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Double Error Correctable Integer Code and Its Application to QAM

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Abstract

A new construction of double error correctable integer codes and their application to QAM scheme are presented. Soft decoding algorithm for multiple error correctable integer code which are suited to any modulation scheme (QAM, PSK, ASK) will be introduced. Comparison of bit error rate (BER) versus signal-to-noise ratio (SNR) between soft and hard decoding using integer coded modulation (ICM) shows us that we can obtain approximately 2 dB coding gain.

1. INTRODUCTION

Coded modulation is an efficiently combined scheme of coding and modulation techniques. It has been investigated extensively by Ungerboeck [1, 2], Imai and Hirakawa [3] and others. In 1982, Ungerboeck constructed a trellis code that maps the input sequence into signal points of a fixed signal constellation by a method referred to as *set partitioning*. This method is called trellis coded modulation (TCM). An alternative which allows us to deal with a variety of constellations is block coded modulation [4, 5]. In block coded modulation, each point of the signal constellation corresponds to a symbol of a finite ring of integers modulo A denoted by $\mathbb{Z}_A$. An information sequence is mapped into a sequence of symbols in $\mathbb{Z}_A$ and coded by a code over $\mathbb{Z}_A$.

Codes over finite rings and in particular codes over finite rings of integers with their applications in coding theory have been studied in numerous papers. The earliest paper is due to I. Blake [6, 7]. Some other works on codes over $\mathbb{Z}_A$ are [8, 9, 10]. M. Nilsson [11] discusses linear block codes over integer rings in order to improve the performance of PSK communication systems. A. Han Vinck, H. Morita [5] investigated these codes with a view to frame synchronization and coded modulation.

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Integer codes are codes defined over finite rings of integers. The original form of integer codes have been found in [12] where an integer code to correct a single insertion/deletion error per codeword was described.

Here, in this article, we are going to present a new construction of integer codes capable of correcting 2 errors of type (± 1). Because of its simple structure we are able to reduce the encoding and decoding complexity compared with TCM. As we shall see later, using integer codes with the introduced soft decoding algorithm, we can gain approximately 2 dB over the hard decoding. The aim of this paper is to show the flexibility of integer codes and their application to QAM scheme for decreasing probability of bit error over AWGN channel. The method we use here to encode and decode QAM scheme gives us an idea how to construct multiple error correcting integer codes with low encoding and decoding complexity and apply them in the communication systems.

In Section 2 we give necessary definitions and new construction of double (± 1) error correcting integer codes. In Section 3 we shall derive a theoretical bound of the bit error rate (BER) for the constructed integer codes. Soft decoding algorithm will be presented in Section 4. Conclusion remarks are given in Section 5.

2. NECESSARY DEFINITIONS AND CONSTRUCTION OF (± 1) DOUBLE ERROR CORRECTING INTEGER CODE

Definition 1. [5] Let $\mathbb{Z}_A$ be the ring of integers modulo A. An *integer code* of length n with check matrix $H \in \mathbb{Z}_A^{n\times m}$, is referred to as a subset of $\mathbb{Z}_A^n$, defined by

$$\mathcal{C}(\boldsymbol{w}, d) = \{\boldsymbol{c} \in \mathbb{Z}_A^n \mid \boldsymbol{c}H^T = \boldsymbol{d} \mod A\}$$

where $\boldsymbol{d} \in \mathbb{Z}_A^m$.

Assume that a signal point s_i is sent through an AWGN-channel. At the other end the detector estimates the received signal r_i and gives signal point s_j

at the output. If $j \neq i$ the detector has taken a wrong decision. In terms of block codes over $\mathbb{Z}_A$ the aforesaid can be described in the following way. When a codeword $\boldsymbol{c} \in \mathcal{C}(\boldsymbol{w}, d)$ is sent through a noisy channel the received vector can be written in the form

$$\boldsymbol{r} = \boldsymbol{c} + \boldsymbol{e},$$

where $\boldsymbol{e} = (e_1, \ldots, e_n) \in \mathbb{Z}_A^n$ denotes the error vector. It is clear that the different signal points have not the same chance to be a result of decision process. The probability signal point s_j to appear at the output of the detector depends on the Euclidean distance between s_j and really-sent signal s_i. In terms of codes over $\mathbb{Z}_A$ it means that the elements of $\mathbb{Z}_A$ are not equally probable as a value taken by e_i. Which elements of $\mathbb{Z}_A$ are more probable depends on the chosen indexing of the signal points by the elements of $\mathbb{Z}_A$. Therefore, it makes sense to consider (there is a point in considering) the next definition.

Definition 2. The code $\mathcal{C}(\boldsymbol{w}, d)$ is said to be a *double* $(\pm e_1, \pm e_2, \ldots, \pm e_s)$*-error correctable* if it can correct any single and double errors with values $\pm e_i$, $i = 1, \ldots, s$.

Using Definition 2. one can easily derive the following bound for A, when $\boldsymbol{d} \in \mathbb{Z}_A$:

$$A \geq 2sn(sn - s + 1) + 1$$

and when $\boldsymbol{d} \in \mathbb{Z}_A^2$:

$$A \geq \sqrt{2sn(sn - s + 1) + 1}. \tag{1}$$

Definition 3. A double $(\pm e_1, \pm e_2, \ldots, \pm e_s)$-error correctable code $\mathcal{C}(\boldsymbol{w}, d)$ of block length n is called *perfect*, when $A = 2sn(sn - s + 1) + 1$ (or $A = \sqrt{2sn(sn - s + 1) + 1}$ when $\boldsymbol{d} \in \mathbb{Z}_A^2$).

We notice that if an integer code is perfect there is a one-to-one correspondence between $\mathbb{Z}_A^m$ and the error vectors. When the code is not perfect we do not have such a correspondence for some of the elements of $\mathbb{Z}_A^m$. Even if the syndrome value is one of those elements, we can say that at least an error(s) which is not of the type $(\pm e_1, \pm e_2, \ldots, \pm e_s)$ appears.

The definition of double $(\pm e_1, \pm e_2, \ldots, \pm e_s)$-error correctable integer code shows us that to construct such a code of length n, with a check matrix H

$$H = \begin{pmatrix} h_{11} & h_{12} & h_{13} & h_{14} & \ldots & h_{1n} \\ h_{21} & h_{22} & h_{23} & h_{24} & \ldots & h_{2n} \end{pmatrix}$$

over $\mathbb{Z}_A^2$ is a task which is equivalent to splitting $\mathbb{Z}_A^2$ into pairwise disjoint subsets each of which contains one of the following subsets:

$$S^1 = \{(h_{1i}e_i^\star, h_{2i}e_i^\star)\} \tag{2}$$

and

$$S^2 = \left\{(h_{1i}e_i^\star + h_{1j}e_j^\star, h_{2i}e_i^\star + h_{2j}e_j^\star)\right\} \tag{3}$$

for $i \neq j, 1 \leq i, j \leq n$ and $e_i^\star, e_j^\star \in \{\pm e_1, \pm e_2, \ldots, \pm e_s\}$ with pairwise different elements.

Here we are going to investigate double (± 1)-error correctable integer codes and their application to QAM. In this case $s = 1$ and using (1) we have

$$A \geq \sqrt{2n^2 + 1}.$$

As we shall see later it is rather difficult to find an exact form of the check matrix H for integer code correcting multiple errors of given type. Theorem 1. gives us one construction for H in case of (± 1) double error correcting integer code.

Let us first define a check matrix H as:

$$H = \begin{pmatrix} 0 & 1 & 2 & 3 & \ldots & n-1 \\ h_0 & h_1 & h_2 & h_3 & \ldots & h_{n-1} \end{pmatrix}$$

where $h_i \in \mathbb{Z}_{n+1}$ for $i = 1, \ldots, n$ and let us consider the $n + 1$ sets in the following way:

$$H_k = \begin{cases} \{0\} & \text{for} \quad k = 0 \\ \{h_k, h_i + h_j, h_l - h_{l-k} : 0 \leq i < j \leq n-1; \\ i + j = k; l = k, \ldots, n-1\} & \text{for} \quad k = 1, 2, 3 \\ \{h_k, h_i + h_j, h_l - h_{l-k}, h_m + h_t : i + j = k; \\ 0 \leq i < j \leq n-1; l = k, \ldots, n; \\ 0 < m < t < n-1; \\ 0 < m + t = 2n - k + 1\} & \text{for} \quad k = 4, \ldots, n-1 \\ \{h_i + h_j, h_m + h_t : 0 \leq i < j \leq n-1; \\ i + j = n + 1; 0 < m < t < n; \\ m + t = n + 1\} & \text{for} \quad k = n. \end{cases}$$

Theorem 1. The integer code of length n over $\mathbb{Z}_{2n+1}$ with a check matrix H is a double (± 1)-error correctable if for every $k : h' \neq h''$, where $h', h'' \in H_k$.

Proof: As we noted above, to proof the theorem we should show that all the elements in the sets S^1 and S^2 are pairwise distinct. Because the code is (± 1)-error correctable the sets $S^1, S^2 \in \mathbb{Z}_{2n+1}$ have the form $S^1 = (s_1^1, s_2^1) = \{(\pm h_{1i}, \pm h_{2i})\}$ and $S^2 = (s_1^2, s_2^2) = \{(\pm h_{1i} + \pm h_{1j}, \pm h_{2i} + \pm h_{2j})\}$ Since we know the exact form of the first row of the matrix H we can calculate the values of s_1^1 and s_1^2.

For a given $k \in \mathbb{Z}_{n+1}$, let us investigate the set $T_k = \left\{s_2^1, s_2^1\right\}$ of the values $s_2^1 \in S^1, s_2^2 \in S^2$ for which the corresponding $s_1^1 \in S^1$ and $s_1^2 \in S^2$ satisfy $s_1^1 = s_1^2 = k$. So, the only thing left to show to complete the proof is that the set T_k consist of pairwise distinct elements. One can see, after some calculations, that $H_k = T_k, k \in \mathbb{Z}_{n+1}$ as sets. But the elements of H_k are pairwise distinct, so are the elements of T_k.

$\square$

3. CALCULATION OF BIT ERROR RATE FOR THE PROPOSED INTEGER CODES

Suppose we use a coded QAM scheme with an integer code C of length n. Let m information bits be coded by a block of n signals. Let $\mathcal{C}$ be capable of correcting up to t-errors of given type and q_u and q_c be the average probability of a correct decision per signal point for uncoded QAM and coded QAM, respectively.

The average probability per symbol Q_s^c of correct decision when the code $\mathcal{C}$ is used is given by [13]

$$Q_s^c = \sqrt[n]{\sum_{i=0}^{t} \binom{n}{i} (q_c - q_u)^i q_u^{n-i}}.$$

From an implementation point of view a better measure for performance is the probability a bit emitted by the source to be erroneously received. It is referred to as *bit-error rate* (BER). Since BER depends also on the chosen mapping of the source bits onto the signals in the constellation this comparison is not an easy task in general. One approach to estimating the bit-error rate P_b is the following:

Let P_s^u and P_s^c be the probabilities for an error per symbol in a uncoded and coded cases, respectively. Let $\mu = m/n$ source bits be coded by a symbol of the chosen constellation. Suppose that the resulting BER is $P_b = p$. Then the probability $(1-p)^\mu$ of correct decoding these μ bits at the receiver should coincide with the average probability per symbol Q_s, i.e.

$$(1-p)^\mu = 1 - P_s,$$

where P_s is the symbol error probability. Therefore

$$\begin{aligned} p &= 1 - (1-P_s)^{\frac{1}{\mu}} \\ &= \frac{1}{\mu} P_s \left(1 + \frac{\mu-1}{2\mu} P_s + \frac{(\mu-1)(2\mu-1)}{6\mu^2} P_s^2 + \cdots \right) \end{aligned}$$

For enough small values of P_s a good approximation is the often given in the literature lower bound

$$P_b \gtrsim \frac{1}{\mu} P_s.$$

Therefore

$$P_b^c \approx \frac{n}{m} \left(1 - Q_s^c\right). \qquad (4)$$

(0, 0)	(0, 1)	(0, 2)	(0, 3)	(0, 4)	(0, 5)	(0, 6)	(0, 7)
(1, 0)	(1, 1)	(1, 2)	(1, 3)	(1, 4)	(1, 5)	(1, 6)	(1, 7)
(2, 0)	(2, 1)	(2, 2)	(2, 3)	(2, 4)	(2, 5)	(2, 6)	(2, 7)
(3, 0)	(3, 1)	(3, 2)	(3, 3)	(3, 4)	(3, 5)	(3, 6)	(3, 7)
(4, 0)	(4, 1)	(4, 2)	(4, 3)	(4, 4)	(4, 5)	(4, 6)	(4, 7)
(5, 0)	(5, 1)	(5, 2)	(5, 3)	(5, 4)	(5, 5)	(5, 6)	(5, 7)
(6, 0)	(6, 1)	(6, 2)	(6, 3)	(6, 4)	(6, 5)	(6, 6)	(6, 7)
(7, 0)	(7, 1)	(7, 2)	(7, 3)	(7, 4)	(7, 5)	(7, 6)	(7, 7)

Figure 1: Indexing of the signal points in 64-QAM

Let us consider M-QAM constellation of square type. In this case we have that $M = 2^{2k},\ k = 1, 2, \ldots$. Let us index each signal point s_{ij} in M-QAM constellation with a pair $(i,j) \in \mathbb{Z}_{2^k} \times \mathbb{Z}_{2^k}$ of elements of $\mathbb{Z}_{2^k}$ where i is the number of the row and j is the number of the column which s_{ij} is placed in. The counting begin from the left bottom corner to up and to right, respectively (see Fig. 1 for the case $M = 16$). A given byte is mapped into signal point s_{ij}, if its left k bits and its right k bits are the binary representation of i and j, respectively.

Example 1. (64-QAM constellation) Let us index each signal point s_{ij} with a pair $(i,j) \in \mathbb{Z}_8 \times \mathbb{Z}_8$ as we described above (see Fig. 1). Using Theorem 1. we can construct a double (± 1)-error correctable code $\mathcal{C}$ of length $n = 4$ over $\mathbb{Z}_9$ with

$$H = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 3 & 1 & 0 & 2 \end{pmatrix}.$$

In this case, any two signal points $s_{i_1 j_1}, s_{i_2 j_2}$ are followed by two additional signals $s_{a_1 b_1}, s_{a_2 b_2}$ such that (i_1, i_2, a_1, a_2) and (j_1, j_2, b_1, b_2) are codewords of $\mathcal{C}$.

Note that we can not use all the codewords of the code $\mathcal{C}$. The error type the code $\mathcal{C}$ can correct is called "square" type of error [13].

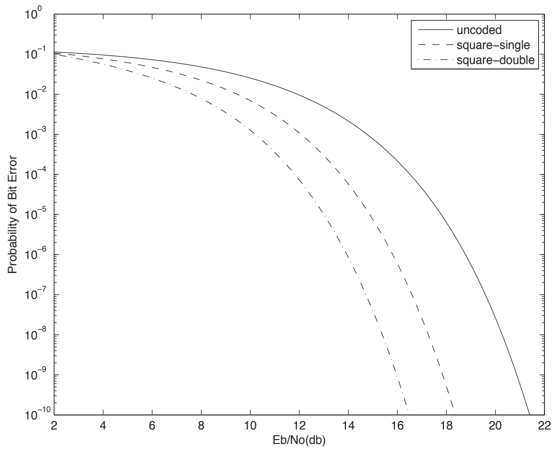

Figure 2: A comparison of bit error probability versus signal-to-noise ratio between uncoded and coded 64QAM (with single and double "square" type error correctable integer codes).

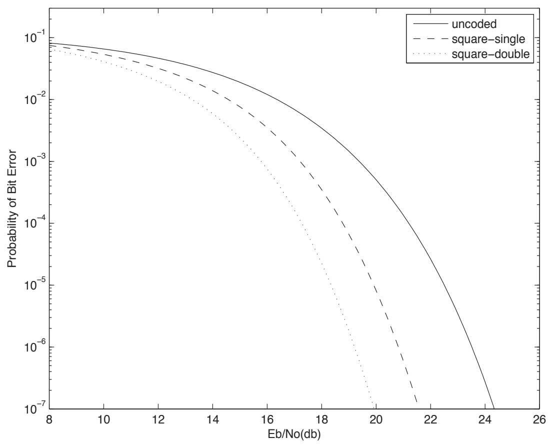

Figure 3: A comparison of bit error probability versus signal-to-noise ratio between uncoded and coded 256QAM (with single and double "square" type error correctable integer codes).

For the probabilities q_u and q_c (type "square") of L^2-QAM over an AWGN channel, Kostadinov et al. [13] obtained

$$q_u = \{1 + (L-1)\,\mathrm{erf}(\gamma)\}^2 / L^2,$$
$$q_c = \left\{(L-2)^2\,\mathrm{erf}^2(3\gamma) + 4(L-2)\,\mathrm{erf}(3\gamma) + 4\right\} / L^2$$

where

$$\gamma = \sqrt{\frac{3\log_2 L}{L^2-1}\frac{E_b}{N_0}}$$

and $\mathrm{erf}(x)$ is the error function.

Hence, for 64-QAM we have:

$$q_u = \frac{1}{64}\left[1 + 7\,\mathrm{erf}(\gamma)\right]^2$$
$$q_c = \frac{1}{64}\left[36\,\mathrm{erf}^2(3\gamma) + 24\,\mathrm{erf}(3\gamma) + 4\right]$$

where $\gamma = \sqrt{E_b/7N_0}$.

Therefore, to obtain P_b^c for the integer code from Example 1 we use (4) , where $n = 4$, $m = 10$ and $t = 2$.

Figure 2. shows that using double (± 1)-error correcting integer code we obtain approximately 2 dB and 5 dB coding gain compare to the single error (type "square") correctable integer code and uncoded 64QAM, respectively.

□

Example 2. (256-QAM constellation) Let us index each signal point s_{ij} with a pair $(i, j) \in \mathbb{Z}_{17} \times \mathbb{Z}_{17}$ as we did in the previous example. Using Theorem 1. we can construct a double (± 1)-error correctable code $\mathcal{C}$ of length $n = 8$ over $\mathbb{Z}_{17}$ with

$$H = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 1 & 5 & 8 & 7 & 3 & 6 & 2 & 0 \end{pmatrix}.$$

In this case, for the probabilities q_u and q_c we obtain

$$q_u = \frac{1}{256}\left[1 + 15\,\mathrm{erf}(\gamma)\right]^2$$
$$q_c = \frac{1}{256}\left[196\,\mathrm{erf}^2(3\gamma) + 56\,\mathrm{erf}(3\gamma) + 4\right]$$

where

$$\gamma = \sqrt{\frac{4}{85}\frac{E_b}{N_0}}.$$

Substituting $n = 8$, $m = 47$ and $t = 2$ in (4) we obtain the probability of bit error. Figure 3. shows us the comparison of bit error probability versus signal-to-noise ratio between uncoded and coded 256QAM with single and double "square" type error correctable integer codes.

□

4. SOFT DECODING ALGORITHM

If we want to obtain more coding gain we should apply soft decoding algorithm instead of hard decoding algorithm (which we used above). In this Section we are going to introduce a soft decoding algorithm for single/multiple $(\pm l_1, \pm l_2, \dots, \pm l_s)$ error correctable integer codes which is based of the algorithm given in [14].

Let us assume that a signal constellation (QAM, PSK, ASK) is coded by single/multiple $(\pm l_1, \pm l_2, \dots, \pm l_s)$ error correctable integer code $\mathcal{C}$ of length n over $\mathbb{Z}_A$.

Let a modulated codeword is represented by $\boldsymbol{x} = (x_1, x_2, \dots, x_n)$ where x_i is a signal point in that constellation and x is transmitted over an AWGN channel. The sampled channel output sequence is given by $\boldsymbol{y} = (y_1, y_2, \dots, y_n)$.

The decoder makes *hard decisions* on each channel output y_i to estimate the transmitted code symbol and selects the nearest signal point x_j to y_i in the constellation. Let $\boldsymbol{r} = (r_1, r_2, \dots, r_n)$, where $r_i \in \mathbb{Z}_A$ is a signal point in the constellation, is the received sequence at the decoder.

In soft decoding we utilize the analog received samples y_i, $1 \le i \le n$, to find the most probable codeword to be transmitted in the sense of the maximum likelihood estimation.

We propose the following algorithm for soft decoding using $(\pm l_1, \pm l_2, \dots, \pm l_s)$ error correctable integer code $\mathcal{C}$ of length n over $\mathbb{Z}_A$.

In: The channel output y and the received sequence $\boldsymbol{r} = (r_1, r_2, \dots, r_n)$.

Out: The decoded codeword $\tilde{\boldsymbol{c}} = (\tilde{c}_1, \tilde{c}_2, \dots, \tilde{c}_n)$.

Step 1. Calculate the squared distance $\Delta^2[i, \varepsilon]$ between y_i and each of the signal points associated with $r_i + \epsilon \pmod M$ where $\epsilon \in \mathcal{L} = \{-l_1, -l_2, \dots, -l_s, 0, l_1, l_2, \dots, l_s\}$.

Step 2. Compute the syndrome value $s = \sum_{i=1}^{n} w_i r_i \pmod A$.

Step 3. Let $\mathcal{E}[s]$ be the set of all the vectors $\boldsymbol{e} = (e_1, e_2, \dots, e_n) \in \mathcal{L}^n$ such that

$$\sum_{i=1}^{n} w_i e_i = s \pmod A.$$

Then find the vector $\boldsymbol{e}^* = (e_1^*, e_2^*, \dots, e_n^*) \in \mathcal{E}[s]$ that minimizes $\sum_{i=1}^{n} \Delta^2[i, e_i^*]$.

Step 4. Output $\tilde{\boldsymbol{c}} = \boldsymbol{r} - \boldsymbol{e}^*$ and stop.

The above algorithm accomplishes maximum likelihood decoding for an AWGN channel. In Step 3 an exhaustive search is performed to find $\boldsymbol{e}^*$ among $\mathcal{E}[s]$. It is reasonable if n is relatively small, say $n = 4$. For a large value of n, we can utilize a trellis with $n+1$ layers, in each of which there are A states. Each state in the ith layer for $i = 0, 1, \dots, n$ is indexed by $k = 0, 1, \dots, A-1$. A pair of numbers $(d_k^{(i)}, e_k^{(i)})$ is attached to the kth state in the ith layer. Here $d_0^{(0)} = 0$ and $d_k^{(0)} = \infty$ for $k \in \mathbb{Z}_A \backslash \{0\}$ and $d_k^{(i)}$ is given by

$$d_k^{(i)} = \min_{\epsilon \in \{\mathcal{L}} \left\{ d_{k+\epsilon w_i}^{(i-1)} + \Delta^2[i, \epsilon] \right\} \qquad (5)$$

for $i = 1, 2, \dots, n$ and $k = 0, 1, \dots, A-1$. Moreover, $e_k^{(i)} = \epsilon^*$ where ϵ^* is an element in $\mathcal{L}$ that achieves the minimum value of (5). The Viterbi algorithm can sequentially calculate $\{(d_k^{(i)}, e_k^{(i)}), k = 0, 1, \dots, A-1\}$ for $i = 1, 2, \dots, n$ in a similar way discussed in [15]. After the calculation is completed, e_n^* is given by the value of $e_s^{(n)}$ associated to the s-th state in the n-th layer where s is the syndrome value obtained in Step 2. The other $e_{n-1}^*, \dots, e_1^*$ are computed in descending order by using the following recursive equations on k_i, $i = n-1, \dots, 1$ with the initial value $k_n = s$:

$$\begin{aligned} k_i &= k_{i+1} + e_{i+1}^* w_{i+1}, \\ e_i^* &= e_{k_i}^{(i)}. \end{aligned}$$

Note that we can apply soft decoding algorithm to any modulation scheme (QAM, ASK, PSK).

In Fig. 4 and Fig. 5 are shown simulation results, using hard and soft decoding algorithms, of a comparison of bit error probability versus signal-to-noise ratio between uncoded and type "square" coded 64-QAM and 256-QAM, respectively.

From the simulation result we can conclude that using double (± 1) error correcting integer code with soft decoding algorithm we gain approximately 2 dB compared to hard decoding and 5 dB to an uncoded QAM.

5. CONCLUSIONS

In this paper we presented double (± 1) error correcting integer code and its application to QAM scheme. We proposed soft decoding algorithm for integer codes. From the experimental results of the comparison on bit error probability versus signal-to-noise ratio between hard and soft decoding algorithms using integer coded modulation we can conclude that the soft decoding algorithm has better performance for small value of code length n. For larger value of n the soft decision decoder is substantially more complex than the hard decision decoder. The using of integer codes capable of correcting more than one error makes it possible to improve the performance, but increases the complexity.

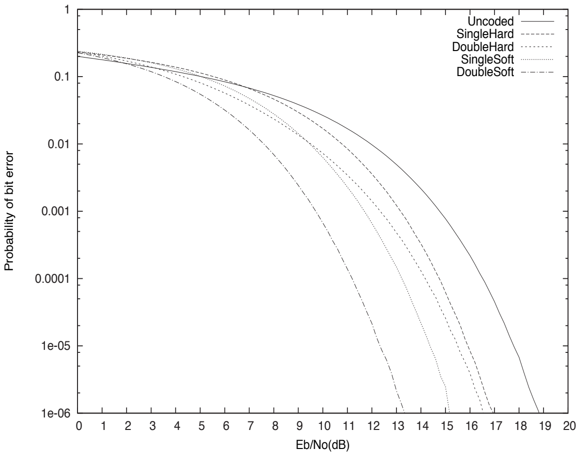

Figure 4: Simulation results of a comparison of bit error probability versus signal-to-noise ratio between uncoded and coded (error type "square") 64QAM using hard and soft decoding algorithms .

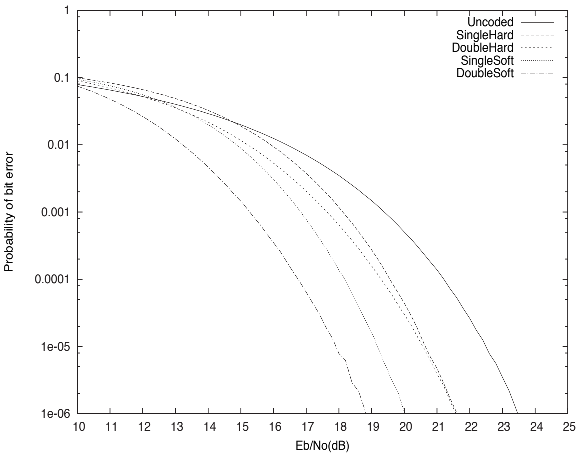

Figure 5: A comparison of bit error probability versus signal-to-noise ratio between uncoded and coded (error type "square") 256QAM using hard and soft decoding algorithms .

References

[1] G. Ungerboeck, "Channel coding with multilevel/phase signals," *IEEE Trans. Inform. Theory*, vol. IT-28, no. 1, pp. 55–66, Jan. 1982.

[2] G. Ungerboeck, "Trellis-coded modulation with redundant signal sets," *IEEE Communications Magazine*, pp. 5–21, Feb. 1987.

[3] H. Imai and S. Hirakawa, "A new multilevel coding method using error-correcting codes," *IEEE Trans. Inform. Theory*, vol. IT-23, No. 1, pp. 656–662, Jan. 1977.

[4] R. Baldini F and P. G. Farrell, "Coded modulation based on rings of integers modulo-q," *IEE Proc. Commun.*, part 1, vol. 141, no. 3, pp. 129-136, 1994.

[5] A.J. Han Vinck and H. Morita, "Codes over the ring of integer modulo m," *IEICE Trans. on Fundamantals*, vol. E81-A, pp. 2013–2018, Oct. 1998.

[6] I. Blake, "Codes over certain rings ," *Information and Control*, v. 20, pp. 396-404, 1972.

[7] I. Blake, "Codes over integer residue rings ," *Information and Control*, v. 29, pp. 295-300, 1975.

[8] A. R. Calderbank, N. J. A. Sloane, "Modular and $p-$adic Cyclic Codes," *Designs, Codes and Cryptography*, vol 6, No 1, pp. 21-36, 1995.

[9] E. Spiegel, "Codes over Z_m," *Information and Control*, vol. 35, pp. 48-51, 1977.

[10] V. I. Levenstein and A. J. Han Vink, "Perfect (d, k) - codes capable of correcting single peak-shifts, " *IEEE Trans. Inform. Theory*, vol. 39, No. 2, pp. 656-662, 1993.

[11] M. Nilsson, Linear block codes over rings for phase shift keying, Thesis no. 331, Linköping University, 1993.

[12] R. R. Varshamov and G. M. Tenengolz , "One asymmetrical error-correctable codes, " (in Russian) *Avtomatika i Telematika*, vol. 26, No. 2, pp. 288-292, 1965.

[13] H. Kostadinov, H. Morita and N. Manev, "Derivation on Bit Error Probability of Coded QAM using Integer Codes," *IEICE Trans. on Fundamentals*, vol. E87-A, No. 12, pp. 3397-3403, Dec. 2004.

[14] H. Morita, A. J. Han Vink and H. Kostadinov, "On Soft Decoding of Coded QAM using Integer Codes," *International Symposium on Information Theory and its Applications (ISITA)*, Parma, Italy, pp. 1321-1325, Oct 2004.

[15] E. Zehavi and J. Wolf, "A new multilevel coding method using error-correcting codes", IEEE Trans. Inform. Theory, vol. IT-33, No. 2, pp. 196-202, March 1987.