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THE CLASS OF ALL SEMIGROUPS RELATED TO SEMIHYPERGROUPS OF ORDER 2

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ABSTRACT. This paper deals with semihypergroups of order two from the point of view of the model theory. We use basic knowledge to show that there are exactly 17 non-isomorphic semihypergroups of order two. Each of them corresponds in a canonical way to a semigroup of order three. We classify all of them by generalized identities a concept introduced by Lyapin. In particular, we classify all non-group semigroups of order three by one generalized identity.

1. Introduction

The concept of algebraic hyperstructures was introduced in 1934 by the French mathematician F. Marty [15]. He introduced the concept of hypergroups. This has been studied in the following decades by many mathematicians. For example, M. M. Zahedi et al. consider categories of several hyperstructures [19]. Algebraic hyperstructures are generalized classical algebraic structures. In a classical algebraic structure, the composition of two elements is an element of the universe of that structure, while in an algebraic hyperstructure, the composition of two elements is a non-empty subset of the universe, i.e., if H is the universe of a hyperstructure then the composition of two elements is an element of the set $\mathcal{P}^*(H) := \{X \subseteq H : X \neq \emptyset\}$. A semihypergroup (also called hypersemigroup or multisemigroup) was first investigated by P. Bonansinga and P. Corsini [2, 4] and later studied by many authors, for example by B. Davvaz (see e.g. [3, 5]), De Salvo et al. [17], D. Freni [7], K. Hila et al. [10], and V. Leoreanu [13]. For a semihypergroup with a universe H, we have two operations. One of them is the operation between the elements of H and the other one is between the elements in $\mathcal{P}^*(H)$. If we show these both operations with the same symbol, as several authors do, a confusion can arise. We will show these both operations by two symbols, namely $\circ$ as the operation between the elements of H and $*$ as the operation between the elements of $\mathcal{P}^*(H)$. The operation $*$ is determined by the operation $\circ$ in the following sense: $*\colon \mathcal{P}^*(H) \times \mathcal{P}^*(H) \to \mathcal{P}^*(H)$ with

$$A * B := \bigcup_{(a,b)\in A\times B} a \circ b.$$

We can easily see that the operation $*$ is well defined (see also [11]). A hyperstructure $(H, \circ)$ is said to be a *semihypergroup* if $\circ\colon H \times H \to \mathcal{P}^*(H)$ with $(a \circ b) * \{c\} = \{a\} * (b \circ c)$ or alternatively, $\bigcup_{x\in(a\circ b)} x\circ c = \bigcup_{x\in b\circ c} a\circ x$, for all $a, b, c \in H$. For convenience, we write also x instead of $\{x\}$, whenever

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$\{x\}$ is a singleton subset of H. So the "associative law" has the form $(a \circ b) * c = a * (b \circ c)$. In [8], M. Golmohamadian and M. M. Zahedi show that each deterministic finite automaton induces an associative hyperoperation $\circ$ on the set S of states. In particular $(S, \circ, 0)$, where 0 is the start state, is a so-called hyper K-algebra, a generalization of the concept of a BCK-algebra. Two semihypergroups $(H_1, \circ_1)$ and $(H_2, \circ_2)$ are called *isomorphic* if there is a bijection $f\colon H_1 \to H_2$ with $f(a \circ_1 b) = f(a) \circ_2 f(b)$ for all $a, b \in H_1$, where $f(a \circ_1 b) := \{f(x) : x \in a \circ_1 b\}$. Note that $*$ is an associative operation on $P^*(H)$ [11], i.e., $(\mathcal{P}^*(H), *)$ is a semigroup. In this sense, each semihypergroup can be regarded as a semigroup. But not conversely, i.e., not each semigroup with universe in $\mathcal{P}^*(H)$, for a suitable set H, corresponds to a semihypergroup in the previous mentioned sense. For example, let us consider the four-element group with the universe $\{\{1\}, \{2\}, \{3\}, \{1, 2\}\}$, where $\{1\}$ is the identity element and $\{3\}$ the element of order 2. Assume that the operation in this group is the operation $*$, which corresponds to the operation $\circ$ belonging to a semihypergroup. Then we have $\{1\} = \{2\} * \{1, 2\} = 2 \circ 1 \cup 2 \circ 2 = \{2\} * \{1\} \cup \{2\} * \{2\} = \{2, 3\}$, a contradiction. So the question arrises, for a given set H, which semigroups with universe in $\mathcal{P}^*(H)$, containing all singleton sets, correspond to a semihypergroup in the previous mentioned sense. More generally, which semigroups correspond to semihypergroups, i.e., given a semigroup S, is there a semihypergroup $(H, \circ)$ such that S is isomorphic to $\mathcal{P}^*(H)$ (in symbols: $S \cong \mathcal{P}^*(H)$). Note that the existence of such a semihypergroup has not be unique, i.e., for a given semigroup S, different semihypergroups $(H, \circ)$ with $S \cong \mathcal{P}^*(H)$ can exist. Take for example, the constant semigroup. The cardinality of the universe of a semihypergroup is denoted by the order of the semihypergroup. For more background about semihypergroups see [11–13].

Semihypergroups of order two are at the center of interest in this paper. In the next section, we will give a complete list of all semihypergroups of order two (up to isomorphism). In the third section, we classify all semigroups which are isomorphic to $\mathcal{P}^*(H)$ for a suitable semihypergroup $(H, \circ)$ of order two, i.e., we will classify semigroups of order three (recall, the order of a semigroup means the cardinality of its universe).

Algebraic structures can be classified by varieties in the sense of Birkhoff. It is known that neither the class of all semigroups of order three nor any of its subclasses forms a variety, except of the trivial variety. Therefore, we will use a concept, introduced by E. S. Lyapin [14] and A. E. Evseev [6], respectively.

Let $X := \{x_1, x_2, x_3, \ldots\}$ be a countable set of variables and let X^+ be the set of all words over the set X. An expression $u \approx v$ (with $u, v \in X^+$) will be called equation and let $Eq(X)$ be the set of all possible equations. A subset $\sigma \subseteq Eq(X)$ is called *disjunction of identities* (for short: DI) (see e.g. [18]). An DI $\sigma \subseteq Eq(X)$ is satisfied by a semigroup S (in symbols: $S \models \sigma$) if for all mappings $h\colon X \to S$, there is an equation $u \approx v \in \sigma$ such that $\overline{h}(u) = \overline{h}(v)$, where $\overline{h}\colon X^+ \to S$ denotes the unique determined homomorphic extension of h to the free word semigroup X^+. Let Σ be a set of $DI's$, i.e., $\Sigma \subseteq \mathcal{P}(Eq(X))$ is a subset of the power set of $Eq(X)$. Then we put $Mod\,\Sigma$ as the class of all semigroups S with $S \models \sigma$ for all $\sigma \in \Sigma$. We will call such a class $Mod\,\Sigma$ an alternative variety. An alternative variety is a generalization of the classical concept of a variety due to Birkhoff, namely taking Σ being a set of singleton sets. In particular, $Mod\,\Sigma$ is closed under isomorphic images. For more background about alternative varieties, we refer the reader to [18]. Recently, alternative varieties were used to describe particular classes of completely regular semigroups [1, 16].

We will show that there are five alternative varieties such that exactly the elements of these classes correspond to semihypergroups in the previous explained sense. In this way, we obtain a classification of a class of semigroups of order three by $DI's$.

In the last section, we classify all non-group semigroups of order three. All of them and exactly these have at least one two-element subsemigroup. But a two-element semigroup can be regarded as a semihypergroup of order two, where any composition of two elements is a singleton set. Such a semihypergroup will be called non-proper and proper otherwise. With other words, we classify semigroups S of order three such that there is a semihypergroup $(H, \circ)$ and a subsemigroup T of S (as usually, we write $T \leq S$ for a subsemigroup T of S) with $T \cong S^*(H)$, where $S^*(H)$ is the subsemigroup of $\mathcal{P}^*(H)$ generated by $\{\{x\} : x \in H\}$. This is stronger as simply saying that S has a subsemigroup of order two.

2. All semihypergroups of order two

The aim of this section is a complete list of all semihypergroups of order two (up to isomorphism). As already mentioned, each semigroup of order two can be regarded as a semihypergroup of order two, namely as a non-proper one. It is well-known that there are exactly five non-isomorphic semigroups of order two (see also [9]).

$\cdot$	a	b
a	a	a
b	a	a

Table S-1

$\cdot$	a	b
a	a	b
b	b	a

Table S-2

$\cdot$	a	b
a	a	b
b	b	b

Table S-3

$\cdot$	a	b
a	a	a
b	b	b

Table S-4

$\cdot$	a	b
a	a	b
b	a	b

Table S-5

It remains to determine all proper semihypergroups of order two. An element $a \in H$ of the universe of a semihypergroup $(H, \circ)$ will be called *idempotent* if $a \circ a = a$.

Lemma 2.1. *Let $(H, \circ)$ be a proper semihypergroup with an idempotent element. Then $(H, \circ)$ is isomorphic to one of the following semihypergroups:*

$\circ$	a	b
a	a	a
b	H	b

Table A_1

$\cdot$	a	b
a	a	b
b	H	b

Table A_2

$\cdot$	a	b
a	a	H
b	H	b

Table A_3

$\cdot$	a	b
a	a	a
b	a	H

Table A_4

$\cdot$	a	b
a	a	a
b	H	H

Table A_5

$\circ$	a	b
a	a	b
b	b	H

Table A_6

$\cdot$	a	b
a	a	b
b	H	H

Table A_7

$\cdot$	a	b
a	a	H
b	a	H

Table A_8

$\cdot$	a	b
a	a	H
b	b	H

Table A_9

$\cdot$	a	b
a	a	H
b	aH	H

Table A_{10}

P r o o f. Let $H = \{a, b\}$ be the universe of the semihypergroup and let a be the idempotent element. If $a \circ b = H$ then $H = (a \circ b) * b = a * (b \circ b)$. This implies that $b \circ b \neq a$, i.e., $(H, \circ)$ is isomorphic to one of the semigroups $A_1^*, A_2^*, A_3, A_8, A_9$, and A_{10}, where

$\circ$	a	b
a	a	H
b	b	b

Table A_1^*

$\circ$	a	b
a	a	H
b	a	b

Table A_2^*

If $b \circ a = H$ then we obtain again $b \circ b \neq a$ and $(H, \circ)$ is isomorphic to one of the semigroups A_1, A_2, A_3, A_7, and A_{10}. It is easy to verify that A_2 and A_2^* are isomorphic. Likewise, A_1 and A_1^* are isomorphic. The remaining semigroups are pairwise non-isomorphic.

Suppose that $a \circ b \neq H$ and $b \circ a \neq H$. Then $b \circ b = H$ and $(b \circ a) * b = b * (a \circ b)$ implies $a \circ b = b \circ a$, i.e., $(H, \circ)$ is isomorphic to A_4 or A_6.

We leave the reader to verify that all the listed structures are semihypergroups. $\square$

Lemma 2.2. *Let $(H, \circ)$ be a proper semihypergroup with the universe $H = \{a, b\}$ such that $a \circ a = b$. Then $(H, \circ)$ is isomorphic to the semihypergroup.*

$\circ$	a	b
a	b	H
b	H	H

Table A_{11}

Proof. We have $a * b = a * (a \circ a) = (a \circ a) * a = b * a$ and $b * b = (a \circ a) * b = a * (a \circ b)$. This shows that $b \circ b \neq H$ if and only if $a * b = b * a \neq H$. Hence, $(H, \circ)$ is isomorphic to A_{11}, which is obviously a semihypergroup. □

Lemma 2.3. *Let $(H, \circ)$ be a proper semihypergroup with the universe $H = \{a, b\}$ such that $a \circ a = b \circ b = H$. Then $(H, \circ)$ is isomorphic to the semihypergroup*

$\circ$	a	b
a	H	H
b	H	H

Table A_{12}

Proof. We have $H = H * b = (a \circ a) * b = a * (a \circ b)$. This implies $a \circ b \neq b$. Likely, $H = a * H = (a \circ b) * b$ implies $a \circ b \neq a$, i.e., $a \circ b = H$. Dually, we can show $b \circ a = H$ and hence, $(H, \circ)$ is isomorphic to A_{12}, which is obviously a semihypergroup. □

Summarizing the previous lemmas, we obtain the complete list of semihypergroups of order two.

Proposition 2.1. *There are exactly* 17 *semihypergroups of order two up to isomorphism.*

Proof. There are five semihypergroups which are non-proper. By Lemma 2.1, there are 10 pairwise non-isomorphic proper semihypergroups with at least one idempotent element. Let $(H, \circ)$ be a proper semihypergroup without an idempotent element and with the universe $H = \{a, b\}$. Then $a \circ a = b$ or $b \circ b = a$ or $a \circ a = b \circ b = H$. Thus, Lemma 2.2 and Lemma 2.3 show that there are exactly two non-isomorphic semihypergroups without idempotent elements. □

3. Classification by $DI's$

In this section, we consider the class $\mathcal{SH}$ of all all semigroups S such that there is a semihypergroup $(H, \circ)$ with $S \cong \mathcal{P}^*(H)$. Of course, the class $\mathcal{SH}$ is neither a variety nor a finite union of varieties. But we can show that $\mathcal{SH}$ is the union of five alternative varieties. All the semigroups in $\mathcal{SH}$ have order at most three, i.e., less than four elements. This can be realized by following DI:

$$\sigma_1 := \{x_i \approx x_j : 1 \leq i < j \leq 4\}.$$

Lemma 3.1. *Let S be a semigroup satisfying σ_1. Then S has at most three elements.*

Proof. Assume that $|S| \geq 4$. Then let a_1, a_2, a_3, and a_4 be pairwise different elements in S. Further, let $h \colon X \to S$ be a mapping with $h(x_i) = a_i$ for $i \in \{1, 2, 3, 4\}$ and $h(x) = a_1$ otherwise. Since $S \models \sigma_1$, we have $\overline{h}(x_i) = \overline{h}(x_j)$ for some $i, j \in \{1, 2, 3, 4\}$ with $i < j$. This gives $a_i = a_j$, a contradiction. □

In the same matter, one can prove that a semigroup S has at most n elements (for a natural number n) if $S \models \{x_i \approx x_j : 1 \leq i < j \leq n+1\}$. So, $Mod\{\{x_i \approx x_j : 1 \leq i < j \leq n+1\}\}$ is the class of all semigroups with order n or less.

We need the following nine DI´s in this section:

$\sigma_2 := \{x_1 x_2 \approx x_3 x_4\}$;

$\sigma_3 := \{x_1 x_2 \approx x_2 x_1\}$;

$$\begin{aligned}
\sigma_4 &:= \{x_1x_2 \approx x_1, x_1x_2 \approx x_2\};\\
\sigma_5 &:= \{x_1x_2 \approx x_1\};\\
\sigma_6 &:= \{x_1x_2 \approx x_2\};\\
\sigma_7 &:= \{x_ix_j \approx x_k : i,j,k \in \{1,2,3\}, i \neq k, j \neq k\} \cup \{x_1 \approx x_2, x_1 \approx x_3, x_2 \approx x_3\};\\
\sigma_8 &:= \sigma_4 \cup \{x_1^2 \approx x_1, x_1^2 \approx x_2, x_1^2x_2 \approx x_1x_2, x_1 \approx x_2\};\\
\sigma_9 &:= \{x_1^3 \approx x_1^2, x_2^2x_1 \approx x_2^2, x_1^2 \approx x_1, x_1^2 \approx x_2\};\\
\sigma_{10} &:= \sigma_4 \cup \{x_1x_2 \approx x_1^2, x_1^2x_2 \approx x_1x_2\}.
\end{aligned}$$

A semigroup S which is isomorphic to $\mathcal{P}^*(H)$ for a non-proper semihypergroup $(H,\circ)$ of order two has rank less than and equals 2 or rank 3, where the rank of S (in symbols: $\operatorname{rank} S$) is the least size of a generating set for S.

Lemma 3.2. *Let S be a semigroup with* rank 3*, isomorphic to a subsemigroup of $\mathcal{P}^*(H)$, for a semihypergroup $(H,\circ)$ of order two. Then $S \in Mod\{\sigma_1,\sigma_2\}$ or $S \in Mod\{\sigma_1,\sigma_3,\sigma_4\}$ or $S \in Mod\{\sigma_1,\sigma_5\}$ or $S \in Mod\{\sigma_1,\sigma_6\}$.*

Proof. Let $H = \{a,b\}$. Since $\operatorname{rank} S = 3$, the set $H \in \mathcal{P}^*(H)$ is not a composition of elements from $\mathcal{P}^*(H)$, i.e., $(H,\circ)$ is a non-proper semihypergroup. Thus, $(H,\circ)$ is a left-zero or right-zero semigroup or $(H,\circ)$ is the constant semigroup or it is the two-element band with zero-element. But, $(H,\circ)$ cannot be the two-element group since otherwise the non-identity element and H generates $\mathcal{P}^*(H)$, i.e., $\operatorname{rank}\mathcal{P}^*(H) \leq 2$ and thus $\operatorname{rank} S \leq 2$, a contradiction.

If $(H,\circ)$ is a left-zero or right-zero semigroup then $\mathcal{P}^*(H)$ is also a left-zero semigroup and a right-zero semigroup, respectively. Since $x * H = x\circ a \cup x \circ b = x$ for all $x \in H$, whenever $(H,\circ)$ is a left-zero semigroup, we can calculate that $\mathcal{P}^*(H) \models \sigma_5$, i.e., $S \in Mod\{\sigma_1,\sigma_5\}$. Dually, we can conclude that $S \in Mod\{\sigma_1,\sigma_6\}$, whenever $(H,\circ)$ is a right-zero semigroup. By similar reasons, we can verify that $S \in Mod\{\sigma_1,\sigma_2\}$, whenever $(H,\circ)$ is the constant semigroup.

If $(H,\circ)$ is the two-element band with zero 0 then $0 * H = 0 \circ a \cup 0 \circ b = 0$ and $H = H * H = x * H = H * x$ for the non-zero element x in H. This shows that $\mathcal{P}^*(H)$ satisfies both σ_3 and σ_4, i.e., $S \in Mod\{\sigma_1,\sigma_3,\sigma_4\}$. $\square$

Lemma 3.3. *Let S be a semigroup with* rank ≤ 2 *such that S is isomorphic to a subsemigroup of $\mathcal{P}^*(H)$ for some semihypergroup $(H,\circ)$ of order two. Then $S \in Mod\{\sigma_1,\sigma_7,\sigma_8,\sigma_9\}$ or $S \in Mod\{\sigma_1,\sigma_3,\sigma_4\}$.*

Proof. First, we suppose that $(H,\circ)$ is a proper semihypergroup and we want to show that σ_7, σ_8, and σ_9 are satisfied in $\mathcal{P}^*(H)$. We have to replace both the variables x_1 and x_2 by elements in $\mathcal{P}^*(H)$, where $H = \{a,b\}$ is assumed.

We start with σ_8. If we replace both variables x_1 and x_2 by the same element, say x, then the equation $x_1 \approx x_2$ provides the equality $x = x$. Let us replace x_1 and x_2 by different elements but not by H, say x_1 by a and x_2 by b. If we suppose that $x_1x_2 \approx x_1$, $x_1x_2 \approx x_2$, $x_1^2 \approx x_1$, and $x_1^2 \approx x_2$ do not provide an equality then we can conclude that $a \circ b = H$ and $a \circ a = H$. This gives $(a\circ a) * b = H * b = a \circ b \cup b \circ b = H = a \circ b$, i.e., we have the equality $(a \circ a) * b = a \circ b$, which is given by the equation $x_1^2x_2 \approx x_1x_2$. Let us replace x_1 by a and x_2 by H and let us assume that $x_1x_2 \approx x_1$, $x_1x_2 \approx x_2$, $x_1^2 \approx x_1$, and $x_1^2 \approx x_2$ do not provide an equality. Then we have $a * H = b$ and $a \circ a = b$. From $a * H$, we conclude that $a \circ a = a \circ b = b$. Thus, $a * (a * H) = a \circ b = b = a * H$, i.e., we have the equality $(a \circ a) * H = a * H = b$, which is given by the equation $x_1^2x_2 \approx x_1x_2$. Dually, the equation $x_1^2x_2 \approx x_1x_2$ provides an equality if we replace x_1 by b (instead by a). Finally, let us replace x_1 by H. Note that $H * H = H$ because at least one of the compositions of the elements in H gives H (since $(H,\circ)$ is a proper semihypergroup), i.e., the equation $x_1^2 \approx x_1$ gives the equality $H * H = H$. Next, we verify that $S \models \sigma_9$. If we replace both variables x_1 and x_2 by the same element x of $\mathcal{P}^*(H)$ then the equation $x_1x_2 \approx x_1^2$ provides the equality $x^2 = x^2$. Let us replace x_1 and x_2 by different elements, but different from H, say x_1 by a and x_2 by b. Suppose that $x_1x_2 \approx x_1^2$, $x_1x_2 \approx x_2^2$, $x_1^2 \approx x_1$, and $x_1^2 \approx x_2$ do not provide an equality. Then we have $a \circ b = H^2$ and $a \circ a = H$ which leads us to $(a \circ a) * a = H * a = a \circ a \cup b \circ a = H \cup b \circ a = H = a \circ a$. So, we have the equality $(a \circ a) * a = a \circ a$, which is given by the equation $x_1^3 \approx x_1^2$. Let us replace x_1 by a and x_2 by H. Admit that $x_1^2 \approx x_1$, and $x_1^2 \approx x_2$ does not provide an equality. Then we get $a \circ a = b$, i.e., $b^2 = a * H = a \circ a \,\cup\, a \circ b \neq a$. If $a * H = b$, then we conclude that $a \circ b = b$ and obtain the equality $a \circ a \circ a = a \circ a$ given by the equation $x_1^3 \approx x_1^2$. If $a * H = H$ then we consider two cases. Admit that $b \circ a \in \{a, H\}$. Using $H^2 = H$, we can conclude $H^2 * a = H * a = a \circ a \cup b \circ a = H = H^2$. Admit now that $b \circ a = b$. Then $H * a = b$ and thus $H * (H * a) = H * b = a \circ a \cup b \circ b = H = H^2$. Thus, we have the equality $H^2 * a = H^2$, given by the equation $x_2^2 x_1 \approx x_2^2$. Dually, we can argue, whenever we replace x_1 by b instead by a. Finally, we replace x_1 by H. Since $H^2 = H$ (since $(H, \circ)$ is a proper semihypergroup), the equation $x_1^3 \approx x_1^2$ provides the equality $H^3 = H^2$.

Note that $S \models \sigma_7$. In fact, there are $x, y \in H$ with $x \circ y = H$ since $(H, \circ)$ is a proper semihypergroup. If we replace the variables x_1, x_2, and x_3 by different elements then there are $i, j, k \in \{1, 2, 3\}$ with $i \neq k$ and $j \neq k$ such that x_i is mapped to x, x_j is mapped to y, and x_k is mapped to H. The equation $x_ix_j \approx x_k$ provides the equality $x \circ y = H$. But if we replace two variables of x_1, x_2, and x_3 by the same element x then one of the equations $x_1 \approx x_2$, $x_1 \approx x_3$ and $x_2 \approx x_3$ provides the equality $x = x$.

It remains to consider the case that $(H, \circ)$ is a non-proper semihypergroup. In this case, $(H, \circ)$ cannot be a band because $\operatorname{rank} \mathcal{P}^*(H) = 2$. Thus, $(H, \circ)$ is the two-element group and we get $x * H = H * x = H$ for all $x \in H$. This shows that $\mathcal{P}^*(H) \in \mathit{Mod}\{\sigma_1, \sigma_3, \sigma_4\}$, i.e., $S \in \mathit{Mod}\{\sigma_1, \sigma_3, \sigma_4\}$. □

Lemma 3.4. *Let S be a semigroup such that $S \cong \mathcal{P}^*(H)$ for some semihypergroup $(H, \circ)$ of order two. Then $S \models \sigma_{10}$.*

Proof. Let $H = \{a, b\}$. If we replace both variables x_1 and x_2 by the same element, say $x \in \mathcal{P}^*(H)$, then we get the equality $x^2 = x^2$ by the equation $x_1x_2 \approx x_1^2$. Let us replace x_1 by a and x_2 by b (if we replace x_1 by b and x_2 by a the we will have the same argumentation). Suppose that $x_1x_2 \approx x_1$ and $x_1x_2 \approx x_2$ give no equality. Then $a \circ b = H$ and $a * H = a \circ a \cup a \circ b = H$, i.e., $a * (a \circ b) = a * H = H = a \circ b$. Thus, the equation $x_1^2x_2 \approx x_1x_2$ provides the equality $a * (a \circ b) = a \circ b$. Let us replace x_1 by a and x_2 by H (if we replace x_1 by b then we can follow the same argumentation). If we admit that $x_1x_2 \approx x_1$ and $x_1x_2 \approx x_2$ give no equality then we obtain $a * H = b$, i.e., $a \circ b = a \circ a = b$. This shows that $a * H = a \circ a$, i.e., the equation $x_1x_2 \approx x_1^2$ provides equality. Finally, we replace x_1 by H and x_2 by a (if we replace x_2 by b then we can follow the same argumentation). Admit that $x_1x_2 \approx x_1^2$, $x_1x_2 \approx x_1$, and $x_1x_2 \approx x_2$ give no equality. Then we conclude that $H * a = b$ and either $H^2 = H$ or $H^2 = a$. The latter case is not possible. Otherwise we have $a \circ a = a$. This gives $a \in a \circ a \cup b \circ a = H * a = b$, a contradiction. Hence, $H^2 = H$, i.e., $H^2 * a = H * a$, which is given by the equation $x_1^2x_2 \approx x_1x_2$.

Altogether, we have shown that $S \models \sigma_{10}$, i.e., $S \in \mathit{Mod}\{\sigma_{10}\}$. □

Remark 1. It is easy to verify that the three-element group does not satisfy σ_{10}. In order to see this, we replace x_1 and x_2 by the both non-identity elements in the group. Since the composition of the both non-identity elements gives the identity element, we can easy verify that non of the four equations in σ_{10} will give an equality in the group.

If T is a two-element semigroup then T can be regarded as a non-proper semihypergroup $(T, \circ)$ of order two and T is isomorphic to a proper subsemigroup of $\mathcal{P}^*(T)$. So, we are interested in semigroups S which are isomorphic to $\mathcal{P}^*(H)$ for a suitable semihypergroup $(H, \circ)$ of order two. A classification of these semigroups will provide the following theorem, the main result of this section.

Theorem 3.1. *Let S be a non-trivial semigroup. Then the following statements are equivalent:*

(i) *There is a semihypergroup $(H, \circ)$ of order two such that S is isomorphic to a subsemigroup of $\mathcal{P}^*(H)$.*

(ii) $S \in Mod\{\sigma_1, \sigma_2\}$ *or* $S \in Mod\{\sigma_1, \sigma_3, \sigma_4\}$ *or* $S \in Mod\{\sigma_1, \sigma_5\}$ *or* $S \in Mod\{\sigma_1, \sigma_6\}$ *or* $S \in Mod\{\sigma_1, \sigma_7, \sigma_8, \sigma_9, \sigma_{10}\}$.

P r o o f. If S is a semigroup of order two then S satisfies both conditions (i) and (ii). In fact, we have already mentioned that (i) is satisfied. In order to see (ii), we have to mention that a two-element semigroup satisfies σ_1. Moreover, we can observe that S satisfies both $DI's$ σ_3 and σ_4 or one of the $DI's$ σ_2, σ_5, and σ_6.

Let us now admit that S has exactly three elements. The direction (i) $\Rightarrow$ (ii) is given by Lemma 3.2 until Lemma 3.4. We consider the converse direction and will show that for any semigroup S in the list below of all three-element semigroups, there is a semihypergroup $(H, \circ)$ of order two such that S is isomorphic to a subsemigroup of $\mathcal{P}^*(H)$, whenever S satisfies (ii).

$\cdot$	x	y	z
x	x	y	z
y	y	z	x
z	x	x	y

Table T_1

$\cdot$	x	y	z
x	y	z	y
y	z	y	z
z	y	z	y

Table T_2

$\cdot$	x	y	z
x	z	z	z
y	z	z	z
z	z	z	z

Table T_3

$\cdot$	x	y	z
x	z	z	z
y	z	y	z
z	z	z	z

Table T_4

$\cdot$	x	y	z
x	x	x	z
y	y	y	z
z	z	z	z

Table T_5

$\cdot$	x	y	z
x	y	z	y
y	z	y	z
z	y	z	y

Table T_6

$\cdot$	x	y	z
x	x	x	x
y	y	y	y
z	x	y	z

Table T_7

$\cdot$	x	y	z
x	x	y	x
y	x	y	y
z	x	y	z

Table T_8

$\cdot$	x	y	z
x	y	z	z
y	z	z	z
z	z	z	z

Table S_1

$\cdot$	x	y	z
x	z	y	x
y	y	y	y
z	x	y	z

Table S_2

$\cdot$	x	y	z
x	z	z	z
y	z	z	z
z	z	z	z

Table S_3

$\cdot$	x	y	z
x	z	z	z
y	z	y	z
z	z	z	z

Table S_4

$\cdot$	x	y	z
x	z	y	z
y	y	y	y
z	z	y	z

Table S_5

$\cdot$	x	y	z
x	z	x	z
y	x	y	z
z	z	z	z

Table S_6

$\cdot$	x	y	z
x	z	z	z
y	y	y	y
z	z	z	z

Table S_7

$\cdot$	x	y	z
x	z	y	z
y	z	y	z
z	z	y	z

Table S_8

$\cdot$	x	y	z
x	z	z	z
y	x	y	z
z	z	z	z

Table S_9

$\cdot$	x	y	z
x	z	x	z
y	z	y	z
z	z	z	z

Table S_{10}

$\cdot$	x	y	z
x	x	y	z
y	y	y	z
z	z	z	z

Table S_{11}

$\cdot$	x	y	z
x	x	z	z
y	z	y	z
z	z	z	z

Table S_{12}

$\cdot$	x	y	z
x	x	x	x
y	y	y	y
z	x	x	z

Table S_{13}

$\cdot$	x	y	z
x	x	y	x
y	x	y	x
z	x	y	z

Table S_{14}

$\cdot$	x	y	z
x	x	x	x
y	y	y	y
z	z	z	z

Table S_{15}

$\cdot$	x	y	z
x	x	y	z
y	x	y	z
z	x	y	z

Table S_{16}

First, we show that $T_1, T_2, T_3, T_4, T_5, T_6, T_7$, and T_8 do not satisfy (ii). Clearly, non of them is a constant semigroup or a left-zero semigroup or a right-zero semigroup. All the four semigroups T_5, T_6, T_7, and T_8 are not commutative. If T is a semigroup isomorphic to T_1 or T_2 or T_3 or T_4 then there are $a, b \in T$ such that $a \circ b \notin \{a, b\}$. This shows that $T \not\models \sigma_4$. It remains to show that all the eight semigroups do not belong to $Mod\{\sigma_1, \sigma_7, \sigma_8, \sigma_9, \sigma_{10}\}$. By Remark 1, we know that $T_1 \not\models \sigma_{10}$. Assume that $T_2 \models \sigma_8$. Then replacing x_1 by x and x_2 by z, we obtain $xz = x$ or $xz = z$ or $x^2 = x$ or $x^2 = z$ or $x^2 z = xz$. But it holds $xz = y$, $x^2 = y$, and $yz = z$ in T_2, a contradiction. Assume that $T_3 \models \sigma_9$. Then replacing x_1 by x and x_2 by y, we obtain $xy = x^2$ or $xy = y^2$ or $x^3 = x^2$ or $y^2 x = y^2$ or $x^2 = x$ or $x^2 = y$. But we have $xy = x$, $x^2 = z$, $y^2 = y$, $x^3 = zx = x$, and $y^2 x = yx = z$ in T_3, a contradiction. Assume that $T_4 \models \sigma_9$. Then replacing x_1 by x and x_2 by y, we obtain again $xy = x^2$ or $xy = y^2$ or $x^3 = x^2$ or $y^2 x = y^2$ or $x^2 = x$ or $x^2 = y$. But in T_4, it holds $x^2 = y^2 = z$, $xy = x$, and $zx = x$, a contradiction. The semigroups T_5, T_6, T_7, and T_8 do not satisfy the *DI* σ_7. We will show it for T_5. The argumentation for the remaining ones is similarly. Assume that $T_5 \models \sigma_7$. Replacing x_1 by x, x_2 by y, and x_3 by z, we obtain $xy = z$ or or $yx = z$ or $yz = x$ or $zy = x$ or $xz = y$ or $zy = x$ or $x^2 \in \{y, z\}$ or $y^2 \in \{x, z\}$ or $z^2 \in \{x, y\}$, i.e., $\operatorname{rank} T_5 \leq 2$, a contradiction.

In the remaining part of this proof, we will verify that for the other three-element semigroups S, there is a semihypergroup $(H, \circ)$ of order two such that $S \cong \mathcal{P}^*(H)$. Let us consider the semihypergroup A_{11}. Then $a \circ b = b \circ a = H$, i.e., all compositions of elements from $\mathcal{P}^*(H)$ are H, except of $a \circ a = b$. This shows that $\mathcal{P}^*(H) \cong S_1$. If $(H, \circ)$ is the semihypergroup A_{10} then we obtain by almost the same arguments that $\mathcal{P}^*(H) \cong S_4$ and if $(H, \circ)$ is the semihypergroup A_3 then $\mathcal{P}^*(H) \cong S_{12}$. If $(H, \circ)$ is the two-element group $S - 2$ then $a \circ a \cup b \circ a = a \circ b \cup b \circ b = a \circ a \cup a \circ b = b \circ a \cup b \circ b = H = H^2$. This shows that $\mathcal{P}^*(H) \cong S_2$, where $\{x, z\}$ is the subgroup of S_2. S_3 is the constant semigroup. It is isomorphic to $\mathcal{P}^*(H)$, where $(H, \circ)$ is the semihypergroup A_{12} since all compositions give H. If $(H, \circ)$ is the semihypergroup A_4 then a is the zero-element in $\mathcal{P}^*(H)$ and the remaining compositions give H since $b \circ b = H$. Hence, $\mathcal{P}^*(H) \cong S_5$, where y is the zero-element. If $(H, \circ)$ is the semihypergroup A_6 then $b \circ b = H$ and $a \circ a \cup a \circ b = a \circ a \cup b \circ a = H$ implies that any compositions of H with any other element from $\mathcal{P}^*(H)$ gives H. Thus, $\mathcal{P}^*(H) \cong S_6$. If $(H, \circ)$ is the semihypergroup A_5 then a is a left-zero from $\mathcal{P}^*(H)$ and the remaining compositions of elements in $\mathcal{P}^*(H)$ give H. This shows that $\mathcal{P}^*(H) \cong S_7$, where y is the left-zero element in S_7. Dually, we obtain $\mathcal{P}^*(H) \cong S_8$, where $(H, \circ)$ is the semihypergroup A_8. If $(H, \circ)$ is the semihypergroup A_7 then $a * H = a \circ a \cup a \circ b = H$ and all the remaining compositions of elements from $\mathcal{P}^*(H)$ give also H. Hence, $\mathcal{P}^*(H) \cong S_9$. Dually, we have $\mathcal{P}^*(H) \cong S_{10}$, where $(H, \circ)$ is the semihypergroup A_9. If $(H, \circ)$ is the semihypergroup S_4 than a is the zero-element in $\mathcal{P}^*(H)$ and the remaining three compositions $b * H$, $H * b$, and $H * H$ give H since $b \circ b = b$. Hence, $\mathcal{P}^*(H) \cong S_{11}$, where z is the zero-element. If $(H, \circ)$ is the semihypergroup A_2 then b is a right-zero in $\mathcal{P}^*(H)$ and since $a * H = a \circ a \cup a \circ b = H$ as well as $b \circ a = H$, we can calculate that the remaining compositions of elements from $\mathcal{P}^*(H)$ give H. Thus, $\mathcal{P}^*(H) \cong S_{13}$. Dually, we obtain $\mathcal{P}^*(H) \cong S_{14}$, where $(H, \circ)$ is the semihypergroup A_1.

It is easy to verify that $\mathcal{P}^*(H)$ is a left-zero semigroup, whenever $(H, \circ)$ is left-zero semigroup, i.e., $\mathcal{P}^*(H) \cong S_{15}$. Dually, $\mathcal{P}^*(H) \cong S_{16}$, whenever $\mathcal{P}^*(H)$ is a right-zero semigroup. $\square$

The proof of Theorem 3.1 shows which semigroups of order three corresponds to semihypergroups of order two explicitly. A description of semihypergroups of order 3 and more in the setting of Theorem 3.1 is a still open problem. Its solution requires another but more complex approach as in the proof of Theorem 3.1, as initial attempts have already shown.

Corollary 3.1.1. *Let S be a semigroup of order three. There is a semihypergroup $(H, \circ)$ of order two such that $\mathcal{P}^*(H) \cong S$ if and only if $S \cong S_i$ for some $i \in \{1, 2, \dots, 16\}$.*

4. All (non-group) semigroups of order three

In this section, we consider the subsemigroup $\widehat{P}^*(H)$ of $\mathcal{P}^*(H)$ generated by the elements $\{x\}$, $x \in H$. Note that $\widehat{P}^*(H)$ is a proper subsemigroup of $\mathcal{P}^*(H)$, whenever $(H, \circ)$ is a non-proper semihypergroup. All semigroups of order three, except of the group, have a two-element subsemigroup. So in the following theorem, all (non-group) semigroups of order 3 are classified by an alternative variety, determined by the following DI:

$$\sigma := \sigma_4 \cup \{x_1x_2 \approx x_1^2, x_1x_2 \approx x_1^2x_2, x_1x_2 \approx x_1x_2^2, x_2^2 \approx x_3, x_1 \approx x_3, x_2 \approx x_3\}.$$

Theorem 4.1. *Let S be a semigroup with three elements. Then the following statements are equivalent:*

(i) *There is a semihypergroup $(H, \circ)$ of order two such that $\widehat{P}^*(H)$ is isomorphic to a subsemigroup of S.*

(ii) $S \in Mod\{\sigma\}$.

P r o o f. (i) $\Rightarrow$ (ii): Let $(H, \circ)$ be the semihypergroup with $H = \{a, b\}$ and let $T \leq S$ be such that $T \cong \widehat{P}^*(H)$. Then there is an isomorphism $f\colon \widehat{P}^*(H) \to T$. Let us put $\alpha := f(a)$, $\beta := f(b)$, and let γ be the remaining element in S. Note that $\gamma = f(H)$, whenever $S = T$. Let us now replace the variables x_1, x_2, and x_3 by elements of S. If we replace both variables x_1 and x_2 by the same element x then the equation $x_1x_2 \approx x_1^2$ provides the equality $x^2 = x^2$.

Let us replace x_1 by α and x_2 by β. Admit that $x_1x_2 \approx x_1$, $x_1x_2 \approx x_2$, and $x_1x_2 \approx x_1^2$ give no equality, i.e., $\alpha\beta \neq \alpha$, $\alpha\beta \neq \beta$, and $\alpha\beta \neq \alpha^2$. Then $\alpha\beta = \gamma$, i.e., $\gamma \in T$. We observe that $f(a \circ b) = f(a)f(b) = \alpha\beta \neq \alpha^2 = f(a)f(a) = f(a \circ a)$, i.e., $a \circ b \neq a \circ a$ and thus $a * H = \{a \circ b, a \circ a\} = H$. Therefore, we have $\alpha^2\beta = \alpha(\alpha\beta) = \alpha\gamma = f(a)f(H) = f(a * H) = f(H) = \gamma = \alpha\beta$, i.e., $x_1^2x_2 \approx x_1x_2$ gives the equality $\alpha^2\beta = \alpha\beta$.

Let us replace x_1 by α and x_2 by γ and admit that $x_1x_2 \approx x_1$, $x_1x_2 \approx x_2$, and $x_1x_2 \approx x_1^2$ give no equality, i.e., $\alpha\gamma \neq \alpha$, $\alpha\gamma \neq \gamma$, and $\alpha\gamma \neq \alpha^2$. Then we have $\alpha\gamma = \beta$. Suppose that $\gamma \notin T$, i.e., $\alpha^2 = \alpha$ or $\alpha^2 = \beta$. The latter is not possible because of $\beta = \alpha\gamma \neq \alpha^2$. Hence, $\alpha^2 = \alpha$, i.e., we have the equality $\alpha^2\gamma = \alpha\gamma$, given by the equation $x_1^2x_2 \approx x_1x_2$. Suppose that $\gamma \in T$. Then $\alpha\gamma = \beta$ implies $f(a)f(H) = f(b)$, i.e., $a * H = b$ and thus $a \circ b = b$. This provides $\alpha\beta = f(a)f(b) = f(a \circ b) = f(b) = \beta$ and hence $\alpha^2\gamma = \alpha(\alpha\gamma) = \alpha\beta = \beta = \alpha\gamma$, i.e., the equation $x_1^2x_2 \approx x_1x_2$ gives the equality $\alpha^2\gamma = \alpha\gamma$.

Finally, let us replace x_1 by γ and x_2 by α. Admit that $x_1x_2 \approx x_1$, $x_1x_2 \approx x_2$, and $x_1x_2 \approx x_1^2$ give no equality, i.e., $\gamma\alpha \neq \gamma$, $\gamma\alpha \neq \alpha$, and $\gamma\alpha \neq \gamma^2$. This provides $\gamma\alpha = \beta$. But $\beta = \gamma\alpha \neq \gamma^2$ implies $\gamma^2 = \gamma$ or $\gamma^2 = \alpha$. If we have $\gamma^2 = \gamma$, then the equation $x_1^2x_2 \approx x_1x_2$ gives the equality $\gamma^2\alpha = \gamma\alpha$. Suppose now that $\gamma^2 = \alpha$. Then $\gamma \notin T$. Otherwise $\alpha = \gamma^2 = f(H)^2 = f(H * H) = f(H) = \gamma$ (since at least one composition of the elements in H gives H), a contradiction. Note that x_3 is replaced by an element x of S. If $x \in \{\alpha, \gamma\}$ then $x_1 \approx x_3$ or $x_2 \approx x_3$ provides the equality $x = x$. Admit now that $x = \beta$ and that $x_2^2 \approx x_3$ gives no equality, i.e., $\alpha^2 \neq \beta$. Then $\alpha^2 = \alpha$ since $\gamma \notin T$ and the equation $x_1x_2^2 \approx x_1x_2$ gives the equality $\gamma\alpha^2 = \gamma\alpha$.

Note that if we replace the variables by β instead by α and conversely, then there is again an equation in σ which gives an equality in S. Altogether, we have shown that $S \models \sigma$.

(ii) $\Rightarrow$ (i): Assume that S has non two-element subsemigroup. This means that S is a group, say with the elements e (the identity in S), a, and b. In S, it holds $ab = e, a^2 = b$, and $b^2 = a$. Hence, $ab \neq a$, $ab \neq b$, $ab \neq a^2$, $ab \neq a^2b$, $ab \neq ab^2$, and $b^2 \neq e$. This contradicts $S \models \sigma$ by the replacement $x_1 \to a, x_2 \to b$, and $x_3 \to e$. Hence there is a two-element subsemigroup $T \leq S$. We can regard T as a non-proper semihypergroup $(H, \circ)$ of order two with the universe $H = T$ and $x \circ y = \{xy\}$ for all $x, y \in H$. Then, $\widehat{P}^*(H)$ is isomorphic to $T \leq S$. $\square$

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