a parity check matrix. Let $\mathbf{H}$ be an $r \times n$ matrix. The subset of $\mathbb{Z}_A^n$, defined by

$$C = \{\mathbf{c} = (c_1, \dots, c_n) \in \mathbb{Z}_A^n \mid \mathbf{c}\mathbf{H}^T = \mathbf{0}\} \tag{1}$$

is a linear code over $\mathbb{Z}_A$.

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Sometimes it is more useful to consider the cosets of C, i.e., to replace $\mathbf{0}$ in (1) with a vector $\mathbf{d} \in \mathbb{Z}_A^r$.

In this paper we restrict ourselves only to the applications of codes with $r = 1$, namely to the codes, which are defined as follows:

Definition 1. [6] Let $\mathbb{Z}_A$ be the ring of integers modulo A. An *integer code* of length n with *weight sequence* $\mathbf{w} = (w_1, w_2, \dots, w_n) \in \mathbb{Z}_A^n$, $w_j \neq 0$, is referred to as a subset of $\mathbb{Z}_A^n$, defined by

$$C(\mathbf{w}, d) = \left\{\mathbf{c} \in \mathbb{Z}_A^n \mid \sum_{i=1}^{n} c_i w_i = d \mod A\right\} \tag{2}$$

where $d \in \mathbb{Z}_A$.

Obviously, $C(\mathbf{w}, d)$ does not exist for any $d \neq 0$. For more details we refer [7].

Assume that a signal point s_i is sent through an AWGN-channel. At the other end the detector estimates the received signal r_i and gives signal point s_j at the output. If $j \neq i$ the detector has taken a wrong decision. In terms of block codes over $\mathbb{Z}_A$ the aforesaid can be described in the following way. When a codeword $\mathbf{c} \in C(\mathbf{w}, d)$ is sent through a noisy channel the received vector can be written in the form

$$\mathbf{r} = \mathbf{c} + \mathbf{e},$$

where $\mathbf{e} = (e_1, \dots, e_n) \in \mathbb{Z}_A^n$ denotes the error vector. It is clear that the different signal points have not the same chance to be a result of decision process. The probability signal point s_j to appear at the output of the detector depends on the Euclidean distance between s_j and really-sent signal s_i. In terms of codes over $\mathbb{Z}_A$ it means that the elements of $\mathbb{Z}_A$ are not equally probable as a value taken by e_i. Which elements of $\mathbb{Z}_A$ are more probable depends on the chosen indexing of the signal points by the elements of $\mathbb{Z}_A$. Therefore, it makes sense to consider (there is a point in considering) the next definition.

Definition 2. [6] The code $C(\mathbf{w}, d)$ is said to be a *single* $(\pm e_1, \pm e_2, \dots, \pm e_s)$*-error correctable* if it can correct any single error with value $\pm e_i$, $i = 1, \dots, s$. □

Obviously, $C(\mathbf{w}, d)$ is a single $(\pm e_1, \pm e_2, \dots, \pm e_s)$-error correctable code if and only if the subsets $\{\pm w_j e_1, \pm w_j e_2, \dots, \pm w_j e_s\} \subset \mathbb{Z}_A$, are pairwise disjoint and of the same cardinality $2s$, for any $j = 1, 2, \dots, n$. Thus, we have

$$A \geq 2sn + 1.$$

Some classes of single error correctable codes are constructed in [7] and the following main results are obtained there.

Theorem 1. Let $Z_A^\star$ be the set of all invertible elements of Z_A and $G = \{1, g_2, \ldots, g_n, -1, -g_2, \ldots, -g_n\}$ be a subgroup of $Z_A^\star$ of even cardinality $|G| = 2n$. If $e_i e_j^{-1} \notin G$, $e_i, e_j \in \mathbb{Z}_A^\star$ or the integer e_i divides A, but $|e_i G| = 2n$ and $e_i \notin e_j G$, then the code with weight sequence $\mathbf{w} = (1, g_2, \ldots, g_n)$ is $(\pm e_1, \pm e_2, \ldots, \pm e_s)$ single error correctable.

Theorem 2. Let $A = t^k + 1$. The integer code over $\mathbb{Z}_A$ with a weight sequence consisting of the elements of the set $\mathbf{W} = \{a_0 t^{k-1} + a_1 t^{k-2} + \ldots + a_{k-1}\}$, satisfying

$$\left|\begin{array}{rcl} 0 & \leq a_0 \leq & \lfloor\frac{t-2}{2}\rfloor \\ a_0 & \leq a_1 \leq & t-2-a_0 \\ \min\{1+a_0, a_1\} & \leq a_2 \leq & t-1-a_0 \\ & \vdots & \\ \min\{1+a_0, a_{k-3}\} & \leq a_{k-2} \leq & t-1-a_0 \\ 1+a_0 & \leq a_{k-1} \leq & t-1-a_0 \end{array}\right.$$

is a single $(\pm 1, \pm t, \ldots, \pm t^{k-1})$-error correctable. Moreover, the length of the code is given by

$$n = \sum_{a=0}^{\lfloor\frac{t-2}{2}\rfloor} \frac{(t-1-2a)^{k-2} - (t-1-2a)}{t-2-2a} + \sum_{a=0}^{\lfloor\frac{t-2}{2}\rfloor} (t-1-2a)^{k-1}.$$

For the proofs and details we refer to [7].

In this paper we investigate the problem of finding exact expressions for the bit error probability over a Gaussian noise channel of coded QAM using integer codes correcting a single error of given type.

We shall compare our codes with trellis codes since the latter are well known and often used in many applications. Our goal is not to make a complete comparison with all existing codes used for code modulation. The codes based on algebraic integers [8], [9] are very elegant and powerful tool for QAM, but they require fixed parameters (nevertheless they are enough for effective implementations) Integers codes considered in the paper are more flexible and they are originally intended for multidimensional modulation. With this paper we want to show that they work well for QAM, too. Our examples are chosen in order to illustrate the flexibility of integer codes (see Example 4).

2. BER for Coded L^2-QAM and Other Examples

Let us consider an M-QAM constellation with average energy E per symbol. Then, in the uncoded case, the average energy per bit is $E_b^u = E/\log_2 M$. Suppose we use a coded M-QAM scheme with an integer code C of length n. Let m information bits be coded by a block of n signals. Then the average energy per bit is $E_b^c = \frac{nE}{m}$. Therefore

$$\frac{E_b^u}{E_b^c} = \frac{m}{n \log_2 M} \stackrel{\text{def}}{=} \mathcal{R}_b.$$

The notation $\mathcal{R}_b$ is used since the above ratio shows not only the loss of energy, but also the *resulting bit rate* of the used code.

Let P_s^u and P_s^c be the probabilities for an error per symbol in a uncoded and coded cases, respectively. We define *code gain*

$$\mathcal{G} \stackrel{\text{def}}{=} \mathcal{R}_b \frac{P_s^u}{P_s^c}$$

as a measure of our gain when we use coded M-QAM. Note that $\mathcal{G}$ is not a constant, but a function of a signal-to-noise ratio.

Let C be capable of correcting up to t-errors of given type. Nevertheless that herein we give examples only for codes correcting one error, we shall consider the general case.

Suppose that in the uncoded case the detector demodulate correctly the received signal with an average probability per symbol q_u. When the code C is used then any t symbols from a sequence of n symbols can be correctly detected even if the received signal is out of the typical decision region $\mathcal{D}$. More precisely, for t of n symbols the decision region $\mathcal{D}'$ is wider than $\mathcal{D}$ and this leads to a probability of correct detection $q_c > q_u$.

The probability for i among n sent symbols the received signals to lie in $\mathcal{D}' \setminus \mathcal{D}$ is

$$\binom{n}{i}(q_c - q_u)^i q_u^{n-i}.$$

Thus, the average probability per symbol Q_s^c of correct decision when the code C is used is given by

$$Q_s^c = \sqrt[n]{\sum_{i=0}^{t} \binom{n}{i}(q_c - q_u)^i q_u^{n-i}}. \qquad (3)$$

In the considered case $t = 1$, we have

$$Q_s^c = q_u \sqrt[n]{1 + n(q_c - q_u)/q_u}. \qquad (4)$$

From an implementation point of view a better measure for performance is the probability a bit emitted by the source to be erroneously received. It is referred to as *bit-error rate* (BER). This probability also allows comparisons among modulation systems with different values of M and different codes used. Since BER depends also on the chosen mapping of the source bits onto the signals in the constellation this comparison is not an easy task in general. One approach to estimating the bit-error rate P_b is the following:

Let $\mu = \mathcal{R}_b \log_2 M$ source bits be coded by a symbol of the chosen M-ary constellation. Suppose that the resulting BER is $P_b = p$. Then the probability $(1-p)^\mu$ of correct decoding these μ bits at the receiver should coincide with the average probability per symbol Q_s, i.e.

$$(1-p)^\mu = 1 - P_s,$$

where P_s is the symbol error probability. Therefore

$$p = 1-(1-P_s)^{\frac{1}{\mu}}$$
$$= \frac{1}{\mu}P_s\left(1+\frac{\mu-1}{2\mu}P_s+\frac{(\mu-1)(2\mu-1)}{6\mu^2}P_s^2+\cdots\right)$$

For enough small values of P_s a good approximation is the often given in the literature lower bound

$$P_b \mathrel{\tilde{\geq}} \frac{1}{\mu}P_s. \tag{5}$$

Therefore

$$P_b^c \approx \frac{1}{\mathcal{R}_b \log_2 M}\left(1-q_u\sqrt[n]{1+n(q_c-q_u)/q_u}\right). \tag{6}$$

The above shows that the code gain $\mathcal{G}$ can be consider as a measure for our gain in bit-error rate since

$$\mathcal{G} = \frac{\mathcal{R}_b \log_2 M}{\log_2 M}\cdot\frac{P_s^u}{P_s^c} \approx \frac{P_b^u}{P_b^c}.$$

The square L^2-QAM constellation

First we shall calculate q_c for a square L^2-QAM constellation in the case of AWGN-channel with two-side power spectral density $N_0/2$. Let $d = \Delta_0/2$ be the half minimum distance between the signal points (along each of in-phase and quadrature axes).

The relation between d and the average energy per bit E_b is given (see e.g. [10, Sect. 7.6]) by

$$d = \sqrt{\frac{3E_b^u \log_2 L}{L^2-1}}. \tag{7}$$

Let denote $\gamma = d/\sqrt{N_0}$. Then

$$\gamma = \sqrt{\frac{3\log_2 L}{L^2-1}\cdot\frac{E_b^u}{N_0}} = \sqrt{\frac{3\log_2 L}{L^2-1}\rho} \tag{8}$$

where $\rho = E_b^u/N_0$.

Since in-phase and quadrature components of the signal are independently detected, when L^2-QAM is used without coding, the probability q_u of correct detection of the received signal is a square of the corresponding probability for one-dimensional case (L-AM). Therefore, the average probability of correct demodulation per signal point (see for example [10, Sect. 7.6] or [11]) is equal to

$$q_u = \frac{1}{L^2}[1+(L-1)\mathrm{erf}(\gamma)]^2 = \left(1-\frac{L-1}{L}\mathrm{erfc}(\gamma)\right)^2, \tag{9}$$

where

$$\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-u^2}\,du$$

is the error function and $\mathrm{erfc}(x) = 1-\mathrm{erf}(x)$ is the complementary error function. ($\mathrm{erf}(-x) = -\mathrm{erf}(x)$, $\lim_{x\to\infty}\mathrm{erf}(x) = 1$)

Therefore

6	5	4	5	6
5	3	2	3	5
4	2	1	2	4
5	3	2	3	5
6	5	4	5	6

Fig. 1 Groups of "cross" type errors on L^2-QAM.

$$P_s^u = 1-q_u = \frac{L^2-1}{L^2}-\frac{(L-1)}{L^2}\cdot\mathrm{erf}(\gamma)\,(2+(L-1)\mathrm{erf}(\gamma)). \tag{10}$$

Let an integer code C be used. Assume that C can correct one error of the type "cross," i.e., when the result of detection $\hat{s}(t)$ is a signal point placed left, right, up or down of the indeed sent signal point $s(t)$. In this case the decision region is a union of decision regions of the five signal points (the sent signal point and its four neighbors).

The signal points are split into 6 groups with different decision regions (see Fig. 1), as follows:

For the signal points in Group 1, i.e. signal points in the inner $(L-4)\times(L-4)$ square, the decision region $\mathcal{D}_1'$ has exactly the form of "cross" and it is defined by

$$\left\{\begin{array}{c}-3d \leq \mathbf{n}_I \leq 3d\\ -d \leq \mathbf{n}_Q \leq d\end{array}\right\} \cup \left\{\begin{array}{c}-d \leq \mathbf{n}_I \leq d\\ -3d \leq \mathbf{n}_Q \leq 3d\end{array}\right\},$$

where $\mathbf{n}_I$ and $\mathbf{n}_Q$ are the in-phase and quadrature components of the noise $\mathbf{n}$, respectively.

Hence the probability of correct detection is

$$q_{11} = 2\Pr(-3d \leq \mathbf{n} \leq 3d)\Pr(-d \leq \mathbf{n} \leq d) - (\Pr(-d \leq \mathbf{n} \leq d))^2,$$

which gives

$$q_{11} = \mathrm{erf}(\gamma)[2\mathrm{erf}(3\gamma)-\mathrm{erf}(\gamma)].$$

In similar way one can obtain the probabilities

$$q_{12} = \frac{1}{2}\mathrm{erf}(\gamma)[3\mathrm{erf}(3\gamma)-2\mathrm{erf}(\gamma)+1]$$
$$q_{13} = \mathrm{erf}(\gamma)[\mathrm{erf}(3\gamma)-\mathrm{erf}(\gamma)+1]$$
$$q_{14} = \frac{1}{2}[2\mathrm{erf}(\gamma)\mathrm{erf}(3\gamma)-\mathrm{erf}^2(\gamma)+\mathrm{erf}(3\gamma)]$$
$$q_{15} = \frac{1}{4}[3\mathrm{erf}(\gamma)\mathrm{erf}(3\gamma)-2\mathrm{erf}^2(\gamma)+\mathrm{erf}(3\gamma)+\mathrm{erf}(\gamma)+1]$$
$$q_{16} = \frac{1}{4}[2\mathrm{erf}(\gamma)\mathrm{erf}(3\gamma)-\mathrm{erf}^2(\gamma)+2\mathrm{erf}(3\gamma)+1]$$

for Group 2, 3, 4, 5 and 6, respectively.

Since $L^2q_c = (L-4)^2q_{11}+4(L-4)q_{12}+4q_{13}+4(L-4)q_{14}+8q_{15}+4q_{16}$, we have

3	3	2	3	3
3	3	2	3	3
2	2	1	2	2
3	3	2	3	3
3	3	2	3	3

Fig. 2 Groups of "square" type errors on L^2-QAM.

$$q_c = \frac{1}{L^2}\Big[2(L-1)(L-2)\mathrm{erf}(\gamma)\mathrm{erf}(3\gamma) - (L-1)^2\mathrm{erf}^2(\gamma) + 2(L-2)\mathrm{erf}(3\gamma) + 2(L-1)\mathrm{erf}(\gamma) + 3\Big]. \quad (11)$$

Let us now consider the case when C can correct one error of the type "square," i.e., when the decision region $\mathcal{D}'_1$ is defined by

$$\left\{ \begin{array}{l} -3d \le \mathbf{n}_I \le 3d \\ -3d \le \mathbf{n}_Q \le 3d \end{array} \right\}.$$

In this case the signal points are split into 3 groups according to their decision region (see Fig. 2).

Similarly to the calculations for the type "cross," we can obtain the following probabilities of correct decision:

$$q_{21} = \mathrm{erf}^2(3\gamma)$$

$$q_{22} = \frac{1}{2}[\mathrm{erf}^2(3\gamma) + \mathrm{erf}(3\gamma)]$$

$$q_{23} = \frac{1}{4}[\mathrm{erf}^2(3\gamma) + 1] + \frac{1}{2}\mathrm{erf}(3\gamma)$$

for the new first, second and third group, respectively.

Since $L^2 q_c = (L-4)^2 q_{21} + 8(L-4)q_{22} + 16q_{23}$, we have

$$q_c = \frac{1}{L^2}\Big[(L-2)^2\mathrm{erf}^2(3\gamma) + 4(L-2)\mathrm{erf}(3\gamma) + 4\Big]. \quad (12)$$

Below we give several examples of L^2-QAM coded with a single error (of type "cross" and type "square") correcting integer codes.

Example 1. (16-QAM constellation) For uncoded 16-QAM modulation according to (8), (9) and (10)

$$\gamma = \sqrt{\frac{2}{5}\rho}, \quad q_u = \frac{1}{16}(1 + 3\mathrm{erf}(\gamma))^2 \quad \text{and}$$

$$P_s^u = \frac{15}{16} - \frac{3}{16}\mathrm{erf}(\gamma)\,(2 + 3\mathrm{erf}(\gamma)).$$

Now let us see what we gain if an integer code is used. Substituting $t = 4$ and $k = 2$ in Theorem 2, we obtain a single $(\pm1, \pm4)$-error correctable code C' over $\mathbb{Z}_{17}$ with a weight sequence $\mathbf{w} = (1, 2, 3, 6)$. Let us assume that the points of the 16-QAM constellation are numbered by the integers from 1 to 16 beginning from the left upper corner to the bottom right corner. To assign each symbol in a codeword with a point in the constellation, we exclude codewords that contain zeros from C'. The new code obtained is denoted by C, where $|C| = 3856$, which implies that 11 information bits can be transmitted per codeword. Then the code C, of rate $\mathcal{R}_b = 11/16$, will be capable to correct any single error of type "cross." But it is necessary to implement the code table of C to map each 11 bit information block to its codeword. Since $2^{11} = 2048 \ll 3856$ we can even choose codewords corresponding to signal points with lowest possible energy. According to (11) we have

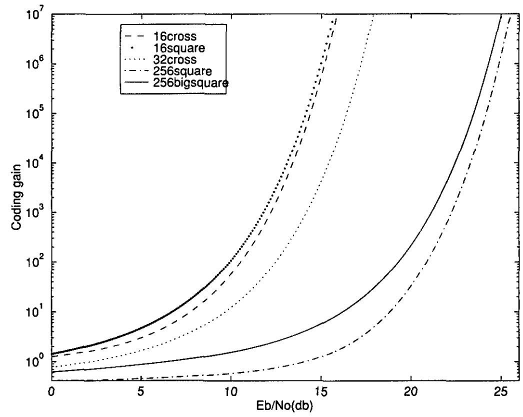

Fig. 3 Code gain $\mathcal{G}$ (in dB) versus SNR for Examples 1–4.

$$q_c = \frac{1}{16}[12\mathrm{erf}(\gamma)\mathrm{erf}(3\gamma) - 9\mathrm{erf}^2(\gamma) + 4\mathrm{erf}(3\gamma) + 6\mathrm{erf}(\gamma) + 3].$$

Since $n = |\mathbf{W}| = 4$, then using (4) we obtain

$$P_s^c = 1 - q_u\sqrt[4]{1 + 4(q_c - q_u)/q_u}.$$

Code gain $\mathcal{G}$ as a function of SNR is plotted in Fig. 3.

Example 2. (16-QAM constellation) Using Theorem 1 and the same indexing as in Example 1 of the points of 16-QAM constellation we can obtain a single $(\pm1, \pm3, \pm4, \pm5)$-error correctable code C' over $\mathbb{Z}_{17}$ with $\mathbf{w} = (1, 2)$. The code C' can correct single error of type "square." The bit rate of the code is $\mathcal{R}_b = 3/8$. If we map 0000 into 16 the code gives rate 1/2 (the value 0 cannot be obtained in a codeword). Note that no code table is required in this example.

According to (12) we have for q_c:

$$q_c = \frac{1}{4}[\mathrm{erf}^2(3\gamma + 2\mathrm{erf}(3\gamma) + 1]$$

The code length $n = 2$ and using (4) we obtain

$$P_s^c = 1 - \sqrt{q_u(2q_c - q_u)}.$$

Example 3. (32-QAM constellation) Let us consider a 32-QAM constellation indexed by the elements of $\mathbb{Z}_{32}$ as shown in Fig. 4. This constellation is a square 6^2-QAM constellation whose 4 vertices are cut. Following the construction given in Theorem 1 let us take $G = \{1, 15, -1, -15\}$.

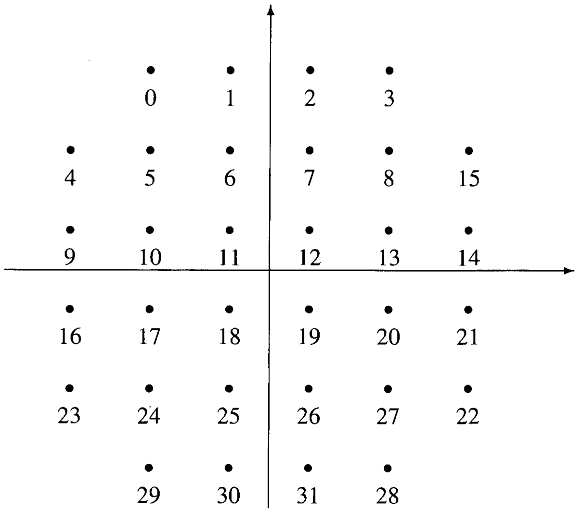

Fig. 4 A 32-QAM signal space constellation.

The cosets G, $3G$, $5G$ and $7G$ are pairwise disjoint. Thus the code C over Z_{32} with a weight sequence $w = (1, 15)$ is a single $(\pm1, \pm3, \pm5, \pm7)$-error correctable. Note that no code table is required in this example. The resulting bit rate $\mathcal{R}_b = 1/2$.

For the considered constellation $\gamma = d/\sqrt{N_0}$ and signal-to-noise ratio $\rho = E_b^u/N_0$ are related by

$$\gamma = \sqrt{\rho/4}.$$

Similar to the case of L^2-QAM constellation, one can calculate that

$$q_u = \frac{1}{32}[23\mathrm{erf}^2(\gamma) + 6\mathrm{erf}(\gamma) + 2\mathrm{erf}(\sqrt{2}\gamma) + 1]$$

and $P_s^u = 1 - q_u$.

Now let us determine the probability of an error when the considered code C is used. This code allows any single error of type "cross" to be corrected but for the chosen constellation the calculations slightly differ from the ones in the case of L^2-QAM. There are 5 types of decision regions: $\mathcal{D}'_1$, $\mathcal{D}'_2$, $\mathcal{D}'_3$, $\mathcal{D}'_4$, $\mathcal{D}'_5$, corresponding to sets of 4, 8, 4, 8 and 8 points, respectively. The corresponding probability of correct detection are respectively given by

$$q'_{11} = \mathrm{erf}(\gamma)[2\mathrm{erf}(3\gamma) - \mathrm{erf}(\gamma)]$$

$$q'_{12} = \frac{1}{2}\mathrm{erf}(\gamma)[3\mathrm{erf}(3\gamma) - 2\mathrm{erf}(\gamma) + 1]$$

$$q'_{13} = \frac{1}{4}[4\mathrm{erf}(\gamma)\mathrm{erf}(3\gamma) - 3\mathrm{erf}^2(\gamma) + 2\mathrm{erf}(\gamma) + 1]$$

$$q'_{14} = \frac{1}{8}\Big[6\mathrm{erf}(\gamma)\mathrm{erf}(3\gamma) - 3\mathrm{erf}^2(\gamma) + 2\mathrm{erf}(3\gamma) + 2\mathrm{erf}\left(\sqrt{2}\gamma\right) + 1\Big]$$

$$q'_{15} = \frac{1}{8}\Big[8\mathrm{erf}(\gamma)\mathrm{erf}(3\gamma) - 4\mathrm{erf}^2(\gamma) + 2\mathrm{erf}(3\gamma) + 2\mathrm{erf}\left(2\sqrt{2}\gamma\right) - \mathrm{erf}^2(2\sqrt{2}\gamma) + 1\Big] + \mathbf{I}_5,$$

where

$$\mathbf{I}_5 = \frac{\gamma}{2\sqrt{\pi}}\int_0^1 \mathrm{erf}(\gamma u)e^{-\gamma^2(4-u^2)}\,du$$

Since $q_c = \frac{1}{32}[4q'_{11} + 8q'_{12} + 4q'_{13} + 8q'_{14} + 8q'_{15}]$, then

$$q_c = \frac{1}{32}[38\mathrm{erf}(\gamma)\mathrm{erf}(3\gamma) - 22\mathrm{erf}^2(\gamma) + 4\mathrm{erf}(3\gamma) + 6\mathrm{erf}(\gamma) + 2\mathrm{erf}(\sqrt{2}\gamma) + 2\mathrm{erf}(2\sqrt{2}\gamma) - \mathrm{erf}^2(2\sqrt{2}\gamma) + 3 + 8(\mathbf{I}_5)]$$

The block length is $n = 2$, hence, the bit error probability is equal to

$$P_s^c = 1 - \sqrt{q_u(2q_c - q_u)}.$$

Example 4. (256-QAM square constellation) In this case, applying (7) and (8) for $L = 16$, we get that

$$\gamma = \sqrt{\frac{4}{85}\rho}$$

is the relation between γ and signal-to-noise ratio ρ. The probabilities q_u and P_s^u for the uncoded case are given by (9) and (10), i.e

$$q_u = \frac{1}{256}[1 + 15\mathrm{erf}(\gamma)]^2$$

and $P_s^u = 1 - q_u$.

Let us index each signal point s_{ij} with a pair $(i, j) \in \mathbb{Z}_{16} \times \mathbb{Z}_{16}$ of elements of $\mathbb{Z}_{16}$, where i is the number of the row and j is the number of the column which s_{ij} is placed in. The counting begin from the left bottom corner to up and to right, respectively. A given byte is mapped into signal point s_{ij}, if its left 4 bits and its right 4 bits are the binary representation of i and j, respectively.

Let us consider two codes over $\mathbb{Z}_{16}$:

- A single ±1-error correctable code C_1 with $\mathbf{W} = (1, 2, 3, 4, 5, 6, 7)$ and
- A single $(\pm1, \pm2)$-error correctable code C_2 with $\mathbf{W} = (1, 3)$.

In the case when the code C_1 is used, any six signal points (corresponding to 6 bytes at the input of the modulator) $s_{i_1j_1}$, $s_{i_2j_2}$, $s_{i_3j_3}$, $s_{i_4j_4}$, $s_{i_5j_5}$, $s_{i_6j_6}$ are followed by such an additional signal s_{ab} that $(a, i_1, i_2, i_3, i_4, i_5, i_6)$ and $(b, j_1, j_2, j_3, j_4, j_5, j_6)$ are codewords of C_1. In this case the probability q_c is given by (11), and namely:

$$q_c = \frac{1}{256}[184\mathrm{erf}^2(3\gamma) + 56\mathrm{erf}(3\gamma) + 16].$$

The code length $n = 7$ and using (4) we have

$$P_s^c = 1 - q_u\sqrt[7]{1 + 7(q_c - q_u)/q_u}.$$

The encoding with C_2 is the same. The unique difference is that the rate of C_2 is not 6/7, but 1/2. However, in this case much more different types of errors are correctable. The region $\mathcal{D}'$ is a "big square" with a side of length $5d$ and with a center - the sent signal point, i.e.,

$$\left\{ \begin{array}{l} -5d \leq \mathbf{n}_I \leq 5d \\ -5d \leq \mathbf{n}_Q \leq 5d \end{array} \right\}.$$

Calculations similar to the ones described above give

$$q_c = \frac{1}{256}[3 + 13\mathrm{erf}(5\gamma)]^2.$$

and

$$P_s^c = 1 - \sqrt{q_u(2q_c - q_u)}.$$

3. Comparison with Trellis Coded Modulation

Integer coded modulation (ICM) has an obvious advantage over trellis coded modulation (TCM): less complexity that results in faster encoding and decoding procedures. It makes ICM better suited for real time applications. But to compare theoretically the performance of ICM and TCM is not a simple task since what is evaluated for TCM is the probability of an error event $P(e)$, and not the probability P_s. An error event of length l occurs when instead of the really-sent sequences of signals the output of the demodulator is another one corresponding to a trellis path that splits from the correct path at a given time, and re-merges exactly l symbol-times later. The Euclidean distance between these two paths is greater or equal to δ_{free}. An upper bound for $P(e)$ and P_b can be found in [12], [13]. A lower bound for $P(e)$ is given (see [14, Sect. 12.4]) by

$$P(e) \geq \frac{1}{2}\mathrm{erfc}\left(\frac{\delta_{free}}{2\sqrt{N_0}}\right). \qquad (13)$$

Since the error event is realized by erroneously detecting l consequence signals (the length of an error event consisting of two paths of distance δ_{free}), the assumption

$$P_s \approx \sqrt[l]{P(e)}$$

gives a base to compare the performance of ICM and TCM. Hence, we use (5) to compare BEP of TCM and ICM.

Another acceptable approach to comparing the two modulation schemes is doing simulation experiments, but the description of such experiments is out of the range of this paper.

16-QAM constellation. Let us compare the ICM schemes from Example 1 to the TCM that maps 3 source bits to a signal of 16-QAM given in [2]. In this case $\delta_{free}^2 = 5\Delta_0^2 = 20d^2$ and according to its trellis diagram it is realized by an error event of length 3. Thus we assume that the probability

$$P_s = \sqrt[3]{\frac{1}{2}\mathrm{erfc}\left(\frac{\sqrt{5}d}{\sqrt{N_0}}\right)}$$

Figure 5 shows the comparison between the symbol error probabilities versus signal-to-noise ratio (SNR).

32-QAM constellation. The comparison between the coded case in Example 3 and TCM with $\delta_{free}^2 = 6\Delta_0^2 = 24d^2$ ([2]) as functions of signal-to-noise ratio is plotted in Fig. 6.

256-QAM constellation. The comparison of ICM schemes from Example 4 to TCM (where $\delta_{free}^2 = 8\Delta_0^2 = 32d^2$ [2]) as functions of signal-to-noise ratio are plotted in Fig. 7.

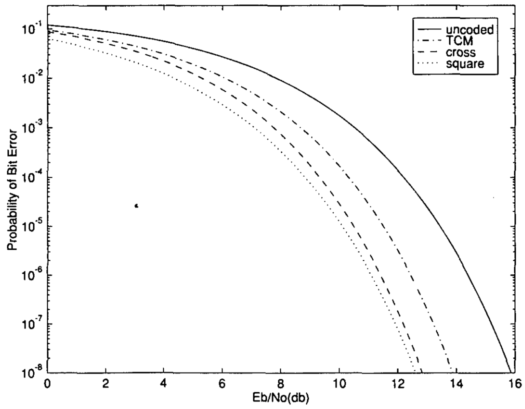

Fig. 5 A comparison of bit error probability versus signal-to-noise ratio between uncoded, coded (type "cross" and "square") 16QAM and TCM.

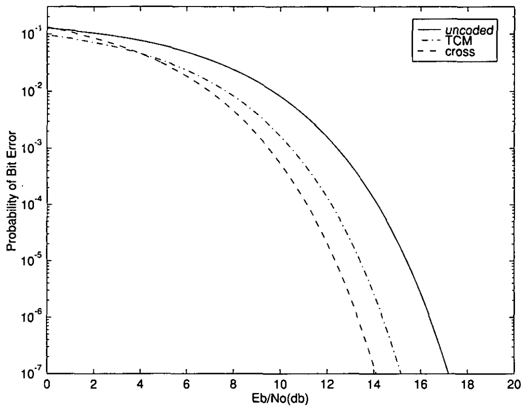

Fig. 6 A comparison of bit error probability versus signal-to-noise ratio between coded 32QAM (from Example 3) and TCM.

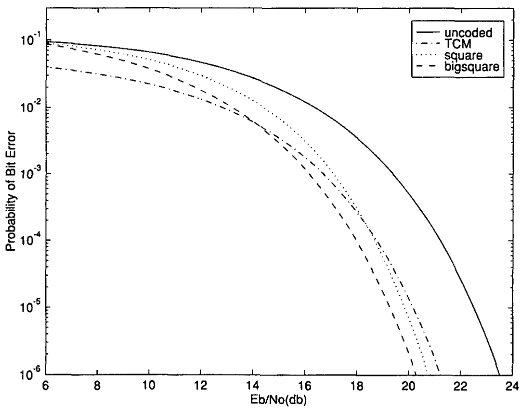

Fig. 7 A comparison of bit error probability versus signal-to-noise ratio between TCM uncoded and coded (type "square" and "big square") 256-QAM.

4. Conclusion

In this paper we derived the exact formulae on bit error probability of L^2-QAM constellation and some other specific constellations coded by single error correctable integer codes. The comparison of bit error probability versus signal-to-noise ratio between integer coded modulation and trellis coded modulation shows that ICM has a better performance and less complexity. As a disadvantage of ICM could be pointed that not for any 2^k-QAM constellation can be found a good code over $\mathbb{Z}_{2^k}$ and corresponding indexing of the points. This results in decreasing the code rate. But the approach used in Example 4 shows how to avoid this disadvantage. Adding points to the constellation is another, nevertheless not so good, approach. The using of integer codes capable of correcting more than one error makes it possible to improve the performance, but increases the decoding complexity.

Therefore we can conclude that ICM is an equal (even better than TCM) opportunity for coded modulation and should be applied by the constructors of modems.

Acknowledgement

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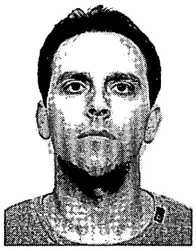

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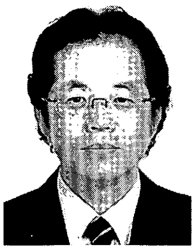

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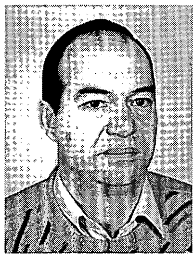

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