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ON RELATIVE RANKS OF FINITE TRANSFORMATION SEMIGROUPS WITH RESTRICTED RANGE

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We determine the relative rank of the semigroup $\mathcal{T}(X,Y)$ of all transformations on a finite chain X with restricted range $Y \subseteq X$ modulo the set $\mathcal{OP}(X,Y)$ of all orientation-preserving transformations in $\mathcal{T}(X,Y)$. Moreover, we determine the relative rank of the semigroup $\mathcal{OP}(X,Y)$ modulo the set $\mathcal{O}(X,Y)$ of all order-preserving transformations in $\mathcal{OP}(X,Y)$. In both cases, we characterize the minimal relative generating sets.

1. Introduction and Preliminaries

Let S be a semigroup. The *rank* of S (denoted by $\operatorname{rank} S$) is defined as the minimal number of elements of the generating set of S. The ranks of various known semigroups have been calculated in [7, 8, 10, 11]. For a set $A \subseteq S$, the *relative rank* of S modulo A, denoted by $\operatorname{rank}(S\colon A)$, is the minimal cardinality of a set $B \subseteq S$ such that $A \cup B$ generates S. It immediately follows from the definition that

$$\operatorname{rank}(S\colon \varnothing) = \operatorname{rank} S, \quad \operatorname{rank}(S\colon S) = 0, \quad \operatorname{rank}(S\colon A) = \operatorname{rank}(S\colon \langle A\rangle), \quad \text{and} \quad \operatorname{rank}(S\colon A) = 0$$

if and only if A is a generating set for S. The relative rank of a semigroup modulo a suitable set was first introduced by Ruškuc in [14] in order to describe the generating sets of semigroups with infinite rank. In [12], Howie, Ruškuc, and Higgins considered the relative ranks of the monoid $\mathcal{T}(X)$ of all full transformations on X, where X is an infinite set modulo some distinguished subsets of $\mathcal{T}(X)$. They showed that $\operatorname{rank}(\mathcal{T}(X)\colon \mathcal{S}(X)) = 2$, $\operatorname{rank}(\mathcal{T}(X)\colon \mathcal{E}(X)) = 2$, and $\operatorname{rank}(\mathcal{T}(X)\colon J) = 0$, where $\mathcal{S}(X)$ is the symmetric group on X, $\mathcal{E}(X)$ is the set of all idempotent transformations on X, and J is the top $\mathcal{J}$-class of $\mathcal{T}(X)$, i.e.,

$$J = \{\alpha \in \mathcal{T}(X)\colon |X\alpha| = |X|\}.$$

However, if, in addition, the rank is finite, then the relative rank gives information about the generating sets. In the present paper, we determine the relative rank for a particular semigroup of transformations on a finite set.

Let X be a finite chain, say, $X = \{1 < 2 < \ldots < n\}$ for a natural number n. A transformation $\alpha \in \mathcal{T}(X)$ is called *order-preserving* if $x \leq y$ implies that $x\alpha \leq y\alpha$ for all $x, y \in X$. We denote by $\mathcal{O}(X)$ the submonoid of $\mathcal{T}(X)$ of all order-preserving full transformations on X. The relative rank of $\mathcal{T}(X)$ modulo $\mathcal{O}(X)$ was considered by Higgins, Mitchell, and Ruškuc in [9]. They showed that $\operatorname{rank}(\mathcal{T}(X)\colon \mathcal{O}(X)) = 1$, if X is an arbitrary countable chain or an arbitrary well-ordered set, whereas $\operatorname{rank}(\mathcal{T}(\mathbb{R})\colon \mathcal{O}(\mathbb{R}))$ is uncountable if we consider the usual order of the set $\mathbb{R}$ of real numbers. In [2], Dimitrova, Fernandes, and Koppitz studied the relative rank of

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the semigroup $\mathcal{O}(X)$ modulo $J = \{\alpha \in \mathcal{O}(X)\colon |X\alpha| = |X|\}$ for an infinite countable chain X. We say that a transformation $\alpha \in \mathcal{T}(X)$ is *orientation-preserving* if there are subsets $X_1, X_2 \subseteq X$ such that $\varnothing \neq X_1 < X_2$ (i.e., $x_1 < x_2$ for $x_1 \in X_1$ and $x_2 \in X_2$), $X = X_1 \cup X_2$, and $x\alpha \leq y\alpha$, whenever either $(x, y) \in X_1^2 \cup X_2^2$ with $x \leq y$ or $(x, y) \in X_2 \times X_1$. Note that $X_2 = \varnothing$ implies $\alpha \in \mathcal{O}(X)$. We denote by $\mathcal{OP}(X)$ the submonoid of $\mathcal{T}(X)$ of all orientation-preserving full transformations on X. An equivalent notion of orientation-preserving transformation was first introduced by McAlister in [13] and, independently, by Catarino and Higgins in [1]. It is clear that $\mathcal{O}(X)$ is a submonoid of $\mathcal{OP}(X)$, i.e., $\mathcal{O}(X) \subset \mathcal{OP}(X) \subset \mathcal{T}(X)$. It is worth noting that the relative rank of $\mathcal{T}(X)$ modulo $\mathcal{OP}(X)$, as well as the relative rank of $\mathcal{OP}(X)$ modulo $\mathcal{O}(X)$, is equal to one (see [1, 12]) but the situation changes if we consider transformations with restricted range.

Let $Y = \{a_1 < a_2 < \ldots < a_m\}$ be a nonempty subset of X for a natural number $m \leq n$. By $\mathcal{T}(X, Y)$ we denote the subsemigroup $\{\alpha \in \mathcal{T}(X)\colon X\alpha \subseteq Y\}$ of $\mathcal{T}(X)$ of all transformations with range (image) restricted to Y. The set $\mathcal{T}(X, Y)$ coincides with $\mathcal{T}(X)$, whenever $Y = X$ (i.e., $m = n$). In 1975, Symons [15] introduced and studied a semigroup $\mathcal{T}(X, Y)$, which is called a semigroup of transformations with restricted range. Recently, the rank of $\mathcal{T}(X, Y)$ was computed by Fernandes and Sanwong [6]. They proved that the rank of $\mathcal{T}(X, Y)$ is the Sterling number $S(n, m)$ of the second kind with $|X| = n$ and $|Y| = m$. The rank of the order-preserving counterpart $\mathcal{O}(X, Y)$ of $\mathcal{T}(X, Y)$ was studied in [4] by Fernandes, Honyam, Quinteiro, and Singha. The authors found that

$$\operatorname{rank} \mathcal{O}(X, Y) = \binom{n-1}{m-1} + \left|Y^{\#}\right|,$$

where $Y^{\#}$ denotes the set of all $y \in Y$ with one of the following properties:

(i) y has no successors in X;

(ii) y is not a successor of any element in X;

(iii) both the successor of Y and the element whose successor is y belong to Y.

Moreover, the regularity and rank of the semigroup $\mathcal{OP}(X, Y)$ were studied by the same authors in [5]. They showed that

$$\operatorname{rank} \mathcal{OP}(X, Y) = \binom{n}{m}.$$

In [16], Tinpun and Koppitz investigated the relative rank of $\mathcal{T}(X, Y)$ modulo $\mathcal{O}(X, Y)$ and proved that

$$\operatorname{rank}\, (\mathcal{T}(X, Y)\colon \mathcal{O}(X, Y)) = S(n, m) - \binom{n-1}{m-1} + a,$$

where $a \in \{0, 1\}$ depending on the set Y. In the present paper, we determine the relative rank of $\mathcal{OP}(X, Y)$ modulo $\mathcal{O}(X, Y)$, as well as the relative rank of $\mathcal{T}(X, Y)$ modulo $\mathcal{OP}(X, Y)$.

Let $\alpha \in \mathcal{T}(X, Y)$. The kernel of α is the equivalence relation $\ker \alpha$ with $(x, y) \in \ker \alpha$ if $x\alpha = y\alpha$. It uniquely corresponds to a partition on X. This enables us to treat $\ker \alpha$ as a partition on X. A block of this partition is called a $\ker \alpha$-class. In particular, the sets $x\alpha^{-1} = \{y \in X : y\alpha = x\}$ for $x \in X\alpha$ are the $\ker \alpha$-classes. We say that a partition P is a subpartition of a partition Q of X if, for all $p \in P$, there is $q \in Q$ with $p \subseteq q$. A set $T \subseteq X$ with $\left|T \cap x\alpha^{-1}\right| = 1$ for all $x \in X\alpha$ is called a transversal of $\ker \alpha$. Let $A \subseteq X$. Then $\alpha|_A : A \to Y$ denotes the restriction of α to A. Moreover, A is called convex if $x < y < z$ with $x, z \in A$ implies that $y \in A$.

Let $l \in \{1, \dots, m\}$. By $\mathcal{P}_l$ we denote the set of all partitions $\{A_1, \dots, A_l\}$ of X such that $A_2 < A_3 < \dots < A_l$ are convex sets (for $l > 1$) and A_1 is the union of two convex sets with $1, n \in A_1$. Further, let $\mathcal{Q}_l$ be the set of all partitions $\{A_1, \dots, A_l\}$ of X such that $A_1 < A_2 < \dots < A_l$ are convex and let $\mathcal{R}_l$ be the set of all partitions of X that do not belong to $\mathcal{Q}_l \cup \mathcal{P}_l$. We observe that $\ker \beta \in \mathcal{Q}_l \cup \mathcal{P}_l$, whenever $\beta \in \mathcal{OP}(X, Y)$ with $|X\beta| = l$. In particular, $\ker \beta \in \mathcal{Q}_l$, whenever $\beta \in \mathcal{O}(X, Y)$.

We consider the case $l = m > 1$. For $P \in \mathcal{P}_m$ with the blocks $A_1,\ A_2 < \dots < A_m$, let α_P be the transformation on X defined by

$$x\alpha_P := a_i, \qquad \text{whenever} \quad x \in A_i \quad \text{for} \quad 1 \le i \le m,$$

in the case where $1 \notin Y$ or $n \notin Y$, and let

$$x\alpha_P := \begin{cases} a_{i+1}, & \text{if} \quad x \in A_i \quad \text{for} \quad 1 \le i < m, \\ a_1, & \text{if} \quad x \in A_m, \end{cases}$$

in the case where $1, n \in Y$. Clearly, $\ker \alpha_P = P$. For $X_1 = \{1, \dots, \max A_m\}$ and $X_2 = \{\max A_m + 1, \dots, n\}$ in the case where $1 \notin Y$ or $n \notin Y$ and $X_1 = \{1, \dots, \max A_{m-1}\}$ and $X_2 = \{\max A_{m-1} + 1, \dots, n\}$ in the case where $1, n \in Y$ [here, $\max A_m$ ($\max A_{m-1}$) denotes the greatest element in the set A_m (A_{m-1}, respectively)], we can easy verify that α_P is orientation-preserving.

Further, let $\eta \in \mathcal{T}(X, Y)$ be defined by

$$x\eta := \begin{cases} a_{i+1}, & \text{if} \quad a_i \le x < a_{i+1}, \quad 1 \le i < m, \\ a_1, & \text{if} \quad x = a_m, \\ a_\Gamma, & \text{otherwise}, \end{cases}$$

with

$$\Gamma := \begin{cases} 1, & \text{if} \quad 1 \notin Y, \\ 2, & \text{otherwise}, \end{cases}$$

in the case where $1 \notin Y$ or $n \notin Y$ and

$$x\eta := \begin{cases} a_{i+1}, & \text{if} \quad a_i \le x < a_{i+1}, \quad 1 \le i < m, \\ a_1 = 1, & \text{if} \quad x = a_m = n, \end{cases}$$

in the case where $1, n \in Y$. Note that $P_0 := \ker \eta \in \mathcal{P}_m$ if $1 \notin Y$ or $n \notin Y$ and $\ker \eta \in \mathcal{Q}_m$ if $1, n \in Y$. In fact, $\eta \in \mathcal{OP}(X, Y)$ with $X_1 = \{1, 2, \dots, a_m - 1\}$ and $X_2 = \{a_m, a_m + 1, \dots, n\}$. Moreover, $\eta|_Y$ is a permutation on Y, namely,

$$\eta|_Y = \begin{pmatrix} a_1 & \dots & a_{m-1} & a_m \\ a_2 & \dots & a_m & a_1 \end{pmatrix}.$$

We denote by $\mathcal{S}(Y)$ the set of all permutations on Y. Note that $\beta \in \mathcal{O}(X, Y)$ implies that either $\beta|_Y$ is the identity mapping on Y or $\beta|_Y \notin \mathcal{S}(Y)$.

2. Relative Rank of $\mathcal{OP}(X,Y)$ Modulo $\mathcal{O}(X,Y)$

In this section, we determine the relative rank of $\mathcal{OP}(X,Y)$ modulo $\mathcal{O}(X,Y)$. Some of these results were presented at the 47th Spring Conference of the Union of Bulgarian Mathematicians in March 2018 and published in the proceedings of this conference [3].

If $m = 1$, then $\mathcal{OP}(X,Y)$ is the set of all constant mappings, which coincides with $\mathcal{O}(X,Y)$, i.e.,

$$\operatorname{rank}\,(\mathcal{OP}(X,Y)\colon\,\mathcal{O}(X,Y)) = 0.$$

Hence, we admit that $m > 1$.

First, we show that

$$\mathcal{A} := \{\alpha_P\colon\, P \in \mathcal{P}_m\} \cup \{\eta\}$$

is a relative generating set of $\mathcal{OP}(X,Y)$ modulo $\mathcal{O}(X,Y)$. Note that $\eta = \alpha_{P_0}$ if $1 \notin Y$ or $n \notin Y$.

Lemma 1. *For each $\alpha \in \mathcal{OP}(X,Y)$ with $\operatorname{rank}\alpha = m$, there is $\widehat{\alpha} \in \{\alpha_P\colon\, P \in \mathcal{P}_m\} \cup \mathcal{O}(X,Y)$ with $\ker\alpha = \ker\widehat{\alpha}$.*

Proof. Let $\alpha \in \mathcal{OP}(X,Y)$ and let $X_1, X_2 \subseteq X$ as in the definition of orientation-preserving transformation. If $X_2 = \varnothing$, then $\alpha \in \mathcal{O}(X,Y)$. Suppose that $X_2 \neq \varnothing$. Let $X_1\alpha = \{x_1 < \ldots < x_r\}$ and let $X_2\alpha = \{y_1 < \ldots < y_s\}$ for suitable natural numbers r and s. We observe that $X_1\alpha$ and $X_2\alpha$ have at most one common element (only $x_1 = y_s$ could be possible). If $x_1 \neq y_s$, then

$$\ker\alpha = \left\{x_1\alpha^{-1} < \ldots < x_r\alpha^{-1} < y_1\alpha^{-1} < \ldots < y_s\alpha^{-1}\right\} = \ker\widehat{\alpha}$$

with

$$\widehat{\alpha} = \begin{pmatrix} x_1\alpha^{-1} & \ldots & x_r\alpha^{-1} & y_1\alpha^{-1} & \ldots & y_s\alpha^{-1} \\ a_1 & \ldots & a_r & a_{r+1} & \ldots & a_{r+s} \end{pmatrix} \in \mathcal{O}(X,Y).$$

If $x_1 = y_s$, then $1, n \in x_1\alpha^{-1} = y_s\alpha^{-1}$ and $\ker\alpha = \ker\alpha_P$ with

$$P = \left\{x_1\alpha^{-1}, x_2\alpha^{-1} < \ldots < x_r\alpha^{-1} < y_1\alpha^{-1} < \ldots < y_{s-1}\alpha^{-1}\right\} \in \mathcal{P}_m.$$

Lemma 1 is proved.

Proposition 1. $\mathcal{OP}(X,Y) = \langle \mathcal{O}(X,Y), \mathcal{A}\rangle$.

Proof. Let $\beta \in \mathcal{OP}(X,Y)$ with $\operatorname{rank}\beta = m$. Then there is $\theta \in \{\alpha_P : P \in \mathcal{P}_m\} \cup \mathcal{O}(X,Y)$ with $\ker\beta = \ker\theta$ by Lemma 1. In particular, there exists $r \in \{0, \ldots, m-1\}$ with $a_1\theta^{-1} = a_{r+1}\beta^{-1}$. Then it is easy to verify that $\beta = \theta\eta^r$, where $\eta^0 = \eta^m$.

Assume now that $i = \operatorname{rank}\beta < m$. Suppose that $\ker\beta \in \mathcal{P}_i$, say, $\ker\beta = \{A_1, A_2 < \ldots < A_i\}$ with $1, n \in A_1$. Then there is a subpartition $P' \in \mathcal{P}_m$ of $\ker\beta$. We set

$$\theta = \alpha_{P'} \qquad \text{and} \qquad a = \min X\beta.$$

Let T be a transversal of $\ker\theta$. In particular, we have

$$Y = \{x(\theta|_T)\eta^k\colon\, x \in T\} \quad \text{for all} \quad k \in \{1, \ldots, m\}.$$

Since both mappings $\theta|_T : T \to Y$ and $\eta|_Y : Y \to Y$ are bijections, there is $k \in \{1, \dots, m\}$ with

$$a_1((\theta|_T)\eta^k)^{-1}\beta = a \quad \text{and} \quad a_1((\theta|_T)\eta^{k+1})^{-1}\beta \neq a.$$

Moreover, since $(\theta|_T)\eta^k$ is a bijection from T to Y and both transformations $\theta\eta^k$ and β are orientation-preserving, it is easy to verify that $f^* = \big((\theta|_T)\eta^k\big)^{-1}\beta$ can be extended to an orientation-preserving transformation f defined by

$$xf = \begin{cases} a_1 f^*, & \text{if} \quad x < a_1, \\ a_i f^*, & \text{if} \quad a_i \leq x < a_{i+1}, \quad 1 \leq i < m, \\ a_m f^*, & \text{if} \quad a_m \leq x, \end{cases}$$

i.e., f and f^* coincide on Y. Moreover,

$$a_1 f = a_1 f^* = a_1\big((\theta|_T)\eta^k\big)^{-1}\beta = a.$$

In order to show that f is order-preserving, it remains to verify that $nf \neq a$. Assume that $nf = a$, where $n \geq a_m$. Then $nf = a_m f^* = a_m f$, i.e., $(n, a_m) \in \ker f$ and $n\eta = a_m\eta = a_1$. Hence, there is $x^* \in T$ such that $x^*\big((\theta|_T)\eta^k\big) = a_m$, i.e., $x^* = a_m\big((\theta|_T)\eta^k\big)^{-1}$. Thus, we get

$$\begin{aligned} a = nf = a_m f^* &= a_m\big((\theta|_T)\eta^k\big)^{-1}\beta \\ &= a_m(\eta^k|_Y)^{-1}(\theta|_T)^{-1}\beta \\ &= a_1(\eta|_Y)^{-1}(\eta^k|_Y)^{-1}(\theta|_T)^{-1}\beta = a_1((\theta|_T)\eta^{k+1})^{-1}\beta \neq a; \end{aligned}$$

a contradiction.

Finally, we show that $\beta = \theta\eta^k f \in \langle \mathcal{O}(X,Y), \mathcal{A}\rangle$. To do this, we assume that $x \in X$. Then there is $\widetilde{x} \in T$ such that $(x, \widetilde{x}) \in \ker \beta$. Hence, we get

$$x\theta\eta^k f = x\theta\eta^k f^* = \widetilde{x}\theta\eta^k\big((\theta|_T)\eta^k\big)^{-1}\beta = \widetilde{x}\beta = x\beta.$$

Further, suppose that $\ker\beta \notin \mathcal{P}_i$ and, thus, $\ker\beta \in \mathcal{Q}_i$. Let $X\beta = \{b_1, \dots, b_i\}$ be such that

$$b_1\beta^{-1} < \ldots < b_i\beta^{-1}.$$

Then we define a transformation φ by $x\varphi = a_j$ for all $x \in b_{j-1}\beta^{-1}$ and $2 \leq j \leq i+1$. Clearly, $\varphi \in \mathcal{O}(X,Y)$. Further, we define a transformation $\nu \in \mathcal{T}(X,Y)$ by

$$x\nu = \begin{cases} b_{j-1}, & \text{if} \quad a_j \leq x < a_{j+1}, \quad 2 \leq j \leq i, \\ b_i, & \text{otherwise}. \end{cases}$$

Since β is orientation-preserving, there is $k \in \{1, \dots, i\}$ such that $k = i$ or $b_1 < \ldots < b_{k-1} < b_k < \ldots < b_i$.

Then $X_1 = \{a_1, \ldots, a_{k+1} - 1\}$ and $X_2 = \{a_{k+1}, \ldots, n\}$ gives a partition of X provided that ν is orientation-preserving. Clearly, $\operatorname{rank} \nu = i$ and $1\nu = n\nu = b_i$. Thus, it is easy to verify that $\ker \nu \in \mathcal{P}_i$. Hence, $\nu \in \langle \mathcal{O}(X,Y), \mathcal{A} \rangle$ by the previous case and it remains to show that $\beta = \varphi\nu \in \langle \mathcal{O}(X,Y), \mathcal{A} \rangle$. To do this, we assume that $x \in X$. Then $x \in b_j\beta^{-1}$ for some $j \in \{1, \ldots, i\}$, i.e., $x\varphi\nu = a_{j+1}\nu = b_j = x\beta$.

Proposition 1 is proved.

This proposition shows that $\mathcal{A}$ is a relative generating set for $\mathcal{OP}(X,Y)$ modulo $\mathcal{O}(X,Y)$. It remains to show that $\mathcal{A}$ has the minimal size.

Lemma 2. *Let $B \subseteq \mathcal{OP}(X,Y)$ be a relative generating set of $\mathcal{OP}(X,Y)$ modulo $\mathcal{O}(X,Y)$. Then*

$$\mathcal{P}_m \subseteq \{\ker \alpha \colon \alpha \in B\}.$$

Proof. Let $P \in \mathcal{P}_m$. Since

$$\alpha_P \in \mathcal{OP}(X,Y) = \langle \mathcal{O}(X,Y), B \rangle\,,$$

there are $\theta_1 \in \mathcal{O}(X,Y) \cup B$ and $\theta_2 \in \mathcal{OP}(X,Y)$ with $\alpha_P = \theta_1\theta_2$. In view of the fact that $\operatorname{rank} \alpha_P = m$, we obtain $\ker \alpha_P = \ker \theta_1$. Since $1\alpha_P = n\alpha_P$, we conclude that $\theta_1 \notin \mathcal{O}(X,Y)$, i.e., $\theta_1 \in B$ with $\ker \theta_1 = \ker \alpha_P = P$.

Lemma 2 is proved.

In order to deduce a formula for the number of elements in $\mathcal{P}_m$, it is necessary to compute the number of possible partitions of X into $m+1$ convex sets. This number is equal to $\binom{n-1}{m}$.

Remark 1. $|\mathcal{P}_m| = \binom{n-1}{m}$.

We are now able to formulate the main result of the section. The relative rank of $\mathcal{OP}(X,Y)$ modulo $\mathcal{O}(X,Y)$ depends of the fact whether both 1 and n belong to Y or not.

Theorem 1. *For each $1 < m < n \in \mathbb{N}$,*

(i) $\operatorname{rank}(\mathcal{OP}(X,Y) \colon \mathcal{O}(X,Y)) = \binom{n-1}{m}$ *if $1 \notin Y$ or $n \notin Y$;*

(ii) $\operatorname{rank}(\mathcal{OP}(X,Y) \colon \mathcal{O}(X,Y)) = 1 + \binom{n-1}{m}$ *if $\{1,n\} \subseteq Y$.*

Proof. 1. Note that $\ker \eta \in \mathcal{P}_m$ and $\eta = \alpha_{P_0}$. Hence, the set $\mathcal{A} = \{\alpha_P \colon P \in \mathcal{P}_m\}$ is a generating set of $\mathcal{OP}(X,Y)$ modulo $\mathcal{O}(X,Y)$ by Proposition 1, i.e., the relative rank of $\mathcal{OP}(X,Y)$ modulo $\mathcal{O}(X,Y)$ is bounded by the cardinality of $\mathcal{P}_m$, which is equal to $\binom{n-1}{m}$ by Remark 1. However, this number cannot be reduced by Lemma 2.

2. Let $B \subseteq \mathcal{OP}(X,Y)$ be a relative generating set of $\mathcal{OP}(X,Y)$ modulo $\mathcal{O}(X,Y)$. By Lemma 2, we know that $\mathcal{P}_m \subseteq \{\ker \alpha \colon \alpha \in B\}$. Assume that the equality holds. Note that $\ker \eta \in \mathcal{Q}_m$ and η is not order-preserving. Hence, there are $\theta_1, \ldots, \theta_l \in \mathcal{O}(X,Y) \cup B$ for a suitable natural number l such that $\eta = \theta_1 \ldots \theta_l$.

From $\operatorname{rank}\eta = m$, we obtain $\ker\theta_1 = \ker\eta$ and $\operatorname{rank}\theta_i = m$ for $i \in \{1, \dots, l\}$. Therefore, $\{1, n\} \subseteq Y$ implies that $(1, n) \notin \ker\theta_i$ for $i \in \{2, \dots, l\}$. This implies that $\theta_2, \dots, \theta_l \in \mathcal{O}(X, Y)$. Since $\ker\theta_1 = \ker\eta \notin \mathcal{P}_m$, we get $\theta_1 \in \mathcal{O}(X, Y)$ and, consequently, $\eta = \theta_1\theta_2 \dots \theta_l \in \mathcal{O}(X, Y)$; a contradiction. Hence, we have verified that $|\mathcal{P}_m| < |B|$, i.e., the relative rank of $\mathcal{OP}(X, Y)$ modulo $\mathcal{O}(X, Y)$ is greater than $\binom{n-1}{m}$. However, it is bounded by $1 + \binom{n-1}{m}$ by virtue of Proposition 1. This proves the assertion.

Theorem 1 is proved.

We complete this section by the characterization of the minimal relative generating sets of $\mathcal{OP}(X, Y)$ modulo $\mathcal{O}(X, Y)$. We recognize that, among these sets, there are sets whose sizes are greater than

$$\operatorname{rank}\bigl(\mathcal{OP}(X, Y) :\, \mathcal{O}(X, Y)\bigr).$$

Theorem 2. *Let $B \subseteq \mathcal{OP}(X, Y)$. Then B is a minimal relative generating set of $\mathcal{OP}(X, Y)$ modulo $\mathcal{O}(X, Y)$ if and only if the following three statements are satisfied for the set $\widetilde{B} = \{\beta \in B :\ \ker\beta \in \mathcal{Q}_m\} \subseteq B$:*

(i) $\mathcal{P}_m \subseteq \bigl\{\ker\beta :\ \beta \in B \setminus \widetilde{B}\bigr\}$,

(ii) $|B \setminus \widetilde{B}| = |\mathcal{P}_m|$,

(iii) $\eta|_Y \in \langle\beta|_Y :\ \beta \in B\rangle$ *but* $\eta|_Y \notin \bigl\langle\beta|_Y :\ \beta \in B \setminus \{\gamma\}\bigr\rangle$ *for any* $\gamma \in \widetilde{B}$.

Proof. Suppose that conditions (i)–(iii) are satisfied for $\widetilde{B} = \{\beta \in B :\ \ker\beta \in \mathcal{Q}_m\}$. We now show that

$$\mathcal{A} \subseteq \langle\mathcal{O}(X, Y), B\rangle\,.$$

Let $\alpha \in \mathcal{A} \setminus \{\eta\}$. Then there is a partition

$$P = \{A_1, A_2 < \ldots < A_m\} \in \mathcal{P}_m$$

such that

$$\alpha = \alpha_P = \begin{pmatrix} A_1 & A_2 & \dots & A_m \\ a_1 & a_2 & \dots & a_m \end{pmatrix}, \qquad \text{if} \quad 1 \notin Y \quad \text{or} \quad n \notin Y,$$

or

$$\alpha = \alpha_P = \begin{pmatrix} A_1 & A_2 & \dots & A_{m-1} & A_m \\ a_2 & a_3 & \dots & a_m & a_1 \end{pmatrix}, \qquad \text{if} \quad 1, n \in Y.$$

Note that, in the last case, $a_1 = 1$ and $a_m = n$.

Further, it follows from (i) that there is $\beta \in B$ with $\ker\beta = \ker\alpha_P$, i.e.,

$$\beta = \alpha_P \quad \text{or} \quad \beta = \begin{pmatrix} A_1 & A_2 & \dots & A_{m-i+1} & A_{m-i+2} & \dots & A_m \\ a_i & a_{i+1} & \dots & a_m & a_1 & \dots & a_{i-1} \end{pmatrix} \ \text{for some} \ \ i \in \{3, \dots, m\}.$$

It is easy to see that $\alpha_P = \beta^k \in \langle B\rangle$ for a suitable natural number k. Hence,

$$\{\alpha_P :\ P \in \mathcal{P}_m\} \subseteq \langle\mathcal{O}(X, Y), B\rangle\,.$$

Further, $\ker\eta \in \mathcal{P}_m$, whenever $1 \notin Y$ or $n \notin Y$ and $\ker\eta \in \mathcal{Q}_m$, otherwise. Thus, there is $\delta \in \langle \mathcal{O}(X,Y), B\rangle$ with $\ker\delta = \ker\eta$. Therefore, as above, we conclude that $\eta = \delta^l \in \langle \mathcal{O}(X,Y), B\rangle$ for a suitable natural number l. Consequently, $\langle \mathcal{O}(X,Y), \mathcal{A}\rangle \subseteq \langle \mathcal{O}(X,Y), B\rangle$. By Proposition 1, we obtain

$$\mathcal{OP}(X,Y) = \langle \mathcal{O}(X,Y), B\rangle .$$

The generating set B is minimal by properties (i) and (ii) together with Lemma 2 and by the property (iii) of $\widetilde{B}$.

Conversely, let B be a minimal relative generating set of $\mathcal{OP}(X,Y)$ modulo $\mathcal{O}(X,Y)$. By Lemma 2, there is a set $\overline{B} \subseteq B$ such that

$$\mathcal{P}_m = \{\ker\beta \colon\ \beta \in \overline{B}\} \quad \text{and} \quad \left|\overline{B}\right| = |\mathcal{P}_m| .$$

Since $\mathcal{OP}(X,Y) = \langle \mathcal{O}(X,Y), B\rangle$, there are $\beta_1, \ldots, \beta_k \in \mathcal{O}(X,Y) \cup B$ such that $\eta = \beta_1 \ldots \beta_k$. Without loss of generality, we can assume that there is no $\gamma \in \left\{\beta_i \colon\ 1 \le i \le k,\ \ker\beta_i \in \mathcal{Q}_m\right\} =\colon \widehat{B}$ such that η is the product of transformations in $\overline{B} \cup (\widehat{B} \setminus \{\gamma\})$. In the first part of the proof, it has been shown that $\overline{B} \cup \widehat{B}$ is a relative generating set of $\mathcal{OP}(X,Y)$ modulo $\mathcal{O}(X,Y)$. Due to the minimality of B, we obtain $B = \overline{B} \cup \widehat{B}$, where

$$\left\{\ker\beta \colon\ \beta \in B \setminus \widehat{B}\right\} \supseteq \mathcal{P}_m, \quad |B \setminus \widehat{B}| = |\overline{B}| = |\mathcal{P}_m|, \quad \text{and} \quad \eta|_Y \in \langle \beta|_Y \colon\ \beta \in B\rangle$$

but

$$\eta|_Y \notin \left\langle \beta|_Y \colon\ \beta \in B \setminus \{\gamma\}\right\rangle \quad \text{for any} \quad \gamma \in \widehat{B}.$$

Theorem 2 is proved.

In particular, for the relative generating sets of the minimal size, we have the following remark.

Remark 2. $B \subseteq \mathcal{OP}(X,Y)$ is a relative generating set of $\mathcal{OP}(X,Y)$ modulo $\mathcal{O}(X,Y)$ of the minimal size if and only if $|\widetilde{B}| = 1$ for $1, n \in Y$ and $\widetilde{B} = \varnothing$, otherwise.

3. Relative Rank of $\mathcal{T}(X,Y)$ Modulo $\mathcal{OP}(X,Y)$

In this section, we determine the relative rank of $\mathcal{T}(X,Y)$ modulo $\mathcal{OP}(X,Y)$ and characterize all minimal relative generating sets of $\mathcal{T}(X,Y)$ modulo $\mathcal{OP}(X,Y)$. Since

$$\mathcal{O}(X,Y) \le \mathcal{OP}(X,Y),$$

we immediately conclude that

$$\operatorname{rank}(\mathcal{T}(X,Y) \colon\ \mathcal{OP}(X,Y)) \le S(n,m) - \binom{n-1}{m-1} + 1.$$

First, we formulate a sufficient condition for a set $B \subseteq \mathcal{T}(X,Y)$ to be a relative generating set of $\mathcal{T}(X,Y)$ modulo $\mathcal{OP}(X,Y)$.

Lemma 3. *Let $B \subseteq \mathcal{T}(X,Y)$. If $\mathcal{R}_m \subseteq \{\ker\beta \colon\ \beta \in B\}$ and $\mathcal{S}(Y) \subseteq \left\langle \{\beta|_Y \colon\ \beta \in B\}, \eta|_Y\right\rangle$, then*

$$\langle \mathcal{OP}(X,Y), B\rangle = \mathcal{T}(X,Y).$$

Proof. Let $\gamma \in \mathcal{T}(X,Y)$ with $\operatorname{rank}\gamma = k \leq m$. We consider two cases.

Case 1. Suppose that $\ker\gamma \in \mathcal{R}_k$. Then $\ker\gamma$ contains a nonconvex set that cannot be decomposed into two convex sets containing 1 and n, respectively. Since $k \leq m$, we can split the partition $\ker\gamma$ into a partition $P \in \mathcal{R}_m$ such that P contains a nonconvex set that cannot be decomposed into two convex sets containing 1 and n, respectively (if $k = m$, then we set $P = \ker\gamma$). Since $\mathcal{R}_m \subseteq \{\ker\beta : \beta \in B\}$, there is $\lambda \in B$ with $\ker\lambda = P$. It is clear that $X\lambda = Y$.

Further, let $X\gamma = \{y_1 < y_2 < \ldots < y_k\}$. We define the sets

$$A_i = \left\{x \in Y : x\lambda^{-1} \subseteq y_i\gamma^{-1}\right\}$$

for $i = 1, \ldots, k$. It is clear that $\{A_1, A_2, \ldots, A_k\}$ is a partition of Y. Moreover, let $\{C_1 < C_2 < \ldots < C_k\} \in \mathcal{Q}_k$ be a partition of X such that $|C_i \cap Y| = |A_i|$ for all $i = 1, \ldots, k$. Let $A_i = \{a_{i_1} < a_{i_2} < \ldots < a_{i_{t_i}}\}$ and let $C_i \cap Y = \{c_{i_1} < c_{i_2} < \ldots < c_{i_{t_i}}\}$ with $t_i \in \{1, \ldots, m\}$ for $i \in \{1, \ldots, k\}$. We define a bijection

$$\sigma : \bigcup_{i=1}^{k} A_i = Y \longrightarrow \bigcup_{i=1}^{k} (C_i \cap Y) = Y$$

on Y with $a_{i_l}\sigma = c_{i_l}$ for $l = 1, \ldots, t_i$ and $i = 1, \ldots, k$. Since $\sigma \in \mathcal{S}(Y)$ and $\mathcal{S}(Y) \subseteq \left\langle\{\beta|_Y : \beta \in B\}, \eta|_Y\right\rangle$, there exists $\mu \in \langle B, \eta\rangle$ with $\mu|_Y = \sigma$.

Finally, we define a transformation $\nu \in \mathcal{O}(X,Y) \subseteq \mathcal{OP}(X,Y)$ with $\ker\nu = \{C_1 < C_2 < \ldots < C_k\}$ and $x\nu = y_i$ for all $x \in C_i$ and $i = 1, \ldots, k$.

Therefore, we have $\lambda, \mu, \nu \in \langle\mathcal{OP}(X,Y), B\rangle$ and it remains to show that $\gamma = \lambda\mu\nu$, i.e., $\gamma \in \langle\mathcal{OP}(X,Y), B\rangle$. Let $x \in X$. Then $x\gamma = y_i$ for some $i \in \{1, \ldots, k\}$ and we get

$$x\gamma = y_i \Rightarrow x\lambda = z \in A_i \Rightarrow z\mu = u \in C_i \cap Y \Rightarrow u\nu = y_i.$$

Hence, $x\gamma = y_i = x(\lambda\mu\nu)$ and we conclude that $\gamma = \lambda\mu\nu$.

Case 2. Suppose that $\ker\gamma \notin \mathcal{R}_k$, i.e., $\ker\gamma \in \mathcal{Q}_k \cup \mathcal{P}_k$ and there is $\rho_1 \in \mathcal{OP}(X,Y)$ with $\ker\rho_1 = \ker\gamma$. Further, there is a partition $P = \{D_y : y \in X\rho_1\} \in \mathcal{R}_k$ such that $y \in D_y$ for all $y \in X\rho_1$. Then we define a transformation $\rho_2 : X \to X\gamma$ with $\ker\rho_2 = P$ and $\{x\rho_2\} = y\rho_1^{-1}\gamma$ for all $x \in D_y$ and $y \in X\rho_1$. Since $\ker\rho_1 = \ker\gamma$, the transformation ρ_2 is well defined, and we have $\gamma = \rho_1\rho_2$. Moreover, $\rho_2 \in \langle\mathcal{OP}(X,Y), B\rangle$ by Case 1 (since $\ker\rho_2 \in \mathcal{R}_k$) and, thus, $\gamma = \rho_1\rho_2 \in \langle\mathcal{OP}(X,Y), B\rangle$.

Lemma 3 is proved.

Lemma 4. $\langle\eta|_Y\rangle = \left\langle\{\beta|_Y : \beta \in \mathcal{OP}(X,Y)\}\right\rangle \cap \mathcal{S}(Y)$.

Proof. The inclusion $\langle\eta|_Y\rangle \subseteq \left\langle\{\beta|_Y : \beta \in \mathcal{OP}(X,Y)\}\right\rangle \cap \mathcal{S}(Y)$ is obvious. Now let $\beta \in \mathcal{OP}(X,Y)$ with $\beta|_Y \in \mathcal{S}(Y)$. Then there is $k \in \{1, \ldots, m\}$ such that

$$\beta = \begin{pmatrix} A_1 & \ldots & A_{m-k+1} & A_{m-k} & \ldots & A_m \\ a_k & \ldots & a_m & a_1 & \ldots & a_{k-1} \end{pmatrix}$$

with

$$\{A_1, A_2 < \ldots < A_m\} \in \mathcal{P}_m \cup \mathcal{Q}_m$$

and $a_i \in A_i$ for $i \in \{1, \dots, m\}$ because Y is a transversal of $\ker\beta$. Thus,

$$\beta|_Y = \begin{pmatrix} a_1 & \dots & a_{m-k+1} & a_{m-k} & \dots & a_m \\ a_k & \dots & a_m & a_1 & \dots & a_{k-1} \end{pmatrix} = (\eta|_Y)^{m-k+1} \in \langle \eta|_Y \rangle .$$

This means that

$$\big\langle \{\beta|_Y \colon \beta \in \mathcal{OP}(X,Y)\} \big\rangle \cap \mathcal{S}(Y) \subseteq \big\{(\eta|_Y)^p \colon p \in \mathbb{N}\big\} = \langle \eta|_Y \rangle .$$

Lemma 4 is proved.

The following lemmas give us necessary conditions for a set $B \subseteq \mathcal{T}(X,Y)$ to be a relative generating set of $\mathcal{T}(X,Y)$ modulo $\mathcal{OP}(X,Y)$.

Lemma 5. *Let $B \subseteq \mathcal{T}(X,Y) \setminus \mathcal{OP}(X,Y)$ with $\langle \mathcal{OP}(X,Y), B\rangle = \mathcal{T}(X,Y)$. Then*

$$\mathcal{S}(Y) \subseteq \big\langle \{\beta|_Y \colon \beta \in B\}, \eta|_Y \big\rangle.$$

Proof. Let $\sigma \in \mathcal{S}(Y)$. We extend σ to a transformation $\gamma\colon X \to Y$, i.e., $\gamma|_Y = \sigma$. Hence, there are $\gamma_1, \dots, \gamma_k \in \mathcal{OP}(X,Y) \cup B$ (for a suitable natural number k) such that $\gamma = \gamma_1 \dots \gamma_k$. Since the image of any transformation in $\mathcal{T}(X,Y)$ belongs to Y, we have

$$\sigma = \gamma|_Y = \gamma_1|_Y \dots \gamma_k|_Y.$$

Moreover, from $\sigma \in \mathcal{S}(Y)$, we conclude that $\gamma_i|_Y \in \mathcal{S}(Y)$ for $1 \le i \le k$. Let $\gamma_i \in \mathcal{OP}(X,Y)$ for some $i \in \{1, \dots, k\}$. Then, by Lemma 4,

$$\gamma_i|_Y = \begin{pmatrix} a_1 & \dots & a_t & a_{t+1} & \dots & a_m \\ a_{m-t+1} & \dots & a_m & a_1 & \dots & a_{m-t} \end{pmatrix} \in \langle \eta|_Y \rangle$$

for a suitable natural number t. This gives $\sigma \in \big\langle \{\beta|_Y \colon \beta \in B\}, \eta|_Y \big\rangle$.

Lemma 5 is proved.

Lemma 6. *Let $B \subseteq \mathcal{T}(X,Y) \setminus \mathcal{OP}(X,Y)$ with $\langle \mathcal{OP}(X,Y), B\rangle = \mathcal{T}(X,Y)$. Then*

$$\mathcal{R}_m \subseteq \{\ker\beta \colon \beta \in B\}.$$

Proof. Assume that there is $P \in \mathcal{R}_m$ with $P \notin \{\ker\beta \colon \beta \in B\}$. Let $\gamma \in \mathcal{T}(X,Y)$ with $\ker\gamma = P$. Then there are $\theta_1 \in \mathcal{OP}(X,Y) \cup B$ and $\theta_2 \in \mathcal{T}(X,Y)$ such that $\gamma = \theta_1\theta_2$. Since $\operatorname{rank}\gamma = m$, we obtain $\ker\gamma = \ker\theta_1 = P$. Thus, $\theta_1 \notin B$, i.e., $\theta_1 \in \mathcal{OP}(X,Y)$ and $\ker\theta_1 \in \mathcal{Q}_m \cup \mathcal{P}_m$, contradicts the fact that $\ker\theta_1 = P \in \mathcal{R}_m$.

Lemma 6 is proved.

Lemma 6 shows that $\operatorname{rank}\left(\mathcal{T}(X,Y) \colon \mathcal{OP}(X,Y)\right) \ge |\mathcal{R}_m|$. We now check the equality.

Lemma 7. $|\mathcal{R}_m| = S(m,n) - \dbinom{n}{m}$.

Proof. The cardinality of the set $\mathcal{D}_m := \mathcal{R}_m \cup \mathcal{P}_m$ was determined in [16]. The authors showed that

$$|\mathcal{D}_m| = S(m,n) - \binom{n-1}{m-1}.$$

In view of $\mathcal{R}_m \cap \mathcal{P}_m = \varnothing$, we obtain $\mathcal{R}_m = \mathcal{D}_m \setminus \mathcal{P}_m$. Since $|\mathcal{P}_m| = \binom{n-1}{m}$ (see Remark 1), we conclude that

$$|\mathcal{R}_m| = |\mathcal{D}_m| - |\mathcal{P}_m| = S(m,n) - \binom{n-1}{m-1} - \binom{n-1}{m} = S(m,n) - \binom{n}{m}.$$

Lemma 7 is proved.

Finally, we can find the relative rank of $\mathcal{T}(X,Y)$ modulo $\mathcal{OP}(X,Y)$.

Theorem 3. $\operatorname{rank}(\mathcal{T}(X,Y)\colon \mathcal{OP}(X,Y)) = S(m,n) - \binom{n}{m}$.

Proof. If $m = 1$ then $\mathcal{T}(X,Y) = \mathcal{OP}(X,Y)$, i.e.,

$$\operatorname{rank}(\mathcal{T}(X,Y)\colon \mathcal{OP}(X,Y)) = 0.$$

On the other hand, we have

$$S(1,n) = n = \binom{n}{1}.$$

Further, suppose that $n \geq 2$. By Lemmas 6 and 7, we obtain

$$\operatorname{rank}(\mathcal{T}(X,Y)\colon \mathcal{OP}(X,Y)) \geq |\mathcal{R}_m| = S(m,n) - \binom{n}{m}.$$

In order to prove the equality, it is necessary to find a relative generating set B of $\mathcal{T}(X,Y)$ modulo $\mathcal{OP}(X,Y)$ with $|B| = |\mathcal{R}_m|$. We observe that, for each $P \in \mathcal{R}_m$, there is $\beta_P \in \mathcal{T}(X,Y)$ with $\ker \beta_P = P$, which will be fixed. Let $\mathcal{B} := \{\beta_P \colon P \in \mathcal{R}_m\}$. If $m = 2$ then $\mathcal{R}_m = \varnothing$ and, clearly,

$$\mathcal{S}(Y) = \{\eta|_Y, (\eta|_Y)^2\} = \langle \eta|_Y \rangle.$$

If $m \geq 3$, then, without loss of generality, we can assume that there is $P' \in \mathcal{R}_m$ such that Y is a transversal of $\ker \beta_{P'}$ and

$$\beta_{P'}|_Y = \begin{pmatrix} a_1 & a_2 & a_3 & \ldots & a_m \\ a_2 & a_1 & a_3 & \ldots & a_m \end{pmatrix}.$$

It is known that $\mathcal{S}(Y) = \langle \beta_{P'}|_Y, \eta|_Y \rangle$. Hence, $\mathcal{B}$ is a relative generating set of $\mathcal{T}(X,Y)$ modulo $\mathcal{OP}(X,Y)$ by Lemma 3. Since $|\mathcal{B}| = |\mathcal{R}_m|$, we obtain the required result.

Theorem 3 is proved.

We now characterize the minimal relative generating sets of $\mathcal{T}(X,Y)$ modulo $\mathcal{OP}(X,Y)$. The minimal relative generating sets do not coincide with the relative generating sets $\operatorname{rank}(\mathcal{T}(X,Y)\colon \mathcal{OP}(X,Y))$ in size.

Theorem 4. *Let $B \subseteq \mathcal{T}(X,Y)$. Then B is a minimal relative generating set of $\mathcal{T}(X,Y)$ modulo $\mathcal{OP}(X,Y)$ if and only if there is a set $\widetilde{B} \subseteq B$ such that the following three statements are satisfied:*

(i) $\mathcal{R}_m \subseteq \big\{ \ker\beta\colon \beta \in B \setminus \widetilde{B} \big\}$,

(ii) $\big|B \setminus \widetilde{B}\big| = |\mathcal{R}_m|$,

(iii) $\mathcal{S}(Y) \subseteq \big\langle \{\beta|_Y\colon \beta \in B\}, \eta|_Y \big\rangle$ *but* $\mathcal{S}(Y) \nsubseteq \big\langle \{\beta|_Y\colon \beta \in B \setminus \{\gamma\}\}, \eta|_Y \big\rangle$ *for any* $\gamma \in B$ *with* $\ker\gamma \in \{\ker\beta\colon \beta \in \widetilde{B}\}$.

Proof. Suppose that conditions (i)–(iii) are satisfied. Then, by Lemma 3, we get

$$\langle \mathcal{OP}(X,Y), B\rangle = \mathcal{T}(X,Y).$$

It remains to show that B is minimal. Assume that there is $\gamma \in B$ such that

$$\langle \mathcal{OP}(X,Y), B \setminus \{\gamma\}\rangle = \mathcal{T}(X,Y).$$

Note that $\alpha\beta|_Y = \alpha|_Y\beta|_Y$ for all $\alpha, \beta \in \mathcal{T}(X,Y)$. Hence, we can conclude that

$$\begin{aligned}\mathcal{S}(Y) &\subseteq \big\langle \big\{\beta|_Y\colon \beta \in \mathcal{T}(X,Y)\big\}\big\rangle \\ &\subseteq \big\langle \big\{\beta|_Y\colon \beta \in \mathcal{OP}(X,Y) \cup (B \setminus \{\gamma\})\big\}\big\rangle \\ &= \big\langle \big\{\beta|_Y\colon \beta \in B \setminus \{\gamma\}\big\}, \eta|_Y \big\rangle\end{aligned}$$

by Lemma 4. Thus, $\ker\gamma \notin \{\ker\beta : \beta \in \widetilde{B}\}$ by (iii). This implies that $\gamma \in B \setminus \widetilde{B}$ and $|(B \setminus \widetilde{B}) \setminus \{\gamma\}| < |\mathcal{R}_m|$ by (ii), i.e.,

$$\mathcal{R}_m \nsubseteq \{\ker\beta\colon \beta \in (B \setminus \widetilde{B}) \setminus \{\gamma\}\}.$$

Since $\ker\gamma \notin \{\ker\beta\colon \beta \in \widetilde{B}\}$, we have $\mathcal{R}_m \nsubseteq \{\ker\beta\colon \beta \in (B \setminus \{\gamma\})\}$. Therefore, by Lemma 6, we conclude that $\langle \mathcal{OP}(X,Y), B \setminus \{\gamma\}\rangle \neq \mathcal{T}(X,Y)$; a contradiction. This shows that B is a minimal relative generating set of $\mathcal{T}(X,Y)$ modulo $\mathcal{OP}(X,Y)$.

Conversely, let B be a minimal relative generating set of $\mathcal{T}(X,Y)$ modulo $\mathcal{OP}(X,Y)$. By Lemmas 5 and 6, we have

$$\mathcal{R}_m \subseteq \{\ker\beta\colon \beta \in B\} \quad \text{and} \quad \mathcal{S}(Y) \subseteq \langle \{\beta|_Y\colon \beta \in B\}, \eta|_Y\rangle,$$

respectively. Then there exists a set $\widetilde{B} \subseteq B$ with

$$|B \setminus \widetilde{B}| = |\mathcal{R}_m| \quad \text{and} \quad \mathcal{R}_m \subseteq \{\ker\beta\colon \beta \in (B \setminus \widetilde{B})\}.$$

For the set $\widetilde{B}$, conditions (i) and (ii) are satisfied. Assume now that there is $\gamma \in B$ with $\ker\gamma \in \{\ker\beta\colon \beta \in \widetilde{B}\}$ such that

$$\mathcal{S}(Y) \subseteq \langle \{\beta|_Y\colon \beta \in B \setminus \{\gamma\}\}, \eta|_Y\rangle.$$

Then, in view of the fact that $\mathcal{R}_m \subseteq \{\ker\beta\colon\ \beta \in (B \setminus \{\gamma\})\}$, the set $B \setminus \{\gamma)$ is a relative generating set of $\mathcal{T}(X,Y)$ modulo $\mathcal{OP}(X,Y)$ by Lemma 3. This contradicts the minimality of B. Hence, (iii) is satisfied.

Theorem 4 is proved.

In particular, for the relative generating sets of the minimal size, we have the following remark:

Remark 3. $B \subseteq \mathcal{T}(X,Y)$ is a relative generating set of $\mathcal{T}(X,Y)$ modulo $\mathcal{OP}(X,Y)$ of the minimal size if and only if $\widetilde{B} = \varnothing$.

REFERENCES

1. P. M. Catarino and P. M. Higgins, "The monoid of orientation-preserving mappings on a chain," *Semigroup Forum*, **58**, 190–206 (1999).
2. I. Dimitrova, V. H. Fernandes, and J. Koppitz, "A note on generators of the endomorphism semigroup of an infinite countable chain," *J. Alg. Appl.*, **16**, No. 2, Article 1750031 (2017).
3. I. Dimitrova, J. Koppitz, and K. Tinpun, "On the relative rank of the semigroup of orientation-preserving transformations with restricted range," *Proc. 47th Spring Conference of the Union of Bulgarian Mathematicians*, 109–114 (2018).
4. V. H. Fernandes, P. Honyam, T. M. Quinteiro, and B. Singha, "On semigroups of endomorphisms of a chain with restricted range," *Semigroup Forum*, **89**, 77–104 (2014).
5. V. H. Fernandes, P. Honyam, T. M. Quinteiro, and B. Singha, "On semigroups of orientation-preserving transformations with restricted range," *Comm. Algebra*, **44**, 253–264 (2016).
6. V. H. Fernandes and J. Sanwong, "On the rank of semigroups of transformations on a finite set with restricted range," *Algebra Colloq.*, **21**, 497–510 (2014).
7. G. M. S. Gomes and J. M. Howie, "On the rank of certain semigroups of order-preserving transformations," *Semigroup Forum*, **51**, 275–282 (1992).
8. G. M. S. Gomes and J. M. Howie, "On the ranks of certain finite semigroups of transformations," *Math. Proc. Cambridge Philos. Soc.*, **101**, 395–403 (1987).
9. P. M. Higgins, J. D. Mitchell, and N. Ruškuc, "Generating the full transformation semigroup using order preserving mappings," *Glasgow Math. J.*, **45**, 557–566 (2003).
10. J. M. Howie, *Fundamentals of Semigroup Theory*, Oxford Univ. Press, Oxford (1995).
11. J. M. Howie and R. B. McFadden, "Idempotent rank in finite full transformation semigroups," *Proc. Roy. Soc. Edinburgh, Sec. A*, **114**, 161–167 (1990).
12. J. M. Howie, N. Ruškuc, and P. M. Higgins, "On relative ranks of full transformation semigroups," *Comm. Algebra*, **26**, 733–748 (1998).
13. D. McAlister, "Semigroups generated by a group and an idempotent," *Comm. Algebra*, **26**, 515–547 (1998).
14. N. Ruškuc, "On the rank of completely 0-simple semigroups," *Math. Proc. Cambridge Philos. Soc.*, **116**, 325–338 (1994).
15. J. S. V. Symons, "Some results concerning a transformation semigroup," *J. Aust. Math. Soc.*, **19**, 413–425 (1975).
16. K. Tinpun and J. Koppitz, "Relative rank of the finite full transformation semigroup with restricted range," *Acta Math. Univ. Comenian. (N.S.)*, **85**, No. 2, 347–356 (2016).