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On a Semigroup of Sets of Transformations with Restricted Range

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Abstract : This paper bases on the well-studied semigroup $T(X,Y)$ of all transformations on X with restricted range $Y \subseteq X$. We introduce the semigroup $T_P(X,Y)$ of all non-empty subsets of $T(X,Y)$ under the operation $AB := \{ab : a \in A, b \in B\}$. We determine the idempotent and regular elements in $T_P(X,Y)$ for the case that $|Y| = 2$. In particular, we characterize the (maximal) regular subsemigroups of $T_P(X,Y)$, the largest semiband, and the (maximal) idempotent subsemigroups of $T_P(X,Y)$.

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1 Introduction

Let $X = \{1, \ldots, n\}$, we denote by $T(X)$ the monoid of all full transformations on X (functions from X to X). The operation is the composition of functions. In the paper, we will write functions from the right, $x\alpha$ rather than $\alpha(x)$ and compose from the left to the right, $x(\alpha\beta) = (x\alpha)\beta$ rather than $(\alpha\beta)(x) = \alpha(\beta(x))$, $\alpha, \beta \in T(X), x \in X$.

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We denote by $im\alpha$ the image (the range) of α, i.e. $im\alpha := X\alpha := \{x\alpha : x \in X\}$ and by $rank\alpha$ the cardinality of $im\alpha$, i.e. $rank\alpha := |im\alpha|$. The kernel of α is the set $\ker\alpha := \{(x,y) : x,y \in X, x\alpha = y\alpha\}$. It is an equivalence relation and thus $\ker\alpha$ corresponds uniquely to a partition of X into blocks. The transformation α is called idempotent if $\alpha\alpha = \alpha$. Notice that α is idempotent if and only if α restricted to $im\alpha$ is the identity mapping on $im\alpha$. For more information about transformation semigroups and semigroups see [1] and [2], respectively.
By several reasons, it can happen that all transformations under consideration have a range in a proper subset of X. Let Y be a non-empty subset of X, say $Y = \{y_1, \ldots, y_m\}$ for some $m \in \{1, \ldots, n\}$. Note that $X = Y$ if $n = m$. Let us consider the set $T(X,Y) := \{\alpha \in T(X) : im\alpha \subseteq Y\}$. In particular, $T(X,Y)$ is a subsemigroup of $T(X)$, which is a semigroup of transformations with restricted range due to J. S. V. Symons [3]. Transformation semigroups with restricted range have been widely investigated (see for example [4, 5, 6, 7, 8, 9]). If Y is a one-element set, say $Y = \{y_1\}$, then $T(X,Y)$ is an one-element set, too, since the only transformation in $T(X,Y)$ is the constant mapping with the image y_1.
Let us now consider a two-element subset Y of X, say $Y = \{y_1, y_2\}$. Then, $T(X,Y)$ consists of the constant mapping with image y_1 (denoted by c_1), the constant mapping with image y_2 (denoted by c_2), and mappings with non-trivial kernel. For any $\alpha \in T(X,Y)$, we define a mapping $\alpha^* \in T(X,Y)$ by

$$x\alpha^* := \begin{cases} y_2 & \text{if} \quad x\alpha = y_1 \\ y_1 & \text{if} \quad x\alpha = y_2. \end{cases}$$

It is easy to verify that $c_1^* = c_2$, $(\alpha^*)^* = \alpha$, $\alpha\beta = \alpha$, and $\alpha\beta^* = \alpha^*$, whenever $\beta \in T(X,Y)$ is an idempotent with rank 2. We observe that $\alpha\beta = c_i$, whenever $y_1\beta = y_2\beta = c_i$ $(i \in \{1,2\})$. For any non-empty set $A \subseteq T(X,Y)$, we put $A^* := \{\alpha^* : \alpha \in A\}$.
This motivates the consideration of the following four subsets of $T(X,Y)$:

$$\begin{aligned} T_1 &:= \{\alpha \in T(X,Y) : \alpha \text{ is idempotent with rank } 2\}; \\ T_2 &:= \{\alpha^* : \alpha \in T_1\}; \\ T_{2+i} &:= \{\alpha \in T(X,Y) : y_1\alpha = y_2\alpha = c_i\} \text{ for } i \in \{1,2\}. \end{aligned}$$

Clearly, $T_4 = \{\alpha^* : \alpha \in T_3\}$. It is easy to verify that $\{T_1, T_2, T_3, T_4\}$ is a partition of $T(X,Y)$. For any non-empty set $A \subseteq T(X,Y)$ and any $i \in \{1,2,3,4\}$, we put

$$A_i := A \cap T_i.$$

Clearly, $A = A_1 \,\dot\cup\, A_2 \,\dot\cup\, A_3 \,\dot\cup\, A_4$.
Let $A, B \subseteq T(X,Y)$ be non-empty sets. If $B_1 \neq \emptyset$, i.e. there is an idempotent $\beta \in B$ with rank 2 such that $\alpha\beta = \alpha$ for all $\alpha \in A$, then $A \subseteq AB$ and $A = AB$ if $B = B_1$. If $B_3 \neq \emptyset$, i.e. there is $\beta \in B$ with $y_1\beta = y_2\beta = c_1$ and $\alpha\beta = c_1$ for all $\alpha \in A$, then $c_1 \in AB$ and $AB = \{c_1\}$ if $B = B_3$. By the same reasons, we have $c_2 \in AB$, whenever $B_4 \neq \emptyset$ and $AB = \{c_2\}$, whenever $B = B_4$. If $B \subseteq \{c_1, c_2\}$, then $AB = B$ since $\alpha c_i = c_i$ for all $i = 1,2$ and all $\alpha \in A$. Finally if $B_2 \neq \emptyset$, i.e.

there is an idempotent β^* with rank 2 such that $\beta \in B$ and $\alpha\beta = \alpha^*$ for all $\alpha \in A$, then $A^* \subseteq AB$ and $A^* = AB$ if $B = B_2$. So, if $B_1 \neq \emptyset$ and $B_2 \neq \emptyset$, then the set $R_A := A \cup A^*$ is contained in AB and $R_A = AB$ if $B = B_1 \cup B_2$. The notation R_A is due to the Green's relation $\mathcal{R}$. Notice that, two transformations $\alpha, \beta \in T(X,Y)$ are $\mathcal{R}$-related, in symbols $\alpha \mathcal{R} \beta$, if and only if $\ker \alpha = \ker \beta$ [2]. We observe that $\ker \alpha = \ker \beta$ if and only if $\alpha = \beta$ or $\alpha = \beta^*$, i.e. the $\mathcal{R}$-class of any $\alpha \in T(X,Y)$ is $\{\alpha, \alpha^*\}$. Throughout this paper, we will use these observations without to refer to them.

Let $T_p(X,Y)$ be the set of all non-empty subsets of $T(X,Y)$ and let us establish $T_p(X,Y)$ with a binary operation $\cdot$ defined by

$$A \cdot B := \{ab : a \in A, b \in B\}$$

for $A, B \in T_p(X,Y)$. Clearly, $\cdot$ is associative. We will write AB rather than $A \cdot B$. Notice the semigroup $T(X,Y)$ can be isomorphic embedded into $T_P(X,Y)$ by $\Phi : T(X,Y) \to T_P(X,Y)$ with $\alpha \mapsto \{\alpha\}$ for $\alpha \in T(X,Y)$. The purpose of this paper is the study of several basic properties of the semigroup $T_P(X,Y)$.

The next section deals with the set E of all idempotents in $T_P(X,Y)$, i.e. with elements $A \in T_P(X,Y)$ satisfying $AA = A$. It is easy to see that $T(X,Y)$ is not a band (take any $\beta \in T_1$ and we have $\beta^*\beta^* = (\beta^*)^* = \beta$). Therefore, $T_P(X,Y)$ is also not a band since $T(X,Y)$ can be isomorphically embedded into $T_P(X,Y)$. We will determine the set E and characterize all maximal subsemigroups consisting entirely of idempotents. Moreover, we determine the semigroup generated by E, i.e. the greatest semiband within $T_P(X,Y)$. The third section is devoted to the regular elements. A set $A \in T_P(X,Y)$ is called regular if there is $B \in T_P(X,Y)$ such that $ABA = A$. We characterize the set of all regular elements in $T_P(X,Y)$. It is a proper subset of $T_P(X,Y)$, i.e. $T_P(X,Y)$ is not regular. But we can provide all maximal regular subsemigroups of $T_P(X,Y)$. Finally, we will obtain the least subsemigroup of $T_P(X,Y)$ containing all regular elements.

2 The Idempotent Elements

Note that an idempotent element $A \in E$ of $T_P(X,Y)$ is a subsemigroup of $T(X,Y)$, i.e. $A \leq T(X,Y)$, since $AA = A$. But conversely, not each subsemigroup of $T(X,Y)$ is idempotent in $T_P(X,Y)$. For example, T_4 is a subsemigroup of $T(X,Y)$ but $T_4 \notin E$ since $T_4T_4 = \{c_2\} \subsetneqq T_4$. The following proposition characterizes all the idempotents in $T_P(X,Y)$.

Proposition 2.1. *Let $A \in T_P(X,Y)$. Then $A \in E$ if and only if the following three conditions are satisfied:*
(i) $R_A = A$ (i.e. $A = A^*$) if $A_2 \neq \emptyset$.
(ii) $A \subseteq \{c_1, c_2\}$ if $A_1 \cup A_2 = \emptyset$.
(iii) $c_i \in A$ if $A_{2+i} \neq \emptyset$, $i = 1, 2$.

Proof. Suppose that (i), (ii), and (iii) are satisfied.
Assume that $A_1 \neq \emptyset$. Then $A \subseteq AA$. Let now $\alpha \in AA$. Then there are $\alpha_1, \alpha_2 \in A$

with $\alpha = \alpha_1\alpha_2$. If $\alpha_2 \in A_1$ then $\alpha = \alpha_1\alpha_2 = \alpha_1 \in A$. If $\alpha_2 \in A_2$ then $A_2 \neq \emptyset$ and $\alpha = \alpha_1\alpha_2 = \alpha_1^* \in A^* = A$ by (i). Let $i \in \{1,2\}$. If $\alpha_2 \in A_{2+i}$ then $A_{2+i} \neq \emptyset$ and $\alpha = \alpha_1\alpha_2 = c_i \in A$ by (iii).
Admit that $A_1 = \emptyset$. Then $A_2 = \emptyset$ by (i), i.e. $A_1 \cup A_2 = \emptyset$. Thus, $A \subseteq \{c_1, c_2\}$ by (ii) and we have $AA = A$. Suppose now that $AA = A$ and we have to show that (i), (ii), and (iii) are satisfied. Admit that $A_2 \neq \emptyset$. Then $A^* \subseteq AA = A$ and thus $A = (A^*)^* \subseteq A^*$. This shows that $A = A^*$ and we have (i). Admit that $A_1 \cup A_2 = \emptyset$, i.e. $A \subseteq A_3 \cup A_4$. Then $A = AA \subseteq \{c_1, c_2\}$. This shows (ii). Let $i \in \{1,2\}$ and suppose that $A_{2+i} \neq \emptyset$. Then we obtain $c_i \in AA = A$, i.e. we have shown (iii). □

By Proposition 2.1, it is easy to verify that the following both sets D_1 and D_2 are subsets of E:

$$\begin{aligned} D_1 \quad &: \quad = \{R_A : \emptyset \neq A \subseteq T_1\} \text{ and} \\ D_2 \quad &: \quad = \{A \cup B \cup \{c_i\} : \emptyset \neq A \subseteq T_1, B \subseteq T_{2+i}, i = 1,2\}. \end{aligned}$$

Lemma 2.2. *We have $D_1 D_2 \cap E = \emptyset$.*

Proof. Let $A \in D_1$ and $A' \in D_2$. Then there are a non-empty set $\widehat{A} \subseteq T_1$ and a set $\widehat{B} \subseteq T_{2+i}$ such that $A' = \widehat{A} \cup \widehat{B} \cup \{c_i\}$ for some $i \in \{1,2\}$. This gives $AA' = A \cup \{c_i\}$. We have $(AA')_2 \neq \emptyset$ (since $A_2 \neq \emptyset$) but $c_i^* \notin AA'$ (since $A \subseteq T_1 \cup T_2$). Thus, $AA' \notin E$ by Proposition 2.1. □

Lemma 2.2 shows that $T_P(X,Y)$ is not orthodox, i.e., its idempotent set does not form a subsemigroup and it arises the question for the (maximal) idempotent subsemigroups of $T_P(X,Y)$. A semigroup S is a maximal idempotent subsemigroup of $T_P(X,Y)$ if $S \subseteq E$ and each subsemigroup of $T_P(X,Y)$, which covers S properly, consists not entirely of idempotents. Let us put

$$E_1 := E \setminus D_1.$$

Lemma 2.3. *E_1 is a maximal idempotent subsemigroup of $T_P(X,Y)$.*

Proof. Let $A, B \in E_1$.
Suppose that $A_1 \neq \emptyset$ and $B_1 \neq \emptyset$. Since both A and B do not belong to D_1, we have $A_2 = \emptyset$ or $\{c_1, c_2\} \subseteq A$ as well as $B_2 = \emptyset$ or $\{c_1, c_2\} \subseteq B$. If $(AB)_2 = \emptyset$ then $B_2 = \emptyset$ since $A_1 \neq \emptyset$. Thus, $AB = A \cup C$, where $C \subseteq \{c_1, c_2\}$. Notice that $A_2 = \emptyset$ since $(AB)_2 = \emptyset$ and $AB = A \cup C$. Thus AB is an idempotent by Proposition 2.1. Because $(AB)_2 = \emptyset$, we have $AB \in E_1$. If $(AB)_2 \neq \emptyset$ then $A_2 \neq \emptyset$ or $B_2 \neq \emptyset$, i.e. $\{c_1, c_2\} \subseteq A$ (if $A_2 \neq \emptyset$) or $\{c_1, c_2\} \subseteq B$ (if $B_2 \neq \emptyset$). This provides $\{c_1, c_2\} \subseteq AB$. In both cases, we obtain $AB = R_A \cup \{c_1, c_2\}$. Thus AB is an idempotent by Proposition 2.1, and in particular, $AB \in E_1$.
Suppose that $A_1 = \emptyset$ or $B_1 = \emptyset$. Then $A \subseteq \{c_1, c_2\}$ (if $A_1 = \emptyset$, i.e. $A_1 \cup A_2 = \emptyset$) or $B \subseteq \{c_1, c_2\}$ (if $B_1 = \emptyset$, i.e. $B_1 \cup B_2 = \emptyset$) by Proposition 2.1. Hence, $AB \subseteq \{c_1, c_2\}$, i.e. AB is idempotent by Proposition 2.1 and in particular, $AB \in E_1$.
We have shown that E_1 is a semigroup and it remains to show that E_1 is maximal. But this fact becomes clear by Lemma 2.2 and the fact $D_2 \subseteq E_1$. □

Now we put

$$E_2 := E \setminus D_2.$$

It is easy to check that $R_A, R_A \cup \{c_1, c_2\} \in E_2$, for any $A \subseteq T_1 \cup T_2$.

Lemma 2.4. *E_2 is a maximal idempotent subsemigroup of $T_P(X, Y)$.*

Proof. Let $A, B \in E_2$. If $A \subseteq \{c_1, c_2\}$ or $B \subseteq \{c_1, c_2\}$, then $AB \subseteq \{c_1, c_2\}$, and $AB \in E_2$. Suppose now that $A, B \nsubseteq \{c_1, c_2\}$. Then $\{c_1, c_2\} \subseteq A$ or $A \cap \{c_1, c_2\} = \emptyset$ and the same for B.
Suppose that $A_1 \neq \emptyset$ and $B_1 \neq \emptyset$. Then $AB = A \cup A^* \cup C$ (if $B_2 \neq \emptyset$) or $AB = A \cup C$ (if $B_2 = \emptyset$), where $C = \emptyset$ or $C = \{c_1, c_2\}$. Since $A \in E_2$, it is easy to verify that AB is idempotent by Proposition 2.1 . Clearly, $A \cup A^* \cup C \notin D_2$ and $A \cup C \notin D_2$. Thus, $AB \in E_2$.
Suppose now that $B_1 = \emptyset$. Then $B \subseteq \{c_1, c_2\}$ by Proposition 2.1 and thus $AB = B \in E_2$.
Suppose that $A_1 = \emptyset$ but $B_1 \neq \emptyset$. Then $A \subseteq \{c_1, c_2\}$ by Proposition 2.1 and thus $AB = A \in E_2$. or $AB = \{c_1, c_2\}$. Notice that $\{c_1, c_2\} \in E_2$. Therefore, we have $AB \in E_2$.
So, we have shown that E_2 is a semigroup. It remains to show that E_2 is maximal, which is clear by Lemma 2.2 and the fact that $D_1 \subseteq E_2$. □

Theorem 2.5. *Let $S \subseteq E$. Then S is a maximal idempotent subsemigroup of $T_P(X, Y)$ if and only if $S = E_1$ or $S = E_2$.*

Proof. One direction is clear by Lemma 2.3 and Lemma 2.4. Suppose now that S is a maximal idempotent subsemigroup. Since $S \subset E$, we obtain $S \subseteq E \setminus D_1 = E_1$ or $S \subseteq E \setminus D_2 = E_2$ by Lemma 2.2. Hence, $S = E_1$ or $S = E_2$ because of the maximality of S. □

Finally, we determine the greatest semiband in $T_P(X, Y)$. Let

$$\begin{aligned} D_3 &:= \{R_A \cup \{c_i\} : \emptyset \neq A \subseteq T_1, i = 1, 2\} \text{ and} \\ E_3 &:= E \cup D_3. \end{aligned}$$

Proposition 2.6. *E_3 is the greatest semiband in $T_P(X, Y)$.*

Proof. We have to show that E_3 is the least subsemigroup of $T_P(X, Y)$ containing E.
Let $A \subseteq T_1$ and let $i \in \{1, 2\}$. Then $R_A(T_1 \cup \{c_i\}) = R_A \cup \{c_i\}$, where both elements R_A and $(T_1 \cup \{c_i\})$ are idempotent. This shows that D_3 belongs to the least subsemigroup of $T_P(X, Y)$ containing E.
For the converse direction, we have to show that the product of elements in E_3 belongs to E_3. First, we check $EE \subseteq E_3$. By Lemma 2.3, Lemma 2.4 and the fact that $D_1 \cap D_2 = \emptyset$, it is enough to check the case that one factor belongs to D_1 and the other factor belongs to D_2. For this let $\emptyset \neq A \subseteq T_1$, $k \in \{1, 2\}$, and $B \subseteq T_{k+2}$. Further, let $R_G \in D_1$, where $\emptyset \neq G \subseteq T_1$. Then we get that

$(A \cup B \cup \{c_k\})R_G = R_{A\cup B\cup\{c_k\}} \in E$ and $R_G(A \cup B \cup \{c_k\}) = R_G \cup \{c_k\} \in D_3$. It is easy to verify that the product of an idempotent with an element from D_3 is equal to a product $e_1e_2e_3$ of three idempotents e_1, e_2, e_3 and thus, it is equal to a product of two idempotents (if e_1e_2 or e_2e_3 is an idempotent) or of two elements in D_3 (if both e_1e_2 and e_2e_3 are in D_3). So, it remains to show that $D_3D_3 \subseteq E$. Indeed, let $\emptyset \neq A, A' \subseteq T_1$ and $i, i' \in \{1, 2\}$. Then $(R_A \cup \{c_i\})(R_{A'} \cup \{c_{i'}\}) = R_A \cup \{c_1, c_2\} \in E$. □

3 The Regular Elements

This section is devoted to the regular subsemigroups of $T_p(X, Y)$. Clearly, each idempotent is regular. Hence, we have still to find the regular elements in $T_p(X, Y)$ which are not idempotent. For this let

$$\widehat{T}_{i+2} := \{A \subseteq T_{i+2} : c_i \in A\} \cup \{\emptyset\} \text{ for } i = 1, 2.$$

Lemma 3.1. *If $\emptyset \neq A \subseteq T_2$, $B \in \widehat{T}_3$, and $C \in \widehat{T}_4$ then $A \cup B \cup C$ is regular in $T_p(X, Y)$.*

Proof. We can calculate $(A \cup B \cup C)T_2 = A^* \cup B^* \cup C^*$ and $(A^* \cup B^* \cup C^*)A = A \cup B \cup C$. Because of $(A^* \cup B^* \cup C^*)B = \{c_1\} \subseteq B$ if $B \neq \emptyset$ and $(A^* \cup B^* \cup C^*)C = \{c_2\} \subseteq C$ if $C \neq \emptyset$, we obtain $(A \cup B \cup C)T_2\ (A \cup B \cup C) = A \cup B \cup C$. Therefore, $A \cup B \cup C$ is regular in $T_p(X, Y)$. □

We observe that, if $\emptyset \neq A \subseteq T_2$, $B \in \widehat{T}_3$, and $C \in \widehat{T}_4$, then $A \cup B \cup C$ is not idempotent by Proposition 2.1 . Hence the set

$$D_4 := \{A \cup B \cup C : \emptyset \neq A \subseteq T_2, B \in \widehat{T}_3, C \in \widehat{T}_4\}$$

is a set of non-idempotent regular elements in $T_p(X, Y)$. Moreover, we have:

Lemma 3.2. *If $A \in T_p(X, Y)$ is regular then $A \in E \cup D_4$.*

Proof. Let $A \in T_p(X, Y)$ be regular. Then there is $B \in T_p(X, Y)$ such that $ABA = A$.
If $A_3 \neq \emptyset$ then $c_1 \in ABA = A$. This shows that $A_3 \in \widehat{T}_3$. By the same reason, we obtain that $A_4 \in \widehat{T}_4$ if $A_4 \neq \emptyset$.
Suppose now that $A_2 = \emptyset$ and $A_1 = \emptyset$. Then $A \subseteq T_3 \cup T_4$ and $A = ABA \subseteq \{c_1, c_2\}$, i.e. $A \in E$ by Proposition 2.1. Suppose that $A_2 = \emptyset$ and $A_1 \neq \emptyset$. Then by the previous observations concerning A_3 and A_4, we obtain $A \in E$ by Proposition 2.1. Admit now that $A_2 \neq \emptyset$. Clearly, then $B_1 \cup B_2 \neq \emptyset$. If $A_1 \neq \emptyset$ then $(BA)_2 \neq \emptyset$. Thus, $A^* \subseteq ABA = A$, i.e. $A = A^*$ (it follows from $A = (A^*)^* \subseteq A^*$) and we obtain $A \in E$ by Proposition 2.1. If $A_1 = \emptyset$ then $A = A_2 \cup A_3 \cup A_4 \in D_4$. □

Proposition 3.3. *Any $A \in T_p(X, Y)$ is regular if and only if $A \in E \cup D_4$.*

Proof. Lemma 3.1 and Lemma 3.2 give the assertion. □

It is easy to verify that $D_3 \cap D_4 = \emptyset$. Hence, the regular elements do not form a semigroup. We are asking for the maximal regular subsemigroups of $T_p(X, Y)$. A semigroup S is a maximal regular subsemigroup of $T_P(X, Y)$ if S is regular and each subsemigroup of $T_P(X, Y)$, which covers S properly, is not regular.

Lemma 3.4. *$E_1 \cup D_4$ is a semigroup.*

Proof. We have to show that the product of two elements in $E_1 \cup D_4$ belongs to $E_1 \cup D_4$. By Lemma 2.3, it is enough to verify the case that at least one of the factors belongs to D_4. For this let $G \in E_1$, $\emptyset \neq A, A' \subseteq T_2$, $B, B' \in \widehat{T}_3$, and $C, C' \in \widehat{T}_4$. Then $(A \cup B \cup C)(A' \cup B' \cup C') = A^* \cup B^* \cup C^* \cup D$ with $D \subseteq \{c_1, c_2\}$, where $A^* \cup B^* \cup C^* \cup D \in E$ by Proposition 2.1. Because $(A^* \cup B^* \cup C^* \cup D) \cap T_2 = \emptyset$, we have $A^* \cup B^* \cup C^* \cup D \in E_1$.
Suppose that $G_2 \neq \emptyset$. Then $G = G^*$ and $\{c_1, c_2\} \subseteq G$. This implies $(A \cup B \cup C)G = R_{A \cup B \cup C} \cup \{c_1, c_2\} \in E_1$ and $G(A \cup B \cup C) = G^* = G \in E_1$.
Suppose that $G_2 = \emptyset$. If $G_1 = \emptyset$, then $G \subseteq \{c_1, c_2\}$ and thus $(A \cup B \cup C)G, G(A \cup B \cup C) \subseteq \{c_1, c_2\}$, i.e. $(A \cup B \cup C)G, G(A \cup B \cup C) \in E_1$. Admit now that $G_1 \neq \emptyset$. Then $(A \cup B \cup C)G = A \cup B \cup C \cup D'$ with $D' := (A \cup B \cup C)(G_3 \cup G_4) \subseteq \{c_1, c_2\}$. Let $B' := B \in \widehat{T}_3$ if $c_1 \notin D'$ and let $B' := B \cup \{c_1\} \in \widehat{T}_3$ if $c_1 \in D'$. In the same matter, we define $C' \in \widehat{T}_4$. Then $A \cup B \cup C \cup D' = A \cup B' \cup C' \in D_4$. Finally, we have $G(A \cup B \cup C) = G^* \cup ((B \cup C) \cap \{c_1, c_2\})$. Notice, we have $G^* = G_1^* \cup G_3^* \cup G_4^*$, where $\emptyset \neq G_1^* \subseteq T_2$ and $G_3^* \in \widehat{T}_4$ as well as $G_4^* \in \widehat{T}_3$. Thus, $G(A \cup B \cup C) = G^* \cup ((B \cup C) \cap \{c_1, c_2\}) \in D_4$ by the same argumentation as above. □

Proposition 3.5. *$E_1 \cup D_4$ is a maximal regular subsemigroup of $T_p(X, Y)$.*

Proof. $E_1 \cup D_4$ is a semigroup by Lemma 3.4. This semigroup is regular since for any $A \in D_4$, we have $AT_2A = A$ (see the proof of Lemma 3.1), where $T_2 \in D_4$.
It remains to show that $E_1 \cup D_4$ is maximal. It is easy to see that D_1 is the set of all regular elements in $T_p(X, Y)$ which not belong to $E_1 \cup D_4$. By Lemma 2.2 and Proposition 2.6, we have $D_1 D_2 \subseteq D_3$, where $D_2 \subseteq E_1$ and $D_3 \cap (E \cup D_4) = \emptyset$. This shows that $E_1 \cup D_4$ is a maximal regular subsemigroup of $T_p(X, Y)$ by Lemma 3.2. □

Let us denote by D_5 the set of all $A \in D_4$ with $A_3 \neq \emptyset$ if and only if $A_4 \neq \emptyset$.

Lemma 3.6. *$E_2 \cup D_5$ is a semigroup.*

Proof. We have to show that the product of two elements in $E_2 \cup D_5$ belongs to $E_2 \cup D_5$ again. It is enough to verify the case that at least one of the both factors belongs to D_5. For this let $G \in E_2$, $\emptyset \neq A, A' \subseteq T_2$, $B, B' \in \widehat{T}_3$, and $C, C' \in \widehat{T}_4$ such that $B \neq \emptyset$ if and only if $C \neq \emptyset$ and $B' \neq \emptyset$ if and only if $C' \neq \emptyset$. Then $(A \cup B \cup C)(A' \cup B' \cup C') = A^* \cup B^* \cup C^*$ or $(A \cup B \cup C)(A' \cup B' \cup C') = A^* \cup B^* \cup C^* \cup \{c_1, c_2\}$, where $A^* \subseteq T_1, C^*, C^* \cup \{c_1\} \in \widehat{T}_3$, and $B^*, B^* \cup \{c_2\} \in \widehat{T}_4$ such that $B^* \neq \emptyset$ if and only if $C^* \neq \emptyset$. Then, $(A \cup B \cup C)(A' \cup B' \cup C')$ is

idempotent by Proposition 2.1 and moreover, $(A\cup B\cup C)(A'\cup B'\cup C')\in E_2$ since $B^*\neq\emptyset$ if and only if $C^*\neq\emptyset$.
If $G_1=\emptyset$, then $G\subseteq\{c_1,c_2\}$. It follows that $G(A\cup B\cup C),(A\cup B\cup C)G\subseteq\{c_1,c_2\}$. Therefore, $G(A\cup B\cup C),(A\cup B\cup C)G\in E_2$. If $G_1\neq\emptyset$, we notice that $G_3\neq\emptyset$ if and only if $G_4\neq\emptyset$. It follows that

$$G(A\cup B\cup C)=\begin{cases} G^* & \text{if } B\cup C=\emptyset\\ G^*\cup\{c_1,c_2\} & \text{otherwise.}\end{cases}$$

We observe that $G^*=G$ (if $G_2\neq\emptyset$) and $G^*\subseteq T_2\cup T_3\cup T_4$ such that $(G^*)_2\neq\emptyset$ and $(G^*)_3\neq\emptyset$ if and only if $(G^*)_4\neq\emptyset$ (if $G_2=\emptyset$). Therefore, $G(A\cup B\cup C)\in E_2$ if $G_2\neq\emptyset$ and $G(A\cup B\cup C)\in D_5$ if $G_2=\emptyset$. On the other hand, we have

$$(A\cup B\cup C)G=\begin{cases} A\cup B\cup C\cup\{c_1,c_2\} & \in D_5 & \text{if} & G_2=\emptyset \text{ and } G_3\neq\emptyset\\ A\cup B\cup C & \in D_5 & \text{if} & G_2=\emptyset \text{ and } G_3=\emptyset\\ R_{A\cup B\cup C} & \in E_2 & \text{if} & G_2\neq\emptyset \text{ and } G_3=\emptyset\\ R_{A\cup B\cup C}\cup\{c_1,c_2\} & \in E_2 & \text{if} & G_2\neq\emptyset \text{ and } G_3\neq\emptyset.\end{cases}$$

This shows that $(A\cup B\cup C)G\in E_2\cup D_5$. □

Proposition 3.7. *$E_2\cup D_5$ is a maximal regular subsemigroup of $T_p(X,Y)$.*

Proof. By Lemma 3.6, we know that $E_2\cup D_5$ is a semigroup. Since $T_2\in D_5$ and $AT_2A=A$ for all $A\in D_5\subseteq D_4$ (see the proof of Lemma 3.1), the semigroup $E_2\cup D_5$ is regular.
It remains to show that $E_2\cup D_5$ is maximal. We show that any semigroup which covers $E_2\cup D_5$ properly, contains non-regular elements. For this let A be a regular element, which not belongs to $E_2\cup D_5$. It is easy to verify that $A\in D_2\cup(D_4\setminus D_5)$. Suppose that $A\in D_2$. Then $BA\in D_3$ for all $B\in D_1$ by Lemma 2.2, where $D_1\subseteq E_2$ and $D_3\cap D_4=\emptyset$, i.e. BA is not regular for all $B\in D_1\subseteq E_2$.
Suppose now that $A\in D_4\setminus D_5$. Then there are $\emptyset\neq A'\subseteq T_2$ and $B\in\widehat{T}_{2+k}$ for some $k\in\{1,2\}$ such that $A=A'\cup B$. We have $T_2\in D_5$, we calculate $AT_2=(A')^*\cup B^*$, where $B^*\in\begin{cases}\widehat{T}_3 & \text{if} \quad k=2\\ \widehat{T}_4 & \text{if} \quad k=1\end{cases}$ and $\emptyset\neq(A')^*\subseteq T_1$, i.e. $AT_2\in D_2$. □

Theorem 3.8. *Let $S\leq T_p(X,Y)$. Then S is a maximal regular subsemigroup of $T_p(X,Y)$ if and only if $S=E_1\cup D_4$ or $S=E_2\cup D_5$.*

Proof. One direction is clear by Proposition 3.5 and Proposition 3.7. Suppose now that S is a maximal regular subsemigroup of $T_p(X,Y)$. By Lemma 3.2, we have $S\subseteq E\cup D_4$. Assume that $S\nsubseteq E_1\cup D_4$ and $S\nsubseteq E_2\cup D_5$. Then there are $A\in S\setminus(E_1\cup D_4)$ and $B\in S\setminus(E_2\cup D_5)$. Since $A,B\in E\cup D_4$, we have $A\in D_1$ and $B\in D_2\cup(D_4\setminus D_5)$. If $B\in D_2$ then $AB\in D_3$ by Lemma 2.2.
Admit now that $B\in D_4\setminus D_5$. Then there are $\emptyset\neq C'\subseteq T_2$ and $C\in\widehat{T}_{2+i}$ for some $i\in\{1,2\}$ such that $B=C\cup C'$. Since $A^*=A=R_{A_1}$ we can calculate $AB=A\cup\{c_i\}$, i.e. $AB\in D_3$, too. But $D_3\cap(E\cup D_4)=\emptyset$. This contradicts $AB\in S\subseteq E\cup D_4$. Consequently, $S\subseteq E_1\cup D_4$ or $S\subseteq E_2\cup D_5$. Finally, the maximality of S provides $S=E_1\cup D_4$ or $S=E_2\cup D_5$. □

Finally, we determine the least semigroup containing all regular elements in $T_p(X,Y)$, i.e. the least semigroup containing $E \cup D_4$.

Proposition 3.9. *The least semigroup containing all regular elements in* $T_p(X,Y)$ *is* $E \cup D_3 \cup D_4$.

Proof. Notice, $E \cup D_3$ is the greatest semiband in $T_p(X,Y)$ by Proposition 2.6. Therefore, it is clear that the least semigroup containing all regular elements of $T_p(X,Y)$ covers $E \cup D_3 \cup D_4$. So, it remains to show that $E \cup D_3 \cup D_4$ is a semigroup. Notice that $E_1 \cup D_4$ is a regular semigroup by Lemma 3.4. Thus, it is enough to check that $D_1 D_4$ and $D_4 D_1$ as well as $D_3 D_4$ and $D_4 D_3$ are subsets of $E \cup D_3 \cup D_4$.
Let $R_{A'} \in D_1$ with $\emptyset \neq A' \subseteq T_1$ and let $A \cup B \cup C \in D_4$ with $\emptyset \neq A \subseteq T_2$, $B \in \widehat{T}_3$, and $C \in \widehat{T}_4$. Then $R_{A'}(A \cup B \cup C) = \begin{cases} R_{A'} \cup \{c_1\} & \in D_3 \quad \text{if } B \neq \emptyset \text{ and } C = \emptyset \\ R_{A'} \cup \{c_2\} & \in D_3 \quad \text{if } B = \emptyset \text{ and } C \neq \emptyset \\ R_{A'} \cup \{c_1, c_2\} & \in E \quad \text{if } B \neq \emptyset \text{ and } C \neq \emptyset \\ R_{A'} & \in E \quad \text{if } B = \emptyset \text{ and } C = \emptyset. \end{cases}$
This shows that $D_1 D_4 \in E \cup D_3 \subseteq E \cup D_3 \cup D_4$. On the other hand, we have $(A \cup B \cup C) R_{A'} = R_{A \cup B \cup C}$. By Proposition 2.1, we can check that $R_{A \cup B \cup C}$ is idempotent. This provides $D_4 D_1 \subseteq E \subseteq E \cup D_3 \cup D_4$. Let additional $i \in \{1, 2\}$. Then $\{c_i\}(A \cup B \cup C) = D$ for some $\emptyset \neq D \subseteq \{c_1, c_2\}$ and $(A \cup B \cup C)\{c_i\} = \{c_i\}$. Hence, $(R_{A'} \cup \{c_i\})(A \cup B \cup C) = R_{A'}(A \cup B \cup C) \cup D$ and $(A \cup B \cup C)(R_{A'} \cup \{c_i\}) = (A \cup B \cup C) R_{A'} \cup \{c_i\}$. By the previous observations, we obtain $(R_{A'} \cup \{c_i\})(A \cup B \cup C) = R_{A'} \cup D' \in E \cup D_3$, where $D \subseteq D' \subseteq \{c_1, c_2\}$, and $(A \cup B \cup C)(R_{A'} \cup \{c_i\}) = R_{A \cup B \cup C} \cup \{c_i\} \in E \cup D_3$. So, we have shown that $D_3 D_4, D_4 D_3 \subseteq E \cup D_3$. ☐

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