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The Rank of the Semigroup of All Order-Preserving Transformations on a Finite Fence

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Abstract A zig-zag (or fence) order is a special partial order on a (finite) set. In this paper, we consider the semigroup $\mathcal{TF}_n$ of all order-preserving transformations on an n-element zig-zag-ordered set. We determine the rank of $\mathcal{TF}_n$ and provide a minimal generating set for $\mathcal{TF}_n$. Moreover, a formula for the number of idempotents in $\mathcal{TF}_n$ is given.

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1 Introduction

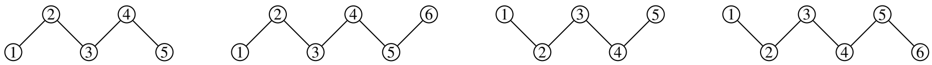
Let $n \in \mathbb{N}$ and denote by $\mathcal{T}_n$ the monoid (under composition) of all full transformations on the set $\overline{n} = \{1, \ldots, n\}$ of the first n natural numbers. Let $\preceq$ be any partial order on $\overline{n}$. Let $\alpha \in \mathcal{T}_n$. We say that α is an *order-preserving* transformation (with respect to $\preceq$) if $x \preceq y$ implies $x\alpha \preceq y\alpha$, for all $x, y \in \overline{n}$. Clearly, the subset of $\mathcal{T}_n$ of all order-preserving transformations (with respect to a fixed partial order) forms a submonoid of $\mathcal{T}_n$.

A very important particular and natural case occurs when a linear order (for instance the one induced by the usual order on the natural numbers) is considered. The monoid $\mathcal{O}_n$ of all order-preserving transformations on $\overline{n}$, endowed with a linear order, has been extensively studied since the early 1960s. In fact, in 1962, Aĭzenštat [1,2] showed that all non-trivial congruences of $\mathcal{O}_n$ are Rees congruences and gave a monoid presentation for $\mathcal{O}_n$, in terms of $2n - 2$ idempotent generators, from which it can be deduced that, for $n > 1$, $\mathcal{O}_n$ only has one non-trivial automorphism. In 1971, Howie [13] calculated the cardinal and the number of idempotents of $\mathcal{O}_n$ and later (1992), jointly with Gomes [11], determined its rank and idempotent rank. More recently, Fernandes et al. [9] described the endomorphisms of the semigroup $\mathcal{O}_n$ by showing that there are three types of endomorphism: automorphisms, constants, and a certain type of endomorphism with two idempotents in the image. The monoid $\mathcal{O}_n$ also played a main role in several other papers [3,7,8,10,12,16,17,19], where the central topic concerns the problem of the decidability of the pseudovariety generated by the family $\{\mathcal{O}_n \mid n \in \mathbb{N}\}$. This question was posed by J.-E. Pin in 1987 in the "Szeged International Semigroup Colloquium" and, as far as we know, is still open.

A nonlinear order (in some sense) *close* to a linear order is the so-called zig-zag order. The pair $(\overline{n}, \preceq)$ is called a *zig-zag poset* or *fence* if

$$\begin{array}{ll}
1 \prec 2 \succ 3 \prec \cdots \prec n-1 \succ n, & \text{if } n \text{ is odd, and} \\
1 \prec 2 \succ 3 \prec \cdots \succ n-1 \prec n, & \text{if } n \text{ is even, or dually} \\
1 \succ 2 \prec 3 \succ \cdots \succ n-1 \prec n, & \text{if } n \text{ is odd, and} \\
1 \succ 2 \prec 3 \succ \cdots \prec n-1 \succ n, & \text{if } n \text{ is even.}
\end{array}$$

The definition of the partial order $\preceq$ is self-explanatory. For instance, for $n = 5$ and $n = 6$, we have the following fences (given by Hasse diagrams):



Observe that, every element in a fence is either minimal or maximal.

Order-preserving transformations of (finite) fences were first investigated by Currie and Visentin [5] and by Rutkowski [18]. In [5], by using generating functions, the authors calculate the number of order-preserving transformations of a fence with an even number of elements. On the other hand, an exact formula for the number of such transformations, for any natural number n, was given in [18].

Recently, several properties of monoids of order-preserving transformations of a fence were studied. In [4] the authors discussed the regular elements in these monoids. So-called coregular elements of this monoids were determined in [15]. On the other hand, in [6] Dimitrova and Koppitz investigated the monoid of all partial permutations preserving a zig-zag order on a set with n elements, by studying Green's relations and generating sets of this monoid.

Without loss of generality, we will assume that $(\overline{n}, \preceq)$ is an *up-fence*, i.e.,

$$1 \prec 2 \succ 3 \prec \cdots .$$

Let $x, y \in \overline{n}$. We say that x and y are *comparable* if $x \prec y$ or $x = y$ or $y \prec x$. Otherwise, x and y are said *incomparable*. Clearly, x and y are comparable if and only if $x \in \{y-1, y, y+1\}$.

Denote by $\mathcal{TF}_n$ the submonoid of $\mathcal{T}_n$ of all order-preserving transformations of the fence $(\overline{n}, \preceq)$.

In this paper, we determine the rank and count the number of idempotents of $\mathcal{TF}_n$.

Recall that the *rank* of a (finite) semigroup S is defined by

$$\operatorname{rank} S = \min\{|A| \mid A \subseteq \text{generates}\},$$

i.e., the rank of S is the minimal size of a generating set of S. For general background on semigroup theory and standard notation, we refer the reader to Howie's book [14].

We begin, in the next section, by giving a characterization of the elements of $\mathcal{TF}_n$. Clearly, the identity mapping id_n on $\overline{n}$ is order-preserving. Also, all the n constant mappings are order-preserving. Moreover, for an even n, id_n is the unique permutation of $\overline{n}$ belonging to $\mathcal{TF}_n$ and, on the other hand, if n is odd, then $\mathcal{TF}_n$ has exactly two permutations, namely the identity mapping and the reflection

$$\gamma_n = \begin{pmatrix} 1 & 2 & \cdots & n \\ n & n-1 & \cdots & 1 \end{pmatrix}.$$

The rest of Sect. 2 is dedicated to counting the idempotents of $\mathcal{TF}_n$. Notice that it is easy to show that an element $\alpha \in \mathcal{T}_n$ is idempotent if and only if $\operatorname{Im}\alpha = \{x \in \overline{n} \mid x\alpha = x\}$, i.e., the image of α coincides with the set of its fix points. In the third section of this paper, we determine the rank of $\mathcal{TF}_n$. In particular, we provide a minimal size generating set for $\mathcal{TF}_n$.

Notice that $\mathcal{TF}_1$ coincides with $\mathcal{T}_1$ and $\mathcal{TF}_2$ coincides with the monoid $\mathcal{O}_2$ of all order-preserving transformations on a two-element chain. Hence, from now on, we always consider $n \geq 3$.

2 Idempotents

The aim of this section is to provide a formula for the number of idempotents of $\mathcal{TF}_n$. In order to accomplish this, it is useful to know the form of the elements of $\mathcal{TF}_n$. We have the following characterization of a transformation in $\mathcal{TF}_n$.

Theorem 2.1 *Let $\alpha \in \mathcal{T}_n$. Then $\alpha \in \mathcal{TF}_n$ if and only if*

(i) $|x\alpha - (x+1)\alpha| \leq 1$, *for all* $x \in \{1, \ldots, n-1\}$;

(ii) x *and* $x\alpha$ *have the same parity or* $(x-1)\alpha = x\alpha = (x+1)\alpha$, *for all* $x \in \{2, \ldots, n-1\}$.

Proof First, suppose that $\alpha \in \mathcal{TF}_n$. Let $x \in \{1, \ldots, n-1\}$. Then, x and $x+1$ are comparable, which implies that $x\alpha$ and $(x+1)\alpha$ are also comparable and so $|x\alpha - (x+1)\alpha| \leq 1$. This shows (i). Now let $x \in \{2, \ldots, n-1\}$. Assume that x is even. Then, $x - 1 \prec x \succ x+1$ and so $(x-1)\alpha \preceq x\alpha \succeq (x+1)\alpha$. If $(x-1)\alpha \neq x\alpha$ or $x\alpha \neq (x+1)\alpha$, then $(x-1)\alpha \prec x\alpha$ or $x\alpha \succ (x+1)\alpha$, which implies in both cases that $x\alpha$ is even. Similarly, if x is odd, we may deduce that $x\alpha$ is also odd or $(x-1)\alpha = x\alpha = (x+1)\alpha$. This shows (ii).

Conversely, suppose that (i) and (ii) are satisfied. Let $x, y \in \overline{n}$ be such that $x \prec y$. Then, x is odd and y is even. Moreover, $y \in \{x-1, x+1\}$. Admit that $x\alpha \neq y\alpha$. If $y = x-1$, then $2 \leq y \leq n-1$ and so $|y\alpha - x\alpha| = |y\alpha - (y+1)\alpha| = 1$ and y and $y\alpha$ have the same parity. If $y = x+1$, then $1 \leq x \leq n-1$ and so $|x\alpha - y\alpha| = |x\alpha - (x+1)\alpha| = 1$. Furthermore, in this last case, if $x > 1$, then x and $x\alpha$ have the same parity; otherwise $y = 2 < n$ and so y and $y\alpha$ have the same parity (since $(y-1)\alpha = x\alpha \neq y\alpha$). Therefore, we have $y\alpha \in \{x\alpha - 1, x\alpha + 1\}$ and, on the other hand, $y\alpha$ is even or $x\alpha$ is odd. Thus, in all cases, $x\alpha \prec y\alpha$, as required. □

As a consequence of Theorem 2.1, we have that the image of a transformation in $\mathcal{TF}_n$ is an interval of $\overline{n}$ (with the usual order).

Corollary 2.2 *Let $\alpha \in \mathcal{TF}_n$. Then* $\operatorname{Im}\alpha = \{k, k+1, \ldots, \ell\}$, *for some* $1 \leq k < \ell \leq n$.

Proof Let $k = \min \operatorname{Im}\alpha$ and $\ell = \max \operatorname{Im}\alpha$ (with respect to the usual order of $\mathbb{N}$). Assume that there exists $p \in \{k, k+1, \ldots, \ell\}$ such that $p \notin \operatorname{Im}\alpha$. Let $x = \max\{i \in \overline{n} \mid i\alpha < p\}$. If $x < n$, then $(x+1)\alpha > p$ and so $|x\alpha - (x+1)\alpha| > 1$, a contradiction. Then, $y = \max\{i \in \overline{n} \mid i\alpha > p\} < n$ and $(y+1)\alpha < p$, whence $|y\alpha - (y+1)\alpha| > 1$, which again is a contradiction. Thus, $\operatorname{Im}\alpha = \{k, k+1, \ldots, \ell\}$, as required. □

Next we will give a formula for the number of idempotents in $\mathcal{TF}_n$. Let $m \in \overline{n}$ and $0 \leq p \leq n - m$. For $r \in \{0, \ldots, m-1\}$, let

$$P(p, r) = \{(p_0, \ldots, p_t) \mid t \in \mathbb{N} \cup \{0\};\ p_1, \ldots, p_t \in \mathbb{N};\ p_0 = 0;\ 0 \leq \sum_{i=1}^{s} (-1)^{i+1} p_i \leq p, \text{ for } 1 \leq s \leq t;\ \sum_{i=1}^{t} p_i = r\}$$

and

$$K(m, r) = \{(k_0, \ldots, k_r) \mid k_0 + r + 2\sum_{i=1}^{r} k_i = m - 1, k_0, \ldots, k_r \in \mathbb{N} \cup \{0\}\}.$$

Further, define

$$A(m,p)=\sum_{r=0}^{m-1}|P(p,r)|\cdot|K(m,r)|.$$

Lemma 2.3 *Let $\alpha\in\mathcal{TF}_n$ with $\operatorname{Im}\alpha=\{k,\ldots,k+p\}$, for some $k\in\overline{n}$ and some $p\in\{0,\ldots,n-k\}$. Let $a_0\in\{k,k+p\}$ and $r\in\{0,\ldots,k-1\}$. Then, there exists a bijection between the set $P(p,r)$ and the set of all sequences $a_0,a_1,\ldots,a_r\in\operatorname{Im}\alpha$ such that $|a_{i-1}-a_i|=1$, for all $i\in\{1,\ldots,r\}$, and there exists a partition $A_0>A_1>\cdots>A_r$ of $\{1,\ldots,k\}$, if $a_0=k$, or a partition $A_0<A_1<\cdots<A_r$ of $\{k+p,\ldots,n\}$, if $a_0=k+p$, verifying $A_i\alpha=\{a_i\}$, for $i\in\{0,\ldots,r\}$.*

Proof Fix a sequence $a_0,a_1,\ldots,a_r\in\operatorname{Im}\alpha$ verifying the conditions of the lemma. Notice that, if $r=0$ then $P(p,0)=\{(0)\}$ and a_0 is the only possible sequence. Then, we may admit that $r>0$. Let $j=1$, if $a_0=k$, or $j=2$, if $a_0=k+p$. Put $p_0=0$ (by technical reasons).

Then, there exists $p_1\in\{1,\ldots,r\}$ such that $(-1)^{1+1}p_1\in\{0,\ldots,p\}$, $a_i=a_0+(-1)^{j+1}i$, for $1\le i\le p_1$, and either $r=p_1$ or $a_{p_1+1}=a_0+(-1)^{j+1}p_1+(-1)^{j+2}$.

If $r>p_1$, then there exists $p_2\in\{1,\ldots,r-p_1\}$ such that $(-1)^{1+1}p_1+(-1)^{2+1}p_2\in\{0,\ldots,p\}$, $a_{p_1+i}=a_0+(-1)^{j+1}p_1+(-1)^{j+2}i$, for $1\le i\le p_2$, and either $r=p_1+p_2$ or $a_{p_1+p_2+1}=a_0+(-1)^{j+1}p_1+(-1)^{j+2}p_2+(-1)^{j+3}$.

Continuing in this way, we obtain t, $p_1\ldots,p_t\in\mathbb{N}$ such that

$$\sum_{i=1}^{t}p_i=r,\quad \sum_{i=1}^{s}(-1)^{i+1}p_i\in\{0,\ldots,p\},\quad \text{for } 1\le s\le t,$$

and

$$a_{i+\sum_{\ell=1}^{q-1}p_\ell}=a_0+\sum_{\ell=1}^{q-1}(-1)^{j+\ell}p_\ell+(-1)^{j+q}i,\quad \text{for } 1\le i\le p_q \text{ and } 1\le q\le t.$$

Hence, the sequence $a_0,a_1,\ldots,a_r$ is uniquely determined by the t-uple $(p_0,\ldots,p_t)$. $\square$

Let us denote by E_m the set of all idempotents of $\mathcal{TF}_m$, for all $m\ge 1$. It is clear that $E_1=\mathcal{TF}_1=\mathcal{T}_1=\left\{\binom{1}{1}\right\}$ and $E_2=\mathcal{TF}_2=\mathcal{T}_2\setminus\left\{\binom{12}{21}\right\}=\left\{\binom{12}{12},\binom{12}{11},\binom{12}{22}\right\}$.

Theorem 2.4 *We have*

$$|E_n|=\sum_{k=1}^{n}\sum_{p=0}^{n-k}A(k,p)\cdot A(n+1-(k+p),p).$$

Proof Let $\alpha\in E_n$. Then, by Corollary 2.2, there exist $k\in\overline{n}$ and $p\in\{0,\ldots,n-k\}$ such that

$$\operatorname{Im}\alpha = \{k, k+1, \ldots, k+p\}.$$

Since α is idempotent, we have $(k+i)\alpha = k+i$, for $i \in \{0, \ldots, p\}$. Let

$$A^- = \{1, \ldots, k\} \quad \text{and} \quad A^+ = \{k+p, \ldots, n\}.$$

First, we consider the set A^-. By Theorem 2.1, we have $|x\alpha - (x+1)\alpha| \leq 1$ for all $x \in \{1, \ldots, k-1\}$. Hence, there exist $r \in \{0, \ldots, k-1\}$, a sequence $a_0, \ldots, a_r \in \operatorname{Im}\alpha$ and a partition $A_0 > A_1 > \cdots > A_r$ of A^- such that $|a_{i-1} - a_i| = 1$, for $1 \leq i < r$, and $A_i\alpha = \{a_i\}$, for $0 \leq i \leq r$. Moreover, $x\alpha$ and x have the same parity or $(x-1)\alpha = x\alpha = (x+1)\alpha$, for all $x \in A^- \setminus \{1, n\}$. It follows that there exist $k_0, k_1, \ldots, k_r \in \mathbb{N} \cup \{0\}$ such that $|A_i| = 1 + 2k_i$, for $0 \leq i \leq r-1$, and $|A_r| = k_r + 1$. Then $k_r + r + 2\sum_{i=0}^{r-1} k_i = k - 1$ and so the sequence $A_0 > A_1 > \cdots > A_r$ is uniquely determined by an element of $K(k, r)$.

If $r = 0$, then $A^- = A_0$ and $P(p, 0) = \{(0)\}$. On the other hand, admit that $r > 0$. Then, by Lemma 2.3 (with $a_0 = k$), we have that the sequence $a_0, \ldots, a_r$ is uniquely determined by an element of the set $P(p, r)$. Hence, $\alpha|_{A^-}$ is uniquely determined by an element of the set

$$B^-(k, p) = \bigcup_{r=0}^{k-1} K(k, r) \times P(p, r) \times \{r\}.$$

Dually, there exist $s \in \{0, \ldots, n-(k+p)\}$, a sequence $a_0, \ldots, a_s \in \operatorname{Im}\alpha$ and a partition $A_0 < A_1 < \cdots < A_s$ of A^+ such that $|a_{i-1} - a_i| = 1$, for $1 \leq i < s$, and $A_i\alpha = \{a_i\}$, for $0 \leq i \leq s$. Also, there exist $\ell_0, \ell_1, \ldots, \ell_s \in \mathbb{N} \cup \{0\}$ such that $|A_i| = 1 + 2\ell_i$, for $0 \leq i \leq s-1$, and $|A_s| = \ell_s + 1$. Then, $\ell_r + r + 2\sum_{i=0}^{s-1} \ell_i = n - (k+p) = (n+1) - (k+p) - 1$, whence the sequence $A_0 < A_1 < \cdots < A_s$ is uniquely determined by an element of $K(n+1-(k+p), s)$.

If $s = 0$, then $A^+ = A_0$ and $P(p, 0) = \{(0)\}$. So, admit that $s > 0$. Then, by Lemma 2.3 (with $a_0 = k+p$), we have that the sequence $a_0, \ldots, a_s$ is uniquely determined by an element of the set $P(p, s)$. Consequently, $\alpha|_{A^+}$ is uniquely determined by an element of the set

$$B^+(k, p) = \bigcup_{s=0}^{n-(k+p)} K(n+1-(k+p), s) \times P(p, s) \times \{s\}.$$

Notice that it is easy to verify that $\left|B^-(k, p)\right| = A(k, p)$ and $\left|B^+(k, p)\right| = A(n+1-(k+p), p)$. Moreover, $\alpha|_{\operatorname{Im}\alpha}$ is the identity mapping on $\operatorname{Im}\alpha$ and $\operatorname{Im}\alpha$ is uniquely determined by an element k of the set $\overline{n}$ and an element p of the set $\{0, \ldots, n-k\}$. Thus, the transformation $\alpha \in E_n$ is uniquely determined by an element of the set

$$\bigcup_{k=1}^{n} \bigcup_{p=0}^{n-k} B^-(k, p) \times B^+(k, p) \times \{(k, p)\}.$$

Conversely, as the construction of this set clearly justifies that each of its elements determines uniquely an idempotent in $\mathcal{TF}_n$, we have

$$\begin{aligned}|E_n| &= \left|\bigcup_{k=1}^{n}\bigcup_{p=0}^{n-k} B^-(k,p)\times B^+(k,p)\times\{(k,p)\}\right|\\ &= \sum_{k=1}^{n}\sum_{p=0}^{n-k}\left|B^-(k,p)\times B^+(k,p)\times\{(k,p)\}\right|\\ &= \sum_{k=1}^{n}\sum_{p=0}^{n-k}\left|B^-(k,p)\right|\cdot\left|B^+(k,p))\right|\cdot|\{(k,p)\}|\\ &= \sum_{k=1}^{n}\sum_{p=0}^{n-k} A(k,p)\cdot A(n+1-(k+p),p),\end{aligned}$$

as required. □

The table below gives us an idea of the size of the monoids $\mathcal{TF}_m$ and of their number of idempotents.

m	$\lvert E_m\rvert$	$\lvert\mathcal{TF}_m\rvert$
1	1	1
2	3	3
3	8	11
4	19	31
5	44	99
6	98	275
7	218	811
8	474	2199

m	$\lvert E_m\rvert$	$\lvert\mathcal{TF}_m\rvert$
9	1039	6187
10	2243	16,459
11	4901	44,931
12	10,591	117,831
13	23,190	315,067
14	50,335	817,323
15	110,651	2,152,915
16	241,457	5,537,839

These numbers were calculated by the formula of Theorem 2.4 and by the formulas given by Rutkowski [18].

3 The Rank of $\mathcal{TF}_n$

This section is devoted to determine the rank of $\mathcal{TF}_n$. In the process, we give an explicit minimal size set of generators of $\mathcal{TF}_n$. The cases n odd and n even will be treated separately.

The following general observation will be frequently used without reference.

Lemma 3.1 *Let $\alpha, \alpha' \in \mathcal{TF}_n$ be such that* $\operatorname{Ker}\alpha = \operatorname{Ker}\alpha'$ *and* $\operatorname{rank}\alpha > 1$. *Then, $x\alpha$ and $x\alpha'$ have the same parity, for all $x \in \overline{n}$.*

Proof Let $x \in \overline{n}$. Since $\operatorname{rank}\alpha > 1$, there exists $y \in x\alpha\alpha^{-1}$ such that $y+1 \in \overline{n}\backslash y\alpha\alpha^{-1}$ or $y-1 \in \overline{n}\backslash y\alpha\alpha^{-1}$. Therefore we may consider four cases. For instance, if $y+1 \in \overline{n}\backslash y\alpha\alpha^{-1}$ and $y \prec y+1$ then $x\alpha = y\alpha \prec (y+1)\alpha$ and $x\alpha' = y\alpha' \prec (y+1)\alpha'$, whence $x\alpha$ and $x\alpha'$ have the same parity. The other three cases are similar. □

Next, we define a series of transformations of $\mathcal{TF}_n$. Let (for any n)

$$\alpha_{1,2} = \begin{pmatrix} \overline{1,2} & 3 & 4 & \cdots & n \\ 2 & 3 & 4 & \cdots & n \end{pmatrix},$$

$$\alpha_{k,k+2} = \begin{pmatrix} 1 & \cdots & k-1 & \overline{k,k+2} & \overline{k+1,k+3} & k+4 & \cdots & n \\ 1 & \cdots & k-1 & k & k+1 & k+2 & \cdots & n-2 \end{pmatrix}, \quad \text{for } \ 2 \le k \le n-4,$$

$$\alpha_{n-2,n} = \begin{pmatrix} 1 & \cdots & n-3 & \overline{n-2,n} & n-1 \\ 1 & \cdots & n-3 & n-2 & n-1 \end{pmatrix}, \text{ for } n \ge 4,$$

$$\alpha_{k,k+1,k+2} = \begin{pmatrix} 1 & \cdots & k-1 & \overline{k,k+1,k+2} & k+3 & \cdots & n \\ 1 & \cdots & k-1 & k & k+1 & \cdots & n-2 \end{pmatrix}, \quad \text{for } 1 \le k \le n-2,$$

$$\alpha_{1,2k+1} = \begin{pmatrix} k+1 & \overline{k,k+2} & \cdots & \overline{2,2k} & \overline{1,2k+1} & 2k+2 & \cdots & n \\ k+1 & k+2 & \cdots & 2k & 2k+1 & 2k+2 & \cdots & n \end{pmatrix}, \quad \text{for } \ 1 \le k \le \left\lfloor \frac{n-1}{2} \right\rfloor, \text{ and}$$

$$\beta_{k,m} = \left(\begin{matrix} 1 & \cdots & k-1 & \overline{k,k+2m} & \overline{k+1,k+2m-1,k+2m+1} & \cdots \\ 1 & \cdots & k-1 & k & k+1 & \cdots \end{matrix} \right.$$
$$\left. \begin{matrix} \cdots & \overline{k+(m-1),k+2m-(m-1),k+2m+(m-1)} & \overline{k+m,k+3m} & k+3m+1 & \cdots & n \\ \cdots & k+(m-1) & k+m & k+m+1 & \cdots & n-2m \end{matrix} \right),$$
for $\ 2 \le k, m \le n$ such that $k+3m \le n-1$.

Moreover, for an odd n, recall that

$$\gamma_n = \begin{pmatrix} 1 & 2 & \cdots & n-1 & n \\ n & n-1 & \cdots & 2 & 1 \end{pmatrix},$$

and, for an even n, let

$$\alpha^e_{1,2} = \begin{pmatrix} \overline{1,2} & 3 & 4 & \cdots & n \\ n & n-1 & n-2 & \cdots & 2 \end{pmatrix},$$

$$\alpha_{n-1,n} = \begin{pmatrix} 1 & 2 & \cdots & n-2 & \overline{n-1,n} \\ n-1 & n-2 & \cdots & 2 & 1 \end{pmatrix},$$

$$\alpha_{2k,n} = \begin{pmatrix} 1 & \cdots & 2k-1 & \frac{n}{2}+k & \overline{\frac{n}{2}+k-1, \frac{n}{2}+k+1} & \overline{\frac{n}{2}+k-2, \frac{n}{2}+k+2} & \cdots & \overline{2k,n} \\ 1 & \cdots & 2k-1 & \frac{n}{2}+k & \frac{n}{2}+k-1 & \frac{n}{2}+k-2 & \cdots & 2k \end{pmatrix},$$
for $1 \le k \le \dfrac{n-4}{2}$, and

$$\alpha^e_{1,2k+1} = \begin{pmatrix} k+1 & \overline{k,k+2} & \cdots & \overline{2,2k} & \overline{1,2k+1} & 2k+2 & \cdots & n \\ k-1 & k & \cdots & 2k-2 & 2k-1 & 2k & \cdots & n-2 \end{pmatrix}, \text{ for } 2 \le k \le \frac{n-2}{2}.$$

Now, for an odd n, define

$$G_n = \{\gamma_n, \alpha_{1,2}\} \cup \{\alpha_{k,k+2} \mid 2 \le k \le \tfrac{n-3}{2}\} \cup \{\alpha_{k,k+1,k+2} \mid 1 \le k \le \tfrac{n-1}{2}\} \cup$$
$$\{\alpha_{1,2k+1} \mid 1 \le k \le \tfrac{n-1}{2}\} \cup \{\beta_{k,m} \mid 2 \le k, m \le \tfrac{n-1}{2} \text{ and } 2k+3m \le n+1\}$$

and, for an even n, define

$$G_n = \{\mathrm{id}_n, \alpha^e_{1,2}, \alpha_{1,3}, \alpha_{n-1,n}, \alpha_{n-2,n}\} \cup \{\alpha_{k,k+2} \mid 2 \le k \le n-4\}$$
$$\cup \{\alpha_{k,k+1,k+2} \mid 2 \le k \le n-3\} \cup$$

$$\{\alpha^e_{1,2k+1} \mid 2 \le k \le \frac{n}{2} - 1\} \cup \{\alpha_{2k,n} \mid 1 \le k \le \frac{n-4}{2}\}$$
$$\cup \{\beta_{k,m} \mid 2 \le k, m \le n \text{ and } k + 3m \le n - 1\}.$$

From now on, our main aim is to prove that G_n is a generating set for $\mathcal{TF}_n$ of minimal size.

The following lemma shows that all the transformations above defined belong to the subsemigroup $\langle G_n\rangle$ of $\mathcal{TF}_n$ generated by G_n. Frequently, we will use it without reference.

Lemma 3.2 *We have:*

(i) $\{\alpha_{k,k+1,k+2} \mid 1 \le k \le n-2\} \subseteq \langle G_n\rangle$;
(ii) $\{\alpha_{1,2k+1} \mid 2 \le k \le \lfloor \frac{n-1}{2} \rfloor\} \subseteq \langle G_n\rangle$;
(iii) $\{\alpha_{k,k+2} \mid 2 \le k \le n-4\} \subseteq \langle G_n\rangle$;
(iv) $\{\beta_{k,m} \mid 2 \le k, m \le n$ *and* $k + 3m \le n - 1\} \subseteq \langle G_n\rangle$;
(v) $\alpha_{2k,n} = \begin{pmatrix} 1 & \cdots & 2k & \overline{2k+1,n} & \cdots & \overline{\frac{n-1}{2}+k, \frac{n+3}{2}+k} & \frac{n+1}{2}+k \\ 1 & \cdots & 2k & 2k+1 & \cdots & \frac{n-1}{2}+k & \frac{n+1}{2}+k \end{pmatrix} \in \langle G_n\rangle$, *for n odd and* $1 \le k \le \frac{n-5}{2}$;
(vi) $\alpha_{n-2,n} \in \langle G_n\rangle$.

Proof (i) For n odd and $\frac{n-1}{2} < k \le n-2$, we have $\alpha_{k,k+1,k+2} = \gamma_n\alpha_{n-k-1,n-k,n-k+1}\gamma_n\alpha_{1,2,3}$. On the other hand, for n even, we have $\alpha_{1,2,3} = \alpha^e_{1,2}\alpha_{n-1,n}$ and $\alpha_{n-2,n-1,n} = \alpha_{n-1,n}\alpha^e_{1,2}\alpha_{1,2,3}$.

(ii) For n even and $2 \le k \le \frac{n-2}{2}$, we have $\alpha_{1,2k+1} = \alpha^e_{1,2k+1}\alpha^e_{1,2}\alpha_{1,2,3}\alpha^e_{1,2}$.

(iii) For n odd and $\frac{n-1}{2} \le k \le n-4$, we have $\alpha_{k,k+2} = \gamma_n\alpha_{n-k-2.n-k}\gamma_n\alpha_{1,2,3}$.

(iv) Let n be an odd number and let $k, m \in \overline{n}$ be such that $k + 3m \le n - 1$ and $2k + 3m > n + 1$. Then $2(n - (k + 3m) + 1) \le n + 1$ and we have $\beta_{k,m} = \gamma_n\beta_{n-(k+3m)+1,m}\gamma_n(\alpha_{1,2,3})^m$.

(v) For $1 \le k \le \frac{n-5}{2}$, we have $\alpha_{2k,n} = \gamma_n\alpha_{1,2(k+1)+1}\gamma_n$.

(vi) Finally, we have $\alpha_{n-2,n} = \gamma_n\alpha_{1,3}\gamma_n$, whenever n is odd.

□

In order to prove that the set G_n generates $\mathcal{TF}_n$, our first step is to show that, for any transformation in $\mathcal{TF}_n$, there exists a transformation in $\langle G_n\rangle$ with the same kernel. For any set $A \subseteq \overline{n}$, define

$$\mathrm{Rel}(A) = \{x \in \overline{n}\backslash A \mid x \text{ and } a \text{ are comparable, for some } a \in A\}.$$

Lemma 3.3 *For any $\alpha \in \mathcal{TF}_n$, there exists $\alpha' \in \langle G_n\rangle$ such that* $\operatorname{Ker}\alpha' = \operatorname{Ker}\alpha$.

Proof Let $\alpha \in \mathcal{TF}_n$. We make the proof by induction on the rank of α.

If $\operatorname{rank}\alpha = n$, then $\operatorname{Ker}\alpha = \operatorname{Ker}\mathrm{id}_n$ and we have $\mathrm{id}_n \in G_n$, for n even, and $\mathrm{id}_n = \gamma_n^2 \in \langle G_n\rangle$, for n odd.

Assume that $\operatorname{rank}\alpha = n - 1$. Then, there exists $i \in \operatorname{Im}\alpha$ such that $|i\alpha^{-1}| = 2$ and $|j\alpha^{-1}| = 1$, for all $j \in \operatorname{Im}\alpha\backslash\{i\}$. This implies $|\mathrm{Rel}(i\alpha^{-1})| \le 2$, i.e., $i\alpha^{-1} = \{1, 2\}$ or $i\alpha^{-1} = \{1, 3\}$ or $i\alpha^{-1} = \{n-2, n\}$ or $i\alpha^{-1} = \{n-1, n\}$. By noticing that, for an odd n, we have $\alpha_{n-1,n} = \gamma_n\alpha_{1,2}$ and $\alpha_{n-2,n} = \gamma_n\alpha_{1,3}\gamma_n$, it follows that there exists $\alpha' \in \langle G_n\rangle$ such that $\operatorname{Ker}\alpha' = \operatorname{Ker}\alpha$.

Admit now that $\operatorname{rank}\alpha = n-2$. Then, for some $i \in \operatorname{Im}\alpha$, we have $2 \le \left|i\alpha^{-1}\right| \le 3$.

If $\left|i\alpha^{-1}\right| = 3$, then there exists $k \in \{1,\dots,n-2\}$ such that $i\alpha^{-1} = \{k, k+1, k+2\}$ and $\left|j\alpha^{-1}\right| = 1$, for all $j \in \operatorname{Im}\alpha\setminus\{i\}$, i.e., $\operatorname{Ker}\alpha = \operatorname{Ker}\alpha_{k,k+1,k+2}$, with $\alpha_{k,k+1,k+2} \in \langle G_n\rangle$.

Now, suppose that $\left|i\alpha^{-1}\right| = 2$. Then, $\left|j\alpha^{-1}\right| = 2$, for some $j \in \operatorname{Im}\alpha\setminus\{i\}$.

Admit that $\left|\operatorname{Rel}(i\alpha^{-1})\right| \le 2$. Then, $i\alpha^{-1} = \{1,2\}$ or $i\alpha^{-1} = \{1,3\}$ or $i\alpha^{-1} = \{n-2, n\}$ or $i\alpha^{-1} = \{n-1, n\}$. Since $\operatorname{rank}\alpha = n-2$, we conclude that $\left|\operatorname{Rel}(j\alpha^{-1})\right| \le 2$ or $i\alpha^{-1} \subseteq \operatorname{Rel}(j\alpha^{-1})$. So, we have $j\alpha^{-1} = \{n-2, n\}$ or $j\alpha^{-1} = \{n-1, n\}$, if $i\alpha^{-1} = \{1,2\}$ or $i\alpha^{-1} = \{1,3\}$, or $j\alpha^{-1} = \{2,4\}$, if $i\alpha^{-1} = \{1,3\}$, or $j\alpha^{-1} = \{n-3, n-1\}$, if $i\alpha^{-1} = \{n-2, n\}$. Hence, we get $\operatorname{Ker}\alpha' = \operatorname{Ker}\alpha$, with $\alpha' = \alpha_{1,2}\alpha_{n-1,n}$ (and $\alpha' = (\alpha_{1,2}\gamma_n)^2$, whenever n is odd) or $\alpha' = \alpha_{1,2}\alpha_{n-2,n}$ or $\alpha' = \alpha_{1,3}\alpha_{n-1,n}$ (and $\alpha' = \alpha_{1,3}\gamma_n\alpha_{1,2}\gamma_n$, whenever n is odd) or $\alpha' = \alpha_{1,3}\alpha_{n-2,n}$ or $\alpha' = \alpha_{1,3}\alpha_{1,5}$ or $\alpha' = \alpha_{n-2,n}\alpha_{n-4,n}$. Observe $\alpha_{n-4,n} = \gamma_n\alpha_{1,5}\gamma_n \in \langle G_n\rangle$, whenever n is odd (since $\alpha_{1,5} \in \langle G_n\rangle$ by Lemma 3.2), and $\alpha_{1,2} = \alpha^e_{1,2}\alpha^e_{1,2}$, whenever n is even. Since all the other transformations used belong to $\langle G_n\rangle$, we have $\alpha' \in \langle G_n\rangle$. Dually, in the case $\left|\operatorname{Rel}(j\alpha^{-1})\right| \le 2$, we can show that there exists $\alpha' \in \langle G_n\rangle$, with $\operatorname{Ker}\alpha' = \operatorname{Ker}\alpha$.

Notice that the case $\left|\operatorname{Rel}(i\alpha^{-1})\right| \ge 4$ or $\left|\operatorname{Rel}(j\alpha^{-1})\right| \ge 4$ is not possible since $\operatorname{rank}\alpha = n-2$. So, it remains the case $\left|\operatorname{Rel}(i\alpha^{-1})\right| = \left|\operatorname{Rel}(j\alpha^{-1})\right| = 3$. This provides $i\alpha^{-1} = \{1,k\}$, for some $k \in 2\mathbb{N}+3$, or $i\alpha^{-1} = \{n-k, n\}$, for some $k \in 2\mathbb{N}+2$, or $i\alpha^{-1} = \{k, k+2\}$ for some $k \in \{2,\dots,n-3\}$. Then, there are two elements in $\operatorname{Rel}(j\alpha^{-1})$ with the same image, which is i since $\operatorname{rank}\alpha = n-2$. This shows that $i\alpha^{-1} \subseteq \operatorname{Rel}(j\alpha^{-1})$. By the same argumentation, we obtain $j\alpha^{-1} \subseteq \operatorname{Rel}(i\alpha^{-1})$.

Suppose that $i\alpha^{-1} = \{1,k\}$, for some $k \in 2\mathbb{N}+3$. Assume that $k \ge 7$. Then, $j\alpha^{-1} \subseteq \operatorname{Rel}(i\alpha^{-1}) = \{2, k-1, k+1\}$ and $i\alpha^{-1} \subseteq \operatorname{Rel}(j\alpha^{-1})$ implies $\left|\operatorname{Rel}(j\alpha^{-1})\right| = 4$, a contradiction. Hence, we have $i\alpha^{-1} = \{1,5\}$. Then, once again $i\alpha^{-1} \subseteq \operatorname{Rel}(j\alpha^{-1})$ and $\left|\operatorname{Rel}(j\alpha^{-1})\right| = 3$ implies $j\alpha^{-1} = \{2,4\}$. Thus, $\operatorname{Ker}\alpha = \operatorname{Ker}\alpha_{1,5}$ and $\alpha_{1,5} \in \langle G_n\rangle$. Dually, we can show the existence of $\alpha' \in \langle G_n\rangle$ with $\operatorname{Ker}\alpha' = \operatorname{Ker}\alpha$, if $i\alpha^{-1} = \{n-k, n\}$, for some $k \in 2\mathbb{N}+2$. Similarly, we obtain $\alpha' \in \langle G_n\rangle$ with $\operatorname{Ker}\alpha' = \operatorname{Ker}\alpha$, if $j\alpha^{-1} = \{1,k\}$, for some $k \in 2\mathbb{N}+3$, or $j\alpha^{-1} = \{n-k, n\}$, for some $k \in 2\mathbb{N}+2$.

Finally, we consider the case $i\alpha^{-1} = \{k, k+2\}$ and $j\alpha^{-1} = \{\ell, \ell+2\}$, for some $k, \ell \in \{2,\dots,n-3\}$. Notice that $\{k, k+2\} = i\alpha^{-1} \subseteq \operatorname{Rel}(j\alpha^{-1}) = \{\ell-1, \ell+1, \ell+3\}$ and so $k = \ell-1$ or $k = \ell+1$. Therefore, we have $\operatorname{Ker}\alpha = \operatorname{Ker}\alpha_{m,m+2}$, with $m = k$, if $k = \ell-1$, or $m = \ell$, if $k = \ell+1$. Hence, $\operatorname{Ker}\alpha = \operatorname{Ker}\alpha_{m,m+2}$ and $\alpha_{m,m+2} \in \langle G_n\rangle$.

Next, we suppose that $p = \operatorname{rank}\alpha < n-2$ and assume that for all $\beta \in \mathcal{TF}_n$ with $\operatorname{rank}\beta > p$, there exists $\beta' \in \langle G_n\rangle$ such that $\operatorname{Ker}\beta' = \operatorname{Ker}\beta$. Further, there exist a unique $m \in \overline{n}$, a sequence $a_1,\dots,a_m \in \operatorname{Im}\alpha$ and a partition $A_1 < \cdots < A_m$ of $\overline{n}$ with $|a_i - a_{i+1}| = 1$, for $1 \le i < m$, and $A_i\alpha = \{a_i\}$, for $1 \le i \le m$. Notice that the elements in the sequence $a_1,\dots,a_m$ have not to be pairwise distinct and $\operatorname{Im}\alpha = \{a_1,\dots,a_m\}$. Put $\chi(\alpha) = m$. Observe that this construction can be applied to any element of $\mathcal{TF}_n$ and so we have a well-defined mapping $\chi : \mathcal{TF}_n \to \overline{n}$.

Let

$$a_0 = \begin{cases} 0 & \text{if } a_1 \text{ is odd} \\ 1 & \text{if } a_1 \text{ is even} \end{cases}$$

and define

$$\beta = \begin{pmatrix} A_1 & A_2 & \cdots & A_m \\ 1 + a_0 & 2 + a_0 & \cdots & m + a_0 \end{pmatrix}.$$

It is clear that $\beta \in \mathcal{TF}_n$.

First, consider the case $m = p$ (i.e., $\operatorname{Ker}\alpha = \operatorname{Ker}\beta$). Take $i \in \{1, \ldots, p\}$ such that $|A_i| \geq 3$ and $A_i = \{k, k+1, \ldots, k+s\}$, with $k \in \{1, \ldots, n-2\}$ and $s \in \{2, \ldots, n-k\}$. Define

$$\alpha_1 = \begin{pmatrix} A_1 & \cdots & A_{i-1} & k & k+1 & \overline{k+2 \cdots k+s} & A_{i+1} & \cdots & A_p \\ 1 + a_0 & \cdots & i - 1 + a_0 & i + a_0 & i + 1 + a_0 & i + 2 + a_0 & i + 3 + a_0 & \cdots & p + 2 + a_0 \end{pmatrix},$$

for $i > 1$, and

$$\alpha_1 = \begin{pmatrix} \overline{1 \cdots k - 2 + s} & k - 1 + s & k + s & A_2 & \cdots & A_p \\ 1 + a_0 & 2 + a_0 & 3 + a_0 & 4 + a_0 & \cdots & p + 2 + a_0 \end{pmatrix},$$

if $i = 1$. Since $p < n-2$, we have $p+2+a_0 \in \overline{n}$. By using Theorem 2.1, we can verify that $\alpha_1 \in \mathcal{TF}_n$. Since $\operatorname{rank}\alpha_1 > p$, there is $\alpha_1^* \in \langle G_n \rangle$ with $\operatorname{Ker}\alpha_1^* = \operatorname{Ker}\alpha_1$. Suppose that $\operatorname{Im}\alpha_1^* = \{a_1^*, \ldots, a_{p+2}^*\}$ such that $a_j^*(\alpha_1^*)^{-1} = (j+a_0)\alpha_1^{-1}$ for $j \in \{1, \ldots, p+2\}$. Let

$$\alpha_2 = \begin{cases} \alpha_{a_i^*, a_{i+1}^*, a_{i+2}^*} & \text{if} \quad a_i^* < a_{i+1}^* \\ \alpha_{a_{i+2}^*, a_{i+1}^*, a_i^*} & \text{if} \quad a_{i+1}^* < a_i^*. \end{cases}$$

It is a routine matter to verify that $\operatorname{Ker}\alpha_1\alpha_2 = \operatorname{Ker}\beta$ and so there exists $\alpha' \in \langle G_n \rangle$ such that $\operatorname{Ker}\alpha' = \operatorname{Ker}\beta = \operatorname{Ker}\alpha$.

Now, admit that $m > p$. Then, there exist $i \in \{1, \ldots, m-1\}$ and $s \in \{i, \ldots, m-i\}$ such that the elements of $\{a_i, \ldots, a_{i+s}\}$ are pairwise distinct, $a_{i+2s} = a_i$ and one of the following five conditions is satisfied:

(a) $i + a_0 = 1$;
(b) $i + a_0 \geq 2$, $i + 2s = m$ and $a_0 + i + 2s = n$;
(c) $i + a_0 \geq 2$, $i + 2s = m$, $a_0 + i + 2s < n$ and $n - m < i$;
(d) $i + a_0 \geq 2$, $i + 2s = m$, $a_0 + i + 2s < n$ and $n - m \geq i$;
(e) $a_{i+3s} = a_{i+s}$ and $i + 3s < n$.

We will define in each of these five cases transformations ρ_1 and ω_1. Let $\rho_1 = \alpha_{1,2s+1}$, in the case (a); let $\rho_1 = \alpha_{2\left\lfloor \frac{n-2s}{2} \right\rfloor, n}$, in the case (b); let $\rho_1 = \alpha_{2\left\lfloor \frac{2(i+s)-n}{2} \right\rfloor, n}$, in the case (c), where $2(i + s) - n = i + m - n > i - i = 0$; let ρ_1^* be defined by

$$x\rho_1^* = \begin{cases} 2(i + s + a_0) - x & \text{if } 1 \leq x \leq i + s + a_0 \\ x & \text{otherwise,} \end{cases}$$

in the case (d); and let $\rho_1 = \beta_{i,s}$, in the case (e). It is easy to verify that $\rho_1 \in \langle G_n \rangle$ in the cases (a), (b), (c) and (e). In the case (d), we observe that $q = \operatorname{rank} \rho_1^* = n - (i + s + a_0) + 1 > p$. Then, there exists $\rho_1 \in \langle G_n \rangle$ such that $\operatorname{Ker} \rho_1 = \operatorname{Ker} \rho_1^*$. Suppose that $\operatorname{Im} \rho_1 = \{d_1, \ldots, d_q\}$ such that $j(\rho_1^*)^{-1} = d_{j-(s+i+a_0)+1}\rho_1^{-1}$ for $i + s + a_0 \le j \le n$. Let ω_1 be defined by

$$x\omega_1 = \begin{cases} a_{1+s} & \text{if } 1 \le x \le 1 + s \\ a_x & \text{if } 1 + s < x < m \\ a_m & \text{otherwise,} \end{cases}$$

in the case (a); let ω_1 be defined by

$$x\omega_1 = \begin{cases} a_{x-a_0} & \text{if } 1 + a_0 \le x < i + s + a_0 \\ a_{i+s} & \text{if } i + s + a_0 \le x \le n \\ a_1 & \text{otherwise,} \end{cases}$$

in the cases (b) and (c). Since ℓ and $a_{\ell-a_0}$ have the same parity for all $1 + a_0 \le \ell \le m + a_0$, we conclude that $\omega_1 \in \mathcal{TF}_n$. Let ω_1 be defined by

$$x\omega_1 = \begin{cases} a_{i+s} & \text{if } 1 \le x \le d_1 < d_2 \text{ or } d_2 < d_1 \le x \le n \\ a_{i+s-\ell+1} & \text{if } x = d_\ell \text{ and } 1 \le \ell \le i + s \\ a_1 & \text{otherwise} \end{cases}$$

in the case (d). Let $l \in \{1, \ldots, i + s\}$. Then, there exists $j \in \{i + s + a_0, \ldots, n\}$ such that $\ell = j - (i + a + a_0) - 1$. From $j(\rho_1^*)^{-1} = d_\ell \rho_1^{-1}$, $d_\ell\omega_1 = a_{i+s-\ell+1}$ and the fact that j and a_{j+a_0} have the same parity, we conclude that d_ℓ and $d_\ell\omega_1$ have the same parity. This shows that $\omega_1 \in \mathcal{TF}_n$. Moreover, $\operatorname{rank} \omega_1 = \operatorname{rank} \alpha = p$ and $\chi(\alpha) = \chi(\omega_1) + s$. Consider now the case (e) and define ω_1 by

$$x\omega_1 = \begin{cases} a_{x-a_0} & \text{if } 1 + a_0 \le x \le i + s + a_0 \\ a_{2s+x-a_0} & \text{if } i + s + a_0 + 1 \le x \le m - 2s + a_0 \\ a_m & \text{if } m - 2s + a_0 < x \le n \\ a_1 & \text{if } x = 1. \end{cases}$$

It is easy to verify that $\operatorname{rank} \alpha = \operatorname{rank} \omega_1$ and $\chi(\alpha) = \chi(\omega_1) + 2s$. Moreover, it is a routine matter to show that $\omega_1 \in \mathcal{TF}_n$ and $\alpha = \beta\rho_1\omega_1$.

Next, we can focus on ω_1 and end up getting a sequence $\rho_1, \ldots, \rho_t \in \langle G_n \rangle$ (for a suitable $t \in \mathbb{N}$) and an element $\omega \in \mathcal{TF}_n$ such that $\operatorname{rank} \alpha = \operatorname{rank} \omega$, $\chi(\omega) = p$ and $\alpha = \beta\rho_1 \cdots \rho_t\omega$.

By the case $m = p$, there exists $\omega' \in \langle G_n \rangle$ such that $\operatorname{Ker} \omega' = \operatorname{Ker} \omega$, whence $\operatorname{Ker} \beta\rho_1 \cdots \rho_t\omega' = \operatorname{Ker} \alpha$.

On the other hand, since $m > p$, there exists $\mu \in \langle G_n \rangle$ such that $\operatorname{Ker} \mu = \{A_1, \ldots, A_m\}$, say

$$\mu = \begin{pmatrix} A_1 & A_2 & \cdots & A_m \\ c_1 & c_2 & \cdots & c_m \end{pmatrix},$$

by our inductive assumption. Clearly, by Theorem 2.1, either $c_1 > \cdots > c_m$ or $c_1 < \cdots < c_m$. If $c_1 > \cdots > c_m$ then we take $\varepsilon_1 = \alpha_{1,2}^e$, if n is even, and we take $\varepsilon_1 = \gamma_n$, if n is odd. Since $\varepsilon_1 \in G_n$, whence $\mu\varepsilon_1 \in \langle G_n\rangle$, we can assume that $c_1 < \cdots < c_m$. If $1 + a_0 < c_1$, then there exists $s \in \overline{n}$ such that $1 + a_0 = c_1 - 2s$. It follows that $\beta = \mu(\alpha_{1,2,3})^s$ and so $\beta \in \langle G_n\rangle$.

Altogether, we have shown that $\beta\rho_1 \cdots \rho_t w' \in \langle G_n\rangle$ and $\operatorname{Ker} \beta\rho_1 \cdots \rho_t w' = \operatorname{Ker} \alpha$, as required. □

Now, we are able to prove that G_n is a generating set for $\mathcal{TF}_n$.

Proposition 3.4 *We have $\langle G_n\rangle = \mathcal{TF}_n$.*

Proof Let $\alpha \in \mathcal{TF}_n$.

Admit that $\operatorname{rank} \alpha = n$. If n is even, then $\alpha = \mathrm{id}_n \in G_n$. If n is odd, then $\alpha = \mathrm{id}_n$ or $\alpha = \gamma_n \in G_n$, with $\gamma_n\gamma_n = \mathrm{id}_n$. Thus, $\alpha \in \langle G_n\rangle$.

Suppose now that $2 \le m = \operatorname{rank} \alpha < n$. By Lemma 3.3, there exists $\alpha' \in \langle G_n\rangle$ such that $\operatorname{Ker} \alpha = \operatorname{Ker} \alpha'$. Take

$$\operatorname{Im} \alpha = \{a_1, \ldots, a_m\} \quad \text{and} \quad \operatorname{Im} \alpha' = \{a'_1, \ldots, a'_m\},$$

with $a_1 < a_2 < \cdots < a_m$ and $a'_1 < a'_2 < \cdots < a'_m$, and define $A_i = a_i\alpha^{-1}$, for $1 \le i \le m$. Observe that $A_i = a'_i\alpha'^{-1}$, for $1 \le i \le m$, or $A_i = a'_{m-i+1}\alpha'^{-1}$, for $1 \le i \le m$.

Let $m = n - 1$. Then, $n \notin \operatorname{Im} \alpha$ or $1 \notin \operatorname{Im} \alpha$ as well as $n \notin \operatorname{Im} \alpha'$ or $1 \notin \operatorname{Im} \alpha'$.

If $A_i = a'_i\alpha'^{-1}$, for $1 \le i \le n-1$, then $a_1 = a'_1$, since a_1 and a'_1 have the same parity, by Lemma 3.1. Hence, $a_i = a'_i$, for $1 \le i \le n-1$, and so $\alpha = \alpha'$.

Next consider the case $A_i = a'_{m-i+1}\alpha'^{-1}$, for $1 \le i \le n-1$. Let

$$k = \begin{cases} 0 & \text{if } a_1 = 1 \\ 1 & \text{if } a_1 = 2. \end{cases}$$

Then, $a_i = i + k$ and

$$a'_{m-i+1} = \begin{cases} n - k - i + 1 & \text{if } n \text{ is odd} \\ n + k - i & \text{if } n \text{ is even,} \end{cases}$$

for $i = 1, \ldots, n-1$. If n is odd, then we have

$$a_i(\alpha'\gamma_n)^{-1} = (i+k)\gamma_n^{-1}\alpha'^{-1} = (n - (i+k) + 1)\alpha'^{-1} = a'_{m-i+1}\alpha'^{-1} = A_i = a_i\alpha^{-1},$$

for $1 \le i \le n-1$. Since $\operatorname{Ker} \alpha = \operatorname{Ker} \alpha' = \operatorname{Ker} \alpha'\gamma_n$, this shows that $\alpha = \alpha'\gamma_n \in \langle G_n\rangle$. If n is even then put $\rho_0 = \alpha_{n-1,n} \in \langle G_n\rangle$ and $\rho_1 = \alpha^e_{1,2} \in \langle G_n\rangle$. Observe that ρ_k restricted to $\operatorname{Im} \alpha'$ is an injection. Hence, we have $\operatorname{Ker} \alpha = \operatorname{Ker} \alpha' = \operatorname{Ker} \alpha'\rho_k$ and

$$a_i(\alpha'\rho_k)^{-1} = (i+k)\rho_k^{-1}\alpha'^{-1} = (n - i + k)\alpha'^{-1} = A_i = a_i\alpha^{-1},$$

for $1 \le i \le n-1$. Thus $\alpha = \alpha'\rho_k \in \langle G_n\rangle$.

Admit now that $2 \le m \le n-2$ and suppose that $\beta \in \langle G_n\rangle$, for all $\beta \in \mathcal{TF}_n$ such that $\operatorname{rank} \beta > m$.

Suppose that $A_i = a'_{m-i+1}\alpha'^{-1}$, for $1 \le i \le m$. Take

$$\rho = \begin{cases} \gamma_n & \text{if } n \text{ is odd} \\ \alpha^e_{1,2} & \text{if } n \text{ is even and } 1 \notin \operatorname{Im}\alpha \\ \alpha_{n-1,n} & \text{if } n \text{ is even and } 1 \in \operatorname{Im}\alpha. \end{cases}$$

Then, we have $\operatorname{Ker}\alpha = \operatorname{Ker}\alpha' = \operatorname{Ker}\alpha'\rho$ and

$$A_i = a'_{m-i+1}\alpha'^{-1} = (a'_{m-i+1}\rho)\rho^{-1}\alpha\prime^{-1} = (a'_{m-i+1}\rho)(\alpha'\rho)^{-1},$$

for $1 \le i \le m$, with $\alpha'\rho \in \langle G_n\rangle$ and $a'_{m-i+1}\rho < a'_{m-j+1}\rho$, for $1 \le i < j \le m$. Thus, we can assume that $A_i = a'_i\alpha'^{-1}$, for $1 \le i \le m$.

If $a_1 = a'_1 = 1$, then we immediately obtain that $a_i = a'_i$, for $1 \le i \le m$, i.e., $\alpha = \alpha' \in \langle G_n\rangle$.

Consider $a_1 = 1$, $a'_1 > 1$ and $a'_m \neq n$. This implies $a'_m, a_m < n$ and so we put

$$\beta_0 = \begin{pmatrix} \overline{1\cdots a'_1} & a'_2 & \cdots & a'_m & \overline{a'_m+1\cdots n} \\ a_1 & a_2 & \cdots & a_m & a_m+1 \end{pmatrix}.$$

It is easy to show that $\beta_0 \in \mathcal{TF}_n$, with $\operatorname{rank}\beta_0 = \operatorname{rank}\alpha + 1$, whence $\beta_0 \in \langle G_n\rangle$. For $1 \le i \le m$, we have

$$a_i(\alpha'\beta_0)^{-1} = a_i\beta_0^{-1}\alpha'^{-1} = a'_i\alpha'^{-1} = A_i = a_i\alpha^{-1},$$

as a_i is the unique element in $\operatorname{Im}\alpha' \cap a_i\beta_0^{-1}$. Since the restriction of β_0 to $\operatorname{Im}\alpha'$ is injective, we also have $\operatorname{Ker}\alpha = \operatorname{Ker}\alpha' = \operatorname{Ker}\alpha'\beta_0$. Thus, $\alpha = \alpha'\beta_0 \in \langle G_n\rangle$.

Next, consider $a_1 = 1$, $a'_1 > 1$ and $a'_m = n$. Then, $a'_1 \ge 3$, since a_1 and a'_1 have the same parity. Further, we have $a_i = i$, for$1 \le i \le m$. So, we obtain

$$\beta_1 = \begin{pmatrix} \overline{1,3} & \overline{2,4} & 5 & \cdots & n \\ 1 & 2 & 3 & \cdots & n-2 \end{pmatrix} = \begin{cases} \alpha_{1,3}\alpha^e_{1,5} \in \langle G_n\rangle & \text{if } n \text{ is even} \\ \alpha_{1,3}\alpha_{1,5}\alpha_{1,2,3} \in \langle G_n\rangle & \text{if } n \text{ is odd.} \end{cases}$$

Moreover, let

$$\beta_2 = \begin{pmatrix} 1 & \overline{2\cdots a'_1-1} & a'_1 & \cdots & a'_{m-1} & \overline{a'_m\cdots n} \\ 1 & 2 & 3 & \cdots & m+1 & m+2 \end{pmatrix}.$$

It is easy to verify that $\beta_2 \in \mathcal{TF}_n$, with $\operatorname{rank}\beta_2 = \operatorname{rank}\alpha + 2 > m$, whence $\beta_2 \in \langle G_n\rangle$. Hence,

$$\begin{aligned} a_1(\alpha'\beta_2\beta_1)^{-1} &= a_1\beta_1^{-1}\beta_2^{-1}\alpha'^{-1} = 1\beta_1^{-1}\beta_2^{-1}\alpha'^{-1} = \{1,3\}\beta_2^{-1}\alpha'^{-1} \\ &= \{1, a'_1\}\alpha'^{-1} = a'_1\alpha'^{-1} = A_1 = a_1\alpha^{-1}, \\ a_2(\alpha'\beta_2\beta_1)^{-1} &= 2\beta_1^{-1}\beta_2^{-1}\alpha'^{-1} = \{2,4\}\beta_2^{-1}\alpha'^{-1} = \{2,\ldots,a'_1-1,a'_2\}\alpha'^{-1} \\ &= a'_2\alpha'^{-1} = A_2 = a_2\alpha^{-1} \end{aligned}$$

and, for $3 \leq i \leq m$,

$$a_i(\alpha'\beta_2\beta_1)^{-1} = i\beta_1^{-1}\beta_2^{-1}\alpha'^{-1} = (i+2)\beta_2^{-1}\alpha'^{-1} = a_i'\alpha'^{-1} = A_i = a_i\alpha^{-1}.$$

Notice that β_2 restricted to $\operatorname{Im}\alpha'$ and β_1 restricted to $\operatorname{Im}\alpha'\beta_2 = \{3, \ldots, m+2\}$ are injective. It follows that $\operatorname{Ker}\alpha = \operatorname{Ker}\alpha'\beta_2\beta_1$ and so $\alpha = \alpha'\beta_2\beta_1 \in \langle G_n \rangle$.

Now, consider $a_1 > 1$. Suppose that $a_1' = 1$. Then, $a_m' < n-1$, since $\operatorname{rank}\alpha' \leq n-2$. Take

$$\beta_3 = \begin{pmatrix} 1 & 2 & \cdots & n-3 & \overline{n-2, n-1, n} \\ 3 & 4 & \cdots & n-1 & n \end{pmatrix}.$$

If n is even, then $\beta_3 = \alpha_{n-1,n}\alpha^e_{1,2}$, whence $\beta_3 \in \langle G_n \rangle$. On the other hand, if n is odd, then $\beta_3 = \gamma_n\alpha_{1,2,3}\gamma_n \in \langle G_n \rangle$. Thus, we have $\alpha'\beta_3 \in \langle G_n \rangle$. Clearly, $1 \notin \operatorname{Im}\beta_3$ and so $1 \notin \operatorname{Im}\alpha'\beta_3$. Since $n, n-1 \notin \operatorname{Im}\alpha'$, we have that β_3 restricted to $\operatorname{Im}\alpha'$ is injective. Hence, $\operatorname{Ker}\alpha' = \operatorname{Ker}\alpha'\beta_3$. Therefore, we can assume that $a_1' > 1$. Take

$$\beta_4 = \begin{pmatrix} \overline{1 \cdots a_1' - 1} & a_1' & \cdots & a_{m-1}' & \overline{a_m' \cdots n} \\ a_1 - 1 & a_1 & \cdots & a_{m-1} & a_m \end{pmatrix}.$$

It is easy to verify that $\beta_4 \in \mathcal{TF}_n$, with $\operatorname{rank}\beta_4 = \operatorname{rank}\alpha + 1 > m$, whence $\beta_4 \in \langle G_n \rangle$. Since β_4 restricted to $\operatorname{Im}\alpha'$ is injective, we obtain $\operatorname{Ker}\alpha = \operatorname{Ker}\alpha' = \operatorname{Ker}\alpha'\beta_4$ and, for $i \in \{1, \ldots, m\}$, we have

$$a_i(\alpha'\beta_4)^{-1} = a_i\beta_4^{-1}\alpha'^{-1} = a_i'\alpha'^{-1} = A_i = a_i\alpha^{-1}.$$

Thus, $\alpha = \alpha'\beta_4 \in \langle G_n \rangle$.

Finally, let $m = 1$, i.e., there exists $a \in \overline{n}$ such that $i\alpha = a$, for all $i \in \overline{n}$. Without loss of generality, suppose that $a > 1$. Clearly, $\beta_5 = \begin{pmatrix} 1 & \overline{2 \cdots n} \\ 1 & 2 \end{pmatrix} \in \langle G_n \rangle$ and either $\beta_6 = \begin{pmatrix} \overline{1, 2} & \overline{3 \cdots n} \\ a & a-1 \end{pmatrix} \in \langle G_n \rangle$ (if a is even) or $\beta_6 = \begin{pmatrix} \overline{1, 2} & \overline{3 \cdots n} \\ a-1 & a \end{pmatrix} \in \langle G_n \rangle$ (if a is odd). Then $\beta_5\beta_6$ is the constant mapping with image $\{a\}$, i.e., $\alpha = \beta_5\beta_6 \in \langle G_n \rangle$, as required. □

It remains to show that G_n is a generating set for $\mathcal{TF}_n$ of minimal size. With this goal in mind, in the next two lemmas, we determine a lower bound for the minimal size of a generating set for $\mathcal{TF}_n$ (for n odd as well as for n even) and find it coincides with the cardinality of G_n (which gives us an upper bound).

First, we consider an odd n.

Lemma 3.5 *Let n be an odd number. Then,* $\operatorname{rank}(\mathcal{TF}_n) \geq \frac{3}{2}(n-1) + \sum_{k=2}^{\frac{n-5}{2}} \left(\left\lfloor \frac{n+1-2k}{3} \right\rfloor - 1 \right) = |G_n|$.

Proof Let A be a generating set of $\mathcal{TF}_n$.

Since $\{\alpha \in \mathcal{TF}_n \mid \operatorname{rank}\alpha = n\} = \{\gamma_n, \operatorname{id}_n\}$, we have $\gamma_n \in A$. Let $A^{(0)} = \{\gamma_n\}$. Then, $\left|A^{(0)}\right| = 1$.

Let $\alpha \in \mathcal{TF}_n$ be such that $\operatorname{rank}\alpha \le n-1$. Then, for some natural number p, there exist $\alpha_1, \dots, \alpha_p \in A\backslash\{\mathrm{id}_n\}$, with $\alpha_1 \neq \gamma_n$, such that $\alpha = \alpha_1 \cdots \alpha_p$ or $\alpha = \gamma_n\alpha_1 \cdots \alpha_p$. Take

$$\alpha_1^* = \begin{cases} \alpha_1 & \text{if } \alpha = \alpha_1 \cdots \alpha_p \\ \gamma_n\alpha_1 & \text{if } \alpha = \gamma_n\alpha_1 \cdots \alpha_p. \end{cases}$$

Clearly, $\operatorname{Ker}\alpha_1^* \subseteq \operatorname{Ker}\alpha$ and $\operatorname{rank}\alpha_1^* \le n-1$.

If $\alpha = \alpha_{1,2}$ then $\operatorname{Ker}\alpha_1^* = \operatorname{Ker}\alpha_{1,2}$ or $\operatorname{Ker}\alpha_1^* = \operatorname{Ker}\gamma_n\alpha_{1,2}$, i.e., there exists $\rho_{1,2} \in A$ with $\operatorname{Ker}\rho_{1,2} = \operatorname{Ker}\alpha_{1,2}$ or $\operatorname{Ker}\rho_{1,2} = \operatorname{Ker}\gamma_n\alpha_{1,2}$ (namely $\rho_{1,2} = \alpha_1$). Take $A^{(1)} = A^{(0)} \cup \{\rho_{1,2}\}$. Then, $\left|A^{(1)}\right| = \left|A^{(0)}\right| + \left|\{\rho_{1,2}\}\right| = 2$. Analogously, there exists $\rho_{1,3} \in A$ with $\operatorname{Ker}\rho_{1,3} = \operatorname{Ker}\alpha_{1,3}$ or $\operatorname{Ker}\rho_{1,3} = \operatorname{Ker}\gamma_n\alpha_{1,3}$. Clearly, $\rho_{1,3} \notin A^{(1)}$ and we take $A^{(2)} = A^{(1)} \cup \{\rho_{1,3}\}$. Then $\left|A^{(2)}\right| = \left|A^{(1)}\right| + \left|\{\rho_{1,3}\}\right| = 2 + 1 = 3$.

Let $\alpha = \alpha_{k,k+2}$, for some $k \in \{2, \dots, \frac{n-3}{2}\}$. Then $(k, k+2) \in \operatorname{Ker}\alpha_1^*$ or $(k+1, k+3) \in \operatorname{Ker}\alpha_1^*$. From $2 \le k \le \frac{n-3}{2}$, it follows that $k+3 < n$. Hence, $|\mathrm{Rel}(\{k, k+2\})| = |\mathrm{Rel}(\{k+1, k+3\})| = 3$ and there exist $a, b \in \overline{n}\backslash\{k, k+2\}$ or $a, b \in \overline{n}\backslash\{k+1, k+3\}$ such that $(a, b) \in \operatorname{Ker}\alpha_1^*$. But $\operatorname{Ker}\alpha_1^* \subseteq \operatorname{Ker}\alpha_{k,k+2}$ implies that $(a, b) \in \operatorname{Ker}\alpha_{k,k+2}$. Since $\operatorname{rank}\alpha_{k,k+2} = n-2$, we have $\operatorname{Ker}\alpha_1^* = \operatorname{Ker}\alpha_{k,k+2}$. Hence, there exists $\rho_{k,k+2} \in A$ with $\operatorname{Ker}\rho_{k,k+2} = \operatorname{Ker}\alpha_{k,k+2}$ or $\operatorname{Ker}\rho_{k,k+2} = \operatorname{Ker}\gamma_n\alpha_{k,k+2}$. Moreover, we have $\rho_{k,k+2} \notin A^{(2)}$. On the other hand, assume there exist $2 \le k < \ell \le \frac{n-3}{2}$ such that $\operatorname{Ker}\alpha_{k,k+2} = \operatorname{Ker}\gamma_n\alpha_{\ell,\ell+2}$. Then $k = n - (\ell+3) + 1$ and so $n = k + \ell + 3 - 1 < \frac{n-3}{2} + \frac{n-3}{2} + 2 = n - 3 + 2 = n - 1$, a contradiction. Hence $\rho_{k,k+2} \neq \rho_{\ell,\ell+2}$, for $2 \le k < \ell \le \frac{n-3}{2}$. Take

$$B^{(3)} = \{\rho_{k,k+2} \mid k \in \{2, \dots, \tfrac{n-3}{2}\}\}$$

and $A^{(3)} = A^{(2)} \cup B^{(3)}$. Since $A^{(2)} \cap B^{(3)} = \emptyset$, we obtain $\left|A^{(3)}\right| = \left|A^{(2)}\right| + \left|B^{(3)}\right| = 3 + \frac{n-5}{2} = \frac{n+1}{2}$.

Let $\alpha = \alpha_{k,k+1,k+2}$, for some $k \in \{2, \dots, \frac{n-1}{2}\}$. Then, $k+2 < n$ and, by Theorem 2.1, there exists no $\beta \in \mathcal{TF}_n$ with $\operatorname{rank}\beta = n-1$ such that $\operatorname{Ker}\beta \subseteq \operatorname{Ker}\alpha_{k,k+1,k+2}$. Hence, $\operatorname{Ker}\alpha_1^* = \operatorname{Ker}\alpha_{k,k+1,k+2}$ and so there exists $\rho_{k,k+1,k+2} \in A$ with $\operatorname{Ker}\rho_{k,k+1,k+2} = \operatorname{Ker}\alpha_{k,k+1,k+2}$ or $\operatorname{Ker}\rho_{k,k+1,k+2} = \operatorname{Ker}\gamma_n\alpha_{k,k+1,k+2}$. Clearly, $\rho_{k,k+1,k+2} \notin A^{(3)}$.

Let $\alpha = \alpha_{1,2,3}$. If $\operatorname{rank}\alpha_1^* = n-2$ then $\operatorname{Ker}\alpha_1^* = \operatorname{Ker}\alpha_{1,2,3}$ or $\operatorname{Ker}\alpha_1^* = \operatorname{Ker}\gamma_n\alpha_{1,2,3}$. Now, admit that $\operatorname{rank}\alpha_1^* = n-1$. Then, there exists $j \in \{2, \dots, p\}$ such that $\operatorname{rank}\alpha_1^*\alpha_2 \dots \alpha_{j-1} = n-1$ and $\operatorname{rank}\alpha_1^*\alpha_2 \dots \alpha_j = n-2$. Observe that either $\operatorname{Im}\alpha_1^*\alpha_2 \dots \alpha_{j-1} = \{1, \dots, n-1\}$, with $\{1,2,3\}\alpha_1^*\alpha_2 \dots \alpha_{j-1} = \{n-2, n-1\}$, or $\operatorname{Im}\alpha_1^*\alpha_2 \dots \alpha_{j-1} = \{2, \dots, n\}$, with $\{1,2,3\}\alpha_1^*\alpha_2 \dots \alpha_{j-1} = \{2,3\}$. Suppose that $\operatorname{Im}\alpha_1^*\alpha_2 \dots \alpha_{j-1} = \{2, \dots, n\}$. Then $\{1,2,3\}\alpha_1^*\alpha_2 \dots \alpha_{j-1} = \{2,3\}$ and we conclude that $(2,3) \in \operatorname{Ker}\alpha_j$. By Theorem 2.1, this implies that $(1,2) \in \operatorname{Ker}\alpha_j$ or $(3,4) \in \operatorname{Ker}\alpha_j$. The case $(3,4) \in \operatorname{Ker}\alpha_j$ is not possible since otherwise $\operatorname{rank}\alpha_1^*\alpha_2 \dots \alpha_j \le n-3$, a contradiction. Thus $(1,2) \in \operatorname{Ker}\alpha_j$ and so $\operatorname{Ker}\alpha_j = \operatorname{Ker}\alpha_{1,2,3}$. If $\operatorname{Im}\alpha_1^*\alpha_2 \dots \alpha_{j-1} = \{1, \dots, n-1\}$ then, similarly, we obtain $\operatorname{Ker}\alpha_j = \operatorname{Ker}\alpha_{n-2,n-1,n} = \operatorname{Ker}\gamma_n\alpha_{1,2,3}$. Therefore, there exists $\rho_{1,2,3} \in A$ with $\operatorname{Ker}\rho_{1,2,3} = \operatorname{Ker}\alpha_{1,2,3}$ or $\operatorname{Ker}\rho_{1,2,3} = \operatorname{Ker}\gamma_n\alpha_{1,2,3}$. Clearly, $\rho_{1,2,3} \notin A^{(3)}$. Assume there exist $1 \le k < \ell \le \frac{n-1}{2}$ such that $\operatorname{Ker}\alpha_{k,k+1,k+2} = \operatorname{Ker}\gamma_n\alpha_{\ell,\ell+1,\ell+2}$. Then

$k = n - (\ell + 2) + 1$ and so $n = \ell + k + 1 < \frac{n-1}{2} + \frac{n-1}{2} + 1 = n - 1 + 1 = n$, a contradiction. Hence $\rho_{k,k+1,k+2} \neq \rho_{\ell,\ell+1,\ell+2}$, for $1 \leq k < \ell \leq \frac{n-1}{2}$. Take

$$B^{(4)} = \{\rho_{k,k+1,k+2} \mid k \in \{1, \ldots, \tfrac{n-1}{2}\}\}$$

and $A^{(4)} = A^{(3)} \cup B^{(4)}$. Since, $A^{(3)} \cap B^{(4)} = \emptyset$, we obtain $\left|A^{(4)}\right| = \left|A^{(3)}\right| + \left|B^{(4)}\right| = \frac{n+1}{2} + \frac{n-1}{2} = n$.

Let $\alpha = \alpha_{1,2k+1}$, for some $k \in \{2, \ldots, \frac{n-1}{2}\}$. Then

$$\operatorname{Ker} \alpha_{1,2k+1} = \{(1 + i, 2k + 1 - i) \mid 0 \leq i \leq k - 1\} \cup \{(x, x) \mid x \in \overline{n}\}.$$

Given $i \in \{1, \ldots, k - 2\}$ such that $(1 + i, 2k + 1 - i) \in \operatorname{Ker} \alpha_1^*$, we have

$$\operatorname{Rel}(\{1 + i, 2k + 1 - i\}) = \{1 + i - 1, 2k + 1 - i - 1, 1 + i + 1, 2k + 1 - i + 1\}.$$

Since $\operatorname{Ker} \alpha_1^* \subseteq \operatorname{Ker} \alpha_{1,2k+1}$, we have $(1 + (i + 1), 2k + 1 - (i + 1)), (1 + (i - 1), 2k + 1 - (i - 1)) \in \operatorname{Ker} \alpha_1^*$. If $(k, k + 2) \in \operatorname{Ker} \alpha_1^*$ then $\operatorname{Rel}(\{k, k + 2\}) = \{k - 1, k + 1, k + 3\}$ and so we have $(k - 1, k + 3) \in \operatorname{Ker} \alpha_1^*$. Now, assume that $(1 + i, 2k+1-i) \notin \operatorname{Ker} \alpha_1^*$, for all $i \in \{1, \ldots, k-1\}$. Then, $\operatorname{Ker} \alpha_1^* \subseteq \operatorname{Ker} \alpha_{1,2k+1}$ implies $(1, 2k + 1) \in \operatorname{Ker} \alpha_1^*$ and $\operatorname{rank} \alpha_1^* = n - 1$, which is not possible by Theorem 2.1. Therefore, $\operatorname{Ker} \alpha_1^* = \operatorname{Ker} \alpha_{1,2k+1}$ and so there exists $\rho_{1,2k+1} \in A$ with $\operatorname{Ker} \rho_{1,2k+1} = \operatorname{Ker} \alpha_{1,2k+1}$ or $\operatorname{Ker} \rho_{1,2k+1} = \operatorname{Ker} \gamma_n \alpha_{1,2k+1}$. Since $(1, 2k + 1) \in \operatorname{Ker} \rho_{1,2k+1}$ or $(n, n - 2k) \in \operatorname{Ker} \rho_{1,2k+1}$, we have $\rho_{1,2k+1} \notin A^{(4)}$. For $k, l \in \{2, \ldots, \frac{n-1}{2}\}$, we have $(1, 2k+1) \in \operatorname{Ker} \alpha_{1,2k+1}$ and $(1, 2k+1) \notin \operatorname{Ker} \gamma_n \alpha_{1,2\ell+1}$. Hence, $\rho_{1,2k+1} \neq \rho_{1,2\ell+1}$, for $2 \leq k < \ell \leq \frac{n-1}{2}$. Take

$$B^{(5)} = \{\rho_{1,2k+1} \mid k \in \{2, \ldots, \tfrac{n-1}{2}\}\}$$

and $A^{(5)} = A^{(4)} \cup B^{(5)}$. Since $A^{(4)} \cap B^{(5)} = \emptyset$, we obtain $\left|A^{(5)}\right| = \left|A^{(4)}\right| + \left|B^{(5)}\right| = n + \frac{n-3}{2} = \frac{3n-3}{2} = \frac{3}{2}(n - 1)$.

Finally, let $\alpha = \beta_{k,m}$, for some $k, m \in \{2, \ldots, \frac{n-1}{2}\}$ such that $2k + 3m \leq n + 1$. It is easy to verify that $\{k + i, k + 2m - i, k + 2m + i\}$, for $0 \leq i \leq m$, are all the non-singleton $\operatorname{Ker} \beta_{k,m}$-classes. If $i \in \{1, \ldots, m - 1\}$ is such that $(k + i)\alpha_1^* = (k + 2m - i)\alpha_1^* = (k + 2m + i)\alpha_1^*$ then

$$\begin{aligned}\operatorname{Rel}(\{k + i, k + 2m - i, k + 2m + i\}) = \{&k + i - 1, k + 2m - i - 1,\\ &k + 2m + i - 1, k + i + 1, k + 2m - i + 1, k + 2m + i + 1\}\end{aligned}$$

implies

$$(k + (i - 1))\alpha_1^* = (k + 2m - (i - 1))\alpha_1^* = (k + 2m + (i - 1))\alpha_1^*$$

and

$$(k + (i + 1))\alpha_1^* = (k + 2m - (i + 1))\alpha_1^* = (k + 2m + (i + 1))\alpha_1^*,$$

since $\operatorname{Ker}\alpha_1^* \subseteq \operatorname{Ker}\beta_{k,m}$. If $(k, k+2m) \in \operatorname{Ker}\alpha_1^*$ then, similarly, we have

$$(k+1)\alpha_1^* = (k+2m-1)\alpha_1^* = (k+2m+1)\alpha_1^*.$$

Moreover, we obtain

$$(k+m-1)\alpha_1^* = (k+2m-(m-1))\alpha_1^* = (k+2m+(m-1))\alpha_1^*,$$

whenever $(k+m, k+3m) \in \operatorname{Ker}\alpha_1^*$. Therefore $\operatorname{Ker}\alpha_1^* = \operatorname{Ker}\beta_{k,m}$ and so there exists $\delta_{k,m} \in A$ with $\operatorname{Ker}\delta_{k,m} = \operatorname{Ker}\beta_{k,m}$ or $\operatorname{Ker}\delta_{k,m} = \operatorname{Ker}\gamma_n\beta_{k,m}$. Moreover, it is easy to verify that $\delta_{k,m} \notin A^{(5)}$. Take

$$B^{(6)} = \{\delta_{k,m} \mid k, m \in \{2, \dots, \tfrac{n-1}{2}\} \text{ and } 2k+3m \leq n+1\}.$$

Assume there exist $k, m, p, q \in \{2, \dots, \frac{n-1}{2}\}$ such that $\beta_{k,m} = \gamma_n\beta_{p,q}$, with $2k+3m, 2p+3q \leq n+1$ and $k \neq p$ or $m \neq q$. Then, $k = n-(p+3q)+1$. If $k < p$ then $n = k+p+3q-1 < 2p+3q-1 \leq n+1-1 = n$, a contradiction. Admit that $p < k$. From $\beta_{k,m} = \gamma_n\beta_{p,q}$, it follows that $\beta_{p,q} = \gamma_n\beta_{k,m}$ and so $p = n-(k+3m)+1$. This provides again $n < n$, as in the previous case. Suppose now that $p = k$. Then, $q \neq m$ and we have $p = n-(p+3m)+1 \neq n-(p+3q)+1 = k$, i.e., $p \neq k$, a contradiction. This allows us to conclude that $\delta_{k,m} \neq \delta_{p,q}$, whenever $k, m, p, q \in \{2, \dots, \frac{n-1}{2}\}$, with $2k+3m, 2p+3q \leq n+1$ and $k \neq p$ or $m \neq q$. Thus, $\left|B^{(6)}\right| = \sum_{k=2}^{\frac{n-5}{2}} \left(\left\lfloor \frac{n+1-2k}{3} \right\rfloor - 1\right)$.

Take $A^{(6)} = A^{(5)} \cup B^{(6)}$. Since $A^{(5)} \cap B^{(6)} = \emptyset$, we obtain

$$\left|A^{(6)}\right| = \left|A^{(5)}\right| + \left|B^{(6)}\right| = \frac{3}{2}(n-1) + \sum_{k=2}^{\frac{n-5}{2}} \left(\left\lfloor \frac{n+1-2k}{3} \right\rfloor - 1\right) = |G_n|.$$

Since $A^{(6)} \subseteq A$, we have $|A| \geq \left|A^{(6)}\right| = \frac{3}{2}(n-1) + \sum_{k=2}^{\frac{n-5}{2}} \left(\left\lfloor \frac{n+1-2k}{3} \right\rfloor - 1\right)$, which allows us to deduce that $\operatorname{rank}(\mathcal{TF}_n) \geq \frac{3}{2}(n-1) + \sum_{k=2}^{\frac{n-5}{2}} \left(\left\lfloor \frac{n+1-2k}{3} \right\rfloor - 1\right) = |G_n|$, as required. ◻

Next, we consider the even case.

Lemma 3.6 *Let n be an even number. Then,* $\operatorname{rank}(\mathcal{TF}_n) \geq 3n-8+\sum_{k=2}^{n-7} \left(\left\lfloor \frac{n-1-k}{3} \right\rfloor - 1\right) = |G_n|$.

Proof Let A be a generating set of $\mathcal{TF}_n$.

Since $\{\alpha \in \mathcal{TF}_n \mid \operatorname{rank}\alpha = n\} = \{\mathrm{id}_n\}$, we have $\mathrm{id}_n \in A$. Let $A^{(0)} = \{\mathrm{id}_n\}$. Then, $\left|A^{(0)}\right| = 1$.

Let $\alpha \in \mathcal{TF}_n$ be such that $\operatorname{rank}\alpha \leq n-1$. Then, there exist $\alpha_1, \ldots, \alpha_p \in A \backslash \{\mathrm{id}_n\}$ such that $\alpha = \alpha_1 \ldots \alpha_p$, for some natural number p. Clearly, $\operatorname{Ker}\alpha_1 \subseteq \operatorname{Ker}\alpha$ and $\operatorname{rank}\alpha_1 \leq n-1$.

If $\alpha \in B^{(1)} = \{\alpha_{1,2}, \alpha_{1,3}, \alpha_{n-1,n}, \alpha_{n-2,n}\}$, then it is easy to verify that $\alpha = \alpha_1$. Hence, $B^{(1)} \subseteq A$ and we define $A^{(1)} = A^{(0)} \cup B^{(1)}$. We have $|A^{(1)}| = |A^{(0)}| + |B^{(1)}| = 1+4=5$.

Let $\alpha = \alpha_{k,k+2}$, for some $2 \leq k \leq n-4$. Then $(k, k+2) \in \operatorname{Ker}\alpha_1$ or $(k+1, k+3) \in \operatorname{Ker}\alpha_1$. Since $2 \leq k < n-3$, we have $\operatorname{Rel}(\{k, k+2\}) = \{k-1, k+1, k+3\} \subseteq \overline{n}$ or $\operatorname{Rel}(\{k+1, k+3\}) = \{k, k+2, k+4\} \subseteq \overline{n}$, respectively. Since $\operatorname{Ker}\alpha_1 \subseteq \operatorname{Ker}\alpha_{k,k+2}$, we obtain $\operatorname{Ker}\alpha_1 = \operatorname{Ker}\alpha_{k,k+2}$. Hence, there exists $\rho_{k,k+2} \in A$ such that $\operatorname{Ker}\rho_{k,k+2} = \operatorname{Ker}\alpha_{k,k+2}$. Thus, being

$$B^{(2)} = \{\rho_{k,k+2} \mid k \in \{2, \ldots, n-4\}\},$$

we have $|B^{(2)}| = n-5$. Take $A^{(2)} = A^{(1)} \cup B^{(2)}$. Since $\operatorname{rank}\rho_{k,k+2} = n-2$, it follows that $\rho_{k,k+2} \notin A^{(1)}$. Then $|A^{(2)}| = |A^{(1)}| + |B^{(2)}| = 5+n-5 = n$.

Let $\alpha = \alpha^e_{k,k+1,k+2}$, for some $k \in \{2, \ldots, n-3\}$. Then, there is no $\beta \in \mathcal{TF}_n$ such that $\operatorname{rank}\beta = n-1$ and $\operatorname{Ker}\beta \subseteq \operatorname{Ker}\alpha^e_{k,k+1,k+2}$. Thus, there exists $\rho_{k,k+1,k+2} \in A$ with $\operatorname{Ker}\rho_{k,k+1,k+2} = \operatorname{Ker}\alpha^e_{k,k+1,k+2}$. Clearly, $\rho_{k,k+1,k+2} \notin A^{(2)}$. Take

$$B^{(3)} = \{\rho_{k,k+1,k+2} \mid k \in \{2, \ldots, n-3\}\}.$$

Then, $|B^{(3)}| = n-4$. Furthermore, being $A^{(3)} = A^{(2)} \cup B^{(3)}$, we have $|A^{(3)}| = |A^{(2)}| + |B^{(3)}| = n+n-4 = 2n-4$.

Let $\alpha = \alpha_{1,2k+1}$, for some $k \in \{2, \ldots, \frac{n}{2}-1\}$. It is clear that

$$\operatorname{Ker}\alpha_{1,2k+1} = \{(1+i, 2k+1-i) \mid 0 \leq i \leq k-1\} \cup \{(x,x) \mid x \in \overline{n}\}.$$

If $i \in \{1, \ldots, k-2\}$ is such that $(1+i, 2k+1-i) \in \operatorname{Ker}\alpha_1$, then

$$\operatorname{Rel}(\{1+i, 2k+1-i\}) = \{1+i-1, 2k+1-i-1, 1+i+1, 2k+1-i+1\}$$

and, as $\operatorname{Ker}\alpha_1 \subseteq \operatorname{Ker}\alpha_{1,2k+1}$, it follows $(1+(i+1), 2k+1-(i+1)) \in \operatorname{Ker}\alpha_1$ and $(1+(i-1), 2k+1-(i-1)) \in \operatorname{Ker}\alpha_1$. If $(k, k+2) \in \operatorname{Ker}\alpha_1$ then $\operatorname{Rel}(\{k, k+2\}) = \{k-1, k+1, k+3\}$, whence $(k-1, k+3) \in \operatorname{Ker}\alpha_1$ (since $\operatorname{Ker}\alpha_1 \subseteq \operatorname{Ker}\alpha_{1,2k+1}$). If $(1, 2k+1) \in \operatorname{Ker}\alpha_1$, then $\operatorname{Rel}(\{1, 2k+1\}) = \{2, 2k, 2k+2\} \subseteq \overline{n}$ (note that $k \leq \frac{n}{2}-1$ implies $2k+2 \leq n$) and, since $\operatorname{Ker}\alpha_1 \subseteq \operatorname{Ker}\alpha_{1,2k+1}$, we have $(2, 2k) \in \operatorname{Ker}\alpha_1$. Therefore, $\operatorname{Ker}\alpha_1 = \operatorname{Ker}\alpha_{1,2k+1}$ and there exists $\rho_{1,2k+1} \in A$ with $\operatorname{Ker}\rho_{1,2k+1} = \operatorname{Ker}\alpha_{1,2k+1}$. Clearly, $\rho_{1,2k+1} \notin A^{(3)}$.

Let $\alpha = \alpha_{2m,n}$, for some $m \in \{1, \ldots, \frac{n-4}{2}\}$. Analogously, we can show there exists $\rho_{2m,n} \in A$ with $\operatorname{Ker}\rho_{2m,n} = \operatorname{Ker}\alpha_{2m,n}$. Moreover, it is easy to verify that $\rho_{2m,n} \notin A^{(3)}$ and $\rho_{2m,n} \neq \rho_{1,2k+1}$, since $(2m, n) \in \operatorname{Ker}\rho_{2m,n}$ and $(2m,n) \notin \operatorname{Ker}\rho_{1,2k+1}$, for $k \in \{2, \ldots, \frac{n}{2}-1\}$.

Take

$$B^{(4)} = \{\rho_{1,2k+1} \mid k \in \{2, \ldots, \tfrac{n}{2}-1\}\} \cup \{\rho_{2m,n} \mid m \in \{1, \ldots, \tfrac{n-4}{2}\}\}.$$

Then, $\left|B^{(4)}\right| = \frac{n-4}{2} + \frac{n-4}{2} = n-4$. Furthermore, define $A^{(4)} = A^{(3)} \cup B^{(4)}$. Since $A^{(3)} \cap B^{(4)} = \emptyset$, it follows that $\left|A^{(4)}\right| = \left|A^{(3)}\right| + \left|B^{(4)}\right| = 2n-4+n-4 = 3n-8$.

Let $\alpha = \beta_{k,m}$, for some $k, m \in \{2, \ldots, n\}$ such that $k + 3m \leq n - 1$. Similarly to the proof of Lemma 3.5, we can prove the existence of an element $\delta_{k,m} \in A$ such that $\operatorname{Ker} \delta_{k,m} = \operatorname{Ker} \beta_{k,m}$. Clearly, we also have $\delta_{k,m} \notin A^{(4)}$. Take

$$B^{(5)} = \{\delta_{k,m} \mid k, m \in \{2, \ldots, n\} \text{ and } k + 3m \leq n - 1\}.$$

Then, $\left|B^{(5)}\right| = \sum_{k=2}^{n-7} \left(\left\lfloor \frac{n-1-k}{3} \right\rfloor - 1\right)$. Moreover, being $A^{(5)} = A^{(4)} \cup B^{(5)}$, since $A^{(4)} \cap B^{(5)} = \emptyset$, we obtain

$$\left|A^{(5)}\right| = \left|A^{(4)}\right| + \left|B^{(5)}\right| = 3n - 8 + \sum_{k=2}^{n-7} \left(\left\lfloor \frac{n-1-k}{3} \right\rfloor - 1\right) = |G_n|.$$

Since $A^{(5)} \subseteq A$, we have $|A| \geq \left|A^{(5)}\right| = 3n - 8 + \sum_{k=2}^{n-7} \left(\left\lfloor \frac{n-1-k}{3} \right\rfloor - 1\right)$, which allows us to conclude that $\operatorname{rank}(\mathcal{TF}_n) \geq 3n - 8 + \sum_{k=2}^{n-7} \left(\left\lfloor \frac{n-1-k}{3} \right\rfloor - 1\right) = |G_n|$, as required. □

As an immediate consequence of Proposition 3.4 and Lemmas 3.5 and 3.6, we can state our main result.

Theorem 3.7 *We have*

$$\operatorname{rank}(\mathcal{TF}_n) = \begin{cases} \frac{3}{2}(n-1) + \sum_{k=2}^{\frac{n-5}{2}} \left(\left\lfloor \frac{n+1-2k}{3} \right\rfloor - 1\right) & \text{if } n \text{ is odd} \\ 3n - 8 + \sum_{k=2}^{n-7} \left(\left\lfloor \frac{n-1-k}{3} \right\rfloor - 1\right) & \text{if } n \text{ is even.} \end{cases}$$

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