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Free rectangular doppelsemigroups

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A doppelsemigroup is a nonempty set equipped with two binary associative operations satisfying certain identities. In this paper, we consider the variety of rectangular doppelsemigroups which are analogs of rectangular semigroups. We construct the free rectangular doppelsemigroup and characterize the least rectangular congruence on the free doppelsemigroup. As a consequence, the free rectangular semigroup is presented. We also describe all (maximal) subdoppelsemigroups, all idempotents and all endomorphisms of the free rectangular doppelsemigroup, and give a criterion for an isomorphism of endomorphism semigroups of free rectangular doppelsemigroups. In addition, we show that the endomorphism semigroup of the free rectangular doppelsemigroup is not regular in general.

Keywords: Doppelsemigroup; free rectangular doppelsemigroup; free doppelsemigroup; interassociativity; doppelalgebra; semigroup; congruence.

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1. Introduction

One of the most useful concepts in algebra is the free object. Every variety contains free algebras and free objects in any variety of algebras are important in the study of that variety. This fact is a component of countless arguments in the subject. Free algebras can be very neatly characterized in terms of identities. One of the key problems that arises is the word problem for free objects in varieties. For solving this problem, it is useful to know the structure of the free object in the variety.

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Doppelsemigroups, a natural generalization of semigroups, form an important variety of algebras arising from doppelalgebras and interassociative semigroups. Recall that a doppelsemigroup [31] is a nonempty set D equipped with two binary operations $\dashv$ and $\vdash$ satisfying the axioms

$$(x \dashv y) \vdash z = x \dashv (y \vdash z), \tag{$D1$}$$

$$(x \vdash y) \dashv z = x \vdash (y \dashv z), \tag{$D2$}$$

$$(x \dashv y) \dashv z = x \dashv (y \dashv z), \tag{$D4$}$$

$$(x \vdash y) \vdash z = x \vdash (y \vdash z). \tag{$D5$}$$

A doppelsemigroup $(D, \dashv, \vdash)$ is strong [34] if it satisfies the axiom

$$x \dashv (y \vdash z) = x \vdash (y \dashv z). \tag{$D3$}$$

Doppelalgebras [22] are linear analogs of doppelsemigroups. It is known that any Lie algebra has a universal enveloping doppelalgebra and there exists a functor from the category of doppelalgebras to the category of associative algebras (see [22]). The notion of interassociativity was introduced in [45] for groupoids and later in [6] for semigroups. Strong interassociativity for semigroups was introduced in [14]. Recall that two semigroups defined on the same set are interassociative provided that they satisfy ($D1$) and ($D2$), and two interassociative semigroups are strongly interassociative if they satisfy ($D3$). The classes of interassociative semigroups are the object of intensive study (see, e.g. [3, 4, 6, 9–14]). Moreover, the definition of interassociativity for semigroups implies that two interassociative semigroups give rise to a doppelsemigroup. This fact allows us to study interassociative semigroups via doppelsemigroups, using methods of universal algebra. Doppelsemigroups as well as interassociative semigroups satisfy the hyperidentity of associativity (see, e.g. [19]).

One of the important motivations for studying doppelsemigroups comes from their connections to duplexes. Recall that a duplex [20] is a nonempty set equipped with two binary associative operations. Free duplexes were constructed in [20]. Good examples of duplexes are dimonoids [17], a prominent class of algebras, that has been widely studied, with many recent developments, see, for example, [25, 30, 35, 40]. It should be noted that every commutative dimonoid [24] is a doppelsemigroup. The g-dimonoids [43] provide another class of duplexes. They first arose in the setting of the theory of associative 0-dialgebras [21]. Dimonoids form a subvariety of the variety of g-dimonoids. Using sets with two binary associative operations and the axiom ($D1$), the notion of a restrictive bisemigroup was considered in [23]. Restrictive bisemigroups play an important role in the theory of binary relations.

Loday and Ronco introduced trioids [18] as a natural generalization of dimonoids. Trioids are used in trialgebra theory [1, 5, 7, 18] and they were actively studied (see, e.g. [27, 36, 41, 42, 44]). It is known that the axioms ($D2$), ($D4$) and ($D5$) of a doppelsemigroup appear in axiomatics of a trioid. Such connections increase the motivation for studying doppelsemigroups. Finally, note that doppelsemigroups have close relationships with n-tuple semigroups [29, 39]. Indeed, every 2-tuple semigroup is a doppelsemigroup. The class of n-tuple semigroups causes the greatest interest from the point of view of applications in the theory of n-tuple algebras of associative type [16].

The problem of constructing doppelsemigroups which are free in the variety V of doppelsemigroups and in subvarieties of V is natural. Relatively free doppelsemigroups have wide applications in doppelalgebra theory and semigroup theory; they especially useful in the development of the variety theory of algebraic systems. We refer the reader to the existing literature [28, 31, 32, 34, 37, 38] for constructions of various relatively free doppelsemigroups.

In this paper, we introduce and study the variety of rectangular doppelsemigroups which are analogs of rectangular semigroups. The paper proceeds as follows. In Sec. 2, we give the material which will be used in the paper and present some new results. The main result of the paper is Theorem 3.1 from Sec. 3, it gives the construction of the free rectangular doppelsemigroup of an arbitrary rank. As a consequence, the free rectangular semigroup is presented (Corollary 3.12). In this section, we also consider separately free rectangular doppelsemigroups of rank 1 (Corollary 3.10), count the cardinality of the free rectangular doppelsemigroup for a finite case, establish that the semigroups of the free rectangular doppelsemigroup are isomorphic and its automorphism group is isomorphic to the symmetric group. In Sec. 4, we characterize the least rectangular congruence on the free doppelsemigroup (Theorem 4.1) and the least rectangular semigroup congruence on the free semigroup (Proposition 4.2). Section 5 is devoted to the study of various properties of free rectangular doppelsemigroups. We begin with a complete description of (maximal) subdoppelsemigroups, idempotents and endomorphisms of the free rectangular doppelsemigroup. Then we give a criterion for an isomorphism of endomorphism semigroups of free rectangular doppelsemigroups. We also point out that the endomorphism semigroup of the free rectangular doppelsemigroup is not regular in general.

2. Preliminaries

A semigroup S is a rectangular band if $x^2 = x$, $xyz = xz$ for all $x, y, z \in S$. It is well known that every rectangular band is isomorphic to the Cartesian product $L \times R$ of the left zero semigroup and the right zero semigroup, and the free rectangular band is a set $X \times X$ with the operation $(x, y)(a, b) = (x, b)$, where X is an arbitrary nonempty set. It is known that operations of a doppelsemigroup $(D, \dashv, \vdash)$ with a rectangular band $(D, \dashv)$ or $(D, \vdash)$ coincide (see [31]).

A semigroup S is called rectangular [35] if $xyz = xz$ foll all $x, y, z \in S$. For example, rectangular bands and zero semigroups are rectangular semigroups. In [8], the lattice of subvarieties of the variety defined by the identity $xyz = xz$ was indicated. This variety is the union of the variety of left zero semigroups, the variety of right zero semigroups and the variety of zero semigroups, and the lattice of its subvarieties is an 8-element Boolean algebra.

For doppelsemigroups, the question about introducing an analog of a rectangular semigroup is natural. A doppelsemigroup $(D, \dashv, \vdash)$ will be called rectangular if both semigroups $(D, \dashv)$ and $(D, \vdash)$ are rectangular. The class of all rectangular doppelsemigroups forms a subvariety of the variety of doppelsemigroups. A doppelsemigroup (semigroup) which is free in the variety of rectangular doppelsemigroups (semigroups) will be called a free rectangular doppelsemigroup (semigroup). If ρ is a congruence on a doppelsemigroup $(D, \dashv, \vdash)$ such that $(D, \dashv, \vdash)/\rho$ is a rectangular doppelsemigroup, we say that ρ is a rectangular congruence. If in the latter definition instead of a doppelsemigroup $(D, \dashv, \vdash)$ take a semigroup S, then S/ρ is a rectangular semigroup and we say that ρ is a rectangular semigroup congruence.

The following two lemmas are needed for the sequel.

Lemma 2.1 ([31, Lemma 3.1]). *In a doppelsemigroup* $(D, \dashv, \vdash)$, *for any* $n > 1$, $n \in \mathbb{N}$, *and any* $x_i \in D$, $1 \le i \le n+1$, *and* $*_j \in \{\dashv, \vdash\}$, $1 \le j \le n$, *any parenthesizing of*

$$x_1 *_1 x_2 *_2 \cdots *_n x_{n+1}$$

gives the same element from D.

Lemma 2.2. *In a rectangular doppelsemigroup* $(D, \dashv, \vdash)$, *for any* $a, b, x, y \in D$, *the following identities are satisfied*:

(i) $a \vdash b \dashv y = a \vdash x \dashv y$.
(ii) $a \dashv b \vdash y = a \dashv x \vdash y$.

Proof. Let $a, b, x, y, z \in D$. Then, using Lemma 2.1 and the identity of a rectangular semigroup, we get

$$\begin{aligned} a \vdash b \dashv y &= a \vdash (b \dashv z \vdash x \dashv y) \\ &= (a \vdash b \dashv z \vdash x) \dashv y = a \vdash x \dashv y, \\ a \dashv b \vdash y &= a \dashv (b \vdash z \dashv x \vdash y) \\ &= (a \dashv b \vdash z \dashv x) \vdash y = a \dashv x \vdash y. \end{aligned}$$ □

The concept of $\mathscr{P}$-related semigroups was introduced by Hewitt and Zuckerman [15]. Let us recall the definition.

Semigroups $(D, \dashv)$ and $(D, \vdash)$ are called $\mathscr{P}$-related if $x \dashv y \dashv z = x \vdash y \vdash z$ for all $x, y, z \in D$. Motivated by problems of algebraic K-theory, Loday introduced the notion of a dimonoid [17]. For details on dimonoids and their linear analogs, see, e.g. [25, 33, 35] and [2, 17], respectively. By [24, Lemma 2], the semigroups $(D, \dashv)$ and $(D, \vdash)$ of a commutative dimonoid $(D, \dashv, \vdash)$ are $\mathscr{P}$-related. Examples of commutative dimonoids can be found in [24, 26].

The following statement establishes necessary and sufficient conditions under which the operations of a rectangular doppelsemigroup coincide.

Proposition 2.3. *The operations of a rectangular doppelsemigroup* $(D, \dashv, \vdash)$ *coincide if and only if* $(D, \dashv)$ *and* $(D, \vdash)$ *are* $\mathscr{P}$*-related semigroups.*

Proof. Let $(D, \dashv, \vdash)$ be a rectangular doppelsemigroup. Assume that $\dashv$ and $\vdash$ coincide. Then

$$x \dashv y \dashv z = x \dashv z = x \vdash z = x \vdash y \vdash z$$

for all $x, y, z \in D$. Hence, $(D, \dashv)$ and $(D, \vdash)$ are $\mathscr{P}$-related semigroups.

Conversely, let $(D, \dashv)$ and $(D, \vdash)$ be $\mathscr{P}$-related semigroups. Since semigroups $(D, \dashv)$ and $(D, \vdash)$ are rectangular, we have

$$x \dashv y \dashv z = x \dashv z, \quad x \vdash y \vdash z = x \vdash z$$

for all $x, y, z \in D$. As $(D, \dashv)$ and $(D, \vdash)$ are $\mathscr{P}$-related semigroups, for all $x, y, z \in D$, we get

$$x \dashv y \dashv z = x \vdash y \vdash z$$

and so, $x \dashv z = x \vdash z$. □

The following proposition shows that there are no rectangular strong doppelsemigroups with distinct operations.

Proposition 2.4. *The operations of a rectangular strong doppelsemigroup coincide.*

Proof. Let $(D, \dashv, \vdash)$ be a rectangular strong doppelsemigroup and $a, x, y \in D$. We have

$$\begin{aligned}
(a \vdash x \dashv y) \dashv (a \vdash x \dashv y) &= a \vdash x \dashv (y \dashv a \vdash x) \dashv y \\
&= a \vdash x \dashv y = (a \vdash x \dashv y) \dashv (a \dashv x \vdash y) \\
&= a \vdash (x \dashv y \dashv a \dashv x) \vdash y = a \vdash y, \\
(a \dashv x \vdash y) \vdash (a \dashv x \vdash y) &= a \dashv x \vdash (y \vdash a \dashv x) \vdash y \\
&= a \dashv x \vdash y = (a \dashv x \vdash y) \vdash (a \vdash x \dashv y) \\
&= a \dashv (x \vdash y \vdash a \vdash x) \dashv y = a \dashv y
\end{aligned}$$

according to Lemma 2.1 and the identity of a rectangular semigroup. By the axiom $(D3)$, $a \vdash y = a \dashv y$ for all $a, y \in D$. □

The free doppelsemigroup is given in [31]. Recall this construction.

Let X be an arbitrary nonempty set, and let w be an arbitrary word in the alphabet X. The length of w is denoted by ℓ_w. Let further $F[X]$ be the free semigroup on X, let T be the free monoid on the two-element set $\{a, b\}$, and let $\theta \in T$ be the empty word. By definition, the length ℓ_θ of θ is equal to 0. Define operations $\dashv$ and $\vdash$ on

$$F = \{(w, u) \in F[X] \times T \mid \ell_w - \ell_u = 1\}$$

by

$$(w_1, u_1) \dashv (w_2, u_2) = (w_1 w_2, u_1 a u_2),$$

$$(w_1, u_1) \vdash (w_2, u_2) = (w_1 w_2, u_1 b u_2)$$

for all $(w_1, u_1), (w_2, u_2) \in F$. The obtained algebra is denoted by $\mathrm{FDS}(X)$.

Theorem 2.5 ([31, Theorem 3.5]). $\mathrm{FDS}(X)$ *is the free doppelsemigroup.*

If $f : D_1 \to D_2$ is a homomorphism of doppelsemigroups, the kernel of f will be denoted by Δ_f. By $\mathrm{End}(D_1)$, we denote the endomorphism semigroup of a doppelsemigroup D_1.

3. Free Objects

In this section, we construct the free rectangular doppelsemigroup of an arbitrary rank and consider separately free rectangular doppelsemigroups of rank 1. As a consequence, the free rectangular semigroup is presented. We also count the cardinality of the free rectangular doppelsemigroup for a finite case, establish that the semigroups of the free rectangular doppelsemigroup are isomorphic and its automorphism group is isomorphic to the symmetric group.

Let X be an arbitrary nonempty set and $B = \{a, b\}$. Define operations $\dashv$ and $\vdash$ on $X \cup (B \times X \times X \times B)$ by

$$(a_1, b_1, c_1, d_1) * (a_2, b_2, c_2, d_2) = (a_1, b_1, c_2, d_2),$$

$$x \dashv (a_1, b_1, c_1, d_1) = (a, x, c_1, d_1), \quad (a_1, b_1, c_1, d_1) \dashv x = (a_1, b_1, x, a),$$

$$x \vdash (a_1, b_1, c_1, d_1) = (b, x, c_1, d_1), \quad (a_1, b_1, c_1, d_1) \vdash x = (a_1, b_1, x, b),$$

$$x \dashv y = (a, x, y, a), \quad x \vdash y = (b, x, y, b)$$

for all $(a_1, b_1, c_1, d_1), (a_2, b_2, c_2, d_2) \in B \times X \times X \times B$, $x, y \in X$ and $* \in \{\dashv, \vdash\}$. The obtained algebra will be denoted by $\mathrm{FRDop}(X)$.

The main result of the paper is the following theorem.

Theorem 3.1. $\mathrm{FRDop}(X)$ *is the free rectangular doppelsemigroup.*

Proof. The proof follows from Lemmas 3.3–3.8. □

Lemma 3.2. *In* FRDop(X), *for any* $x \in X, \omega \in$ FRDop(X), *the following equalities are satisfied*:

$$x \dashv \omega = (a, x, x, a) \dashv \omega, \quad x \vdash \omega = (b, x, x, b) \vdash \omega,$$

$$\omega \dashv x = \omega \dashv (a, x, x, a), \quad \omega \vdash x = \omega \vdash (b, x, x, b).$$

Proof. The statement is proved by a direct calculation. □

Lemma 3.3. *The operation* $\dashv$ *of* FRDop(X) *is associative.*

Proof. Let (a_1, b_1, c_1, d_1), (a_2, b_2, c_2, d_2), $(a_3, b_3, c_3, d_3) \in B \times X \times X \times B$. Using the definition of $\dashv$, obtain

$$\begin{aligned}
&((a_1, b_1, c_1, d_1) \dashv (a_2, b_2, c_2, d_2)) \dashv (a_3, b_3, c_3, d_3) \\
&\quad = (a_1, b_1, c_2, d_2) \dashv (a_3, b_3, c_3, d_3) = (a_1, b_1, c_3, d_3) \\
&\quad = (a_1, b_1, c_1, d_1) \dashv (a_2, b_2, c_3, d_3) \\
&\quad = (a_1, b_1, c_1, d_1) \dashv ((a_2, b_2, c_2, d_2) \dashv (a_3, b_3, c_3, d_3)). \qquad (3.1)
\end{aligned}$$

Applying Lemma 3.2 and (3.1), one can check the remaining cases.
Thus, the operation $\dashv$ is associative. □

Lemma 3.4. *The operation* $\vdash$ *of* FRDop(X) *is associative.*

Proof. The proof is similar to the proof of Lemma 3.3. □

Lemma 3.5. FRDop(X) *satisfies the axiom* $(D1)$ *of a doppelsemigroup.*

Proof. For all (a_1, b_1, c_1, d_1), (a_2, b_2, c_2, d_2), $(a_3, b_3, c_3, d_3) \in B \times X \times X \times B$, we have

$$\begin{aligned}
&((a_1, b_1, c_1, d_1) \dashv (a_2, b_2, c_2, d_2)) \vdash (a_3, b_3, c_3, d_3) \\
&\quad = (a_1, b_1, c_2, d_2) \vdash (a_3, b_3, c_3, d_3) = (a_1, b_1, c_3, d_3) \\
&\quad = (a_1, b_1, c_1, d_1) \dashv (a_2, b_2, c_3, d_3) \\
&\quad = (a_1, b_1, c_1, d_1) \dashv ((a_2, b_2, c_2, d_2) \vdash (a_3, b_3, c_3, d_3)). \qquad (3.2)
\end{aligned}$$

Using Lemma 3.2 and (3.2), the remaining cases can be checked.
So, FRDop(X) satisfies the axiom $(D1)$. □

Lemma 3.6. FRDop(X) *satisfies the axiom* $(D2)$ *of a doppelsemigroup.*

Proof. The proof is similar to the proof of Lemma 3.5. □

Lemma 3.7. FRDop(X) *is a rectangular doppelsemigroup.*

Proof. By Lemmas 3.3–3.6, FRDop(X) is a doppelsemigroup. Show that it is rectangular.

Let (a_1, b_1, c_1, d_1), (a_2, b_2, c_2, d_2), $(a_3, b_3, c_3, d_3) \in B \times X \times X \times B$. Then, using (3.1), we get

$$\begin{aligned}(a_1, b_1, c_1, d_1) &\dashv (a_2, b_2, c_2, d_2) \dashv (a_3, b_3, c_3, d_3)\\ &= (a_1, b_1, c_3, d_3) = (a_1, b_1, c_1, d_1) \dashv (a_3, b_3, c_3, d_3). \qquad (3.3)\end{aligned}$$

The remaining cases for $\dashv$ can be checked due to Lemma 3.2 and (3.3). It means that $(X \cup (B \times X \times X \times B), \dashv)$ is a rectangular semigroup. Rectangularity of the semigroup $(X \cup (B \times X \times X \times B), \vdash)$ is proved in a similar way.

Thus, FRDop(X) is a rectangular doppelsemigroup. □

Lemma 3.8. FRDop(X) *is free in the variety of rectangular doppelsemigroups.*

Proof. By Lemma 3.7, FRDop(X) is a rectangular doppelsemigroup. Moreover, FRDop(X) is generated by X. Indeed,

$$(a_1, b_1, c_1, d_1) \in \{b_1 \dashv c_1, b_1 \vdash c_1, (b_1 \dashv c_1) \vdash (b_1 \vdash c_1), (b_1 \vdash c_1) \vdash (b_1 \dashv c_1)\}$$

for $(a_1, b_1, c_1, d_1) \in B \times X \times X \times B$, and hence any element of $B \times X \times X \times B$ can be expressed by elements from X.

Let $(D, \dashv', \vdash')$ be an arbitrary rectangular doppelsemigroup, and let $\alpha : X \to D$ be an arbitrary map. Fix $\varepsilon \in D$ and define a map $\psi : \mathrm{FRDop}(X) \to (D, \dashv', \vdash')$ by the rule

$$\omega\psi = \begin{cases} b_1\alpha \dashv' c_1\alpha, & \text{if } \omega = (a, b_1, c_1, a),\\ b_1\alpha \vdash' c_1\alpha, & \text{if } \omega = (b, b_1, c_1, b),\\ b_1\alpha \dashv' \varepsilon \vdash' c_1\alpha, & \text{if } \omega = (a, b_1, c_1, b),\\ b_1\alpha \vdash' \varepsilon \dashv' c_1\alpha, & \text{if } \omega = (b, b_1, c_1, a),\\ \omega\alpha, & \text{if } \omega \in X.\end{cases}$$

By Lemmas 2.1 and 2.2, ψ is well-defined. Show that ψ is a homomorphism. We will use Lemmas 2.1, 2.2 and the identities of a rectangular doppelsemigroup.

Let (a_1, b_1, c_1, d_1), $(a_2, b_2, c_2, d_2) \in B \times X \times X \times B$ and $x, y \in X$. We should check twenty five cases for $\dashv$.

Case (1):

$$\begin{aligned}((a, b_1, c_1, a) &\dashv (a, b_2, c_2, a))\psi\\ &= (a, b_1, c_2, a)\psi = b_1\alpha \dashv' c_2\alpha = (b_1\alpha \dashv' c_1\alpha) \dashv' (b_2\alpha \dashv' c_2\alpha)\\ &= (a, b_1, c_1, a)\psi \dashv' (a, b_2, c_2, a)\psi.\end{aligned}$$

Case (2):

$$\begin{aligned}
&((a, b_1, c_1, a) \dashv (b, b_2, c_2, b))\psi \\
&\quad = (a, b_1, c_2, b)\psi = b_1\alpha \dashv' \varepsilon \vdash' c_2\alpha = b_1\alpha \dashv' (c_1\alpha \dashv' b_2\alpha) \vdash' c_2\alpha \\
&\quad = (b_1\alpha \dashv' c_1\alpha) \dashv' (b_2\alpha \vdash' c_2\alpha) = (a, b_1, c_1, a)\psi \dashv' (b, b_2, c_2, b)\psi.
\end{aligned}$$

Case (3):

$$\begin{aligned}
&((a, b_1, c_1, a) \dashv (a, b_2, c_2, b))\psi \\
&\quad = (a, b_1, c_2, b)\psi = b_1\alpha \dashv' \varepsilon \vdash' c_2\alpha = (b_1\alpha \dashv' c_1\alpha) \dashv' (b_2\alpha \dashv' \varepsilon \vdash' c_2\alpha) \\
&\quad = (a, b_1, c_1, a)\psi \dashv' (a, b_2, c_2, b)\psi.
\end{aligned}$$

Case (4):

$$\begin{aligned}
&((a, b_1, c_1, a) \dashv (b, b_2, c_2, a))\psi \\
&\quad = (a, b_1, c_2, a)\psi = b_1\alpha \dashv' c_2\alpha = (b_1\alpha \dashv' c_1\alpha) \dashv' (b_2\alpha \vdash' \varepsilon \dashv' c_2\alpha) \\
&\quad = (a, b_1, c_1, a)\psi \dashv' (b, b_2, c_2, a)\psi.
\end{aligned}$$

Case (5):

$$(x \dashv y)\psi = (a, x, y, a)\psi = x\alpha \dashv' y\alpha = x\psi \dashv' y\psi.$$

Case (6):

$$\begin{aligned}
((a, b_1, c_1, a) \dashv x)\psi &= (a, b_1, x, a)\psi = b_1\alpha \dashv' x\alpha \\
&= (b_1\alpha \dashv' c_1\alpha) \dashv' x\alpha = (a, b_1, c_1, a)\psi \dashv' x\psi.
\end{aligned}$$

Case (7):

$$\begin{aligned}
((b, b_1, c_1, b) \dashv x)\psi &= (b, b_1, x, a)\psi = b_1\alpha \vdash' \varepsilon \dashv' x\alpha \\
&= (b_1\alpha \vdash' c_1\alpha) \dashv' x\alpha = (b, b_1, c_1, b)\psi \dashv' x\psi.
\end{aligned}$$

Case (8):

$$\begin{aligned}
((a, b_1, c_1, b) \dashv x)\psi &= (a, b_1, x, a)\psi = b_1\alpha \dashv' x\alpha \\
&= (b_1\alpha \dashv' \varepsilon \vdash' c_1\alpha) \dashv' x\alpha = (a, b_1, c_1, b)\psi \dashv' x\psi.
\end{aligned}$$

Case (9):

$$\begin{aligned}
&((b, b_1, c_1, a) \dashv x)\psi \\
&\quad = (b, b_1, x, a)\psi = b_1\alpha \vdash' \varepsilon \dashv' x\alpha = (b_1\alpha \vdash' \varepsilon \dashv' c_1\alpha) \dashv' x\alpha \\
&\quad = (b, b_1, c_1, a)\psi \dashv' x\psi.
\end{aligned}$$

Case (10):

$$\begin{aligned}
&(x \dashv (a, b_2, c_2, a))\psi = (a, x, c_2, a)\psi \\
&\quad = x\alpha \dashv' c_2\alpha = x\alpha \dashv' (b_2\alpha \dashv' c_2\alpha) = x\psi \dashv' (a, b_2, c_2, a)\psi.
\end{aligned}$$

Case (11):

$$\begin{aligned}(x \dashv (b, b_2, c_2, b))\psi &= (a, x, c_2, b)\psi = x\alpha \dashv' \varepsilon \vdash' c_2\alpha = x\alpha \dashv' (b_2\alpha \vdash' c_2\alpha)\\ &= x\psi \dashv' (b, b_2, c_2, b)\psi.\end{aligned}$$

Case (12):

$$\begin{aligned}(x \dashv (a, b_2, c_2, b))\psi &= (a, x, c_2, b)\psi = x\alpha \dashv' \varepsilon \vdash' c_2\alpha = x\alpha \dashv' (b_2\alpha \dashv' \varepsilon \vdash' c_2\alpha)\\ &= x\psi \dashv' (a, b_2, c_2, b)\psi.\end{aligned}$$

Case (13):

$$\begin{aligned}(x \dashv (b, b_2, c_2, a))\psi &= (a, x, c_2, a)\psi = x\alpha \dashv' c_2\alpha = x\alpha \dashv' (b_2\alpha \vdash' \varepsilon \dashv' c_2\alpha)\\ &= x\psi \dashv' (b, b_2, c_2, a)\psi.\end{aligned}$$

The remaining cases for $\dashv$ are considered in a similar way. The fact that ψ is a homomorphism from $(X \cup (B \times X \times X \times B), \vdash)$ into $(D, \vdash')$ is proved analogously. So, ψ is a homomorphism of doppelsemigroups.

Since $x\psi = x\alpha$ for all $x \in X$ and X generates $\mathrm{FRDop}(X)$, the uniqueness of the homomorphism ψ is obvious. Thus, $\mathrm{FRDop}(X)$ is free in the variety of rectangular doppelsemigroups. □

Corollary 3.9. *The free rectangular doppelsemigroup* $\mathrm{FRDop}(X)$ *generated by a finite set* X *is finite. Specifically, if* $|X| = n$, *then* $|\mathrm{FRDop}(X)| = n + 4n^2$.

Now we construct a doppelsemigroup which is isomorphic to the free rectangular doppelsemigroup of rank 1.

Let e be an arbitrary symbol. Define operations $\dashv$ and $\vdash$ on $(B \times B) \cup \{e\}$ by

$$\begin{aligned}&(a_1, b_1) * (a_2, b_2) = (a_1, b_2),\\ &e \dashv (a_1, b_1) = (a, b_1), \quad (a_1, b_1) \dashv e = (a_1, a),\\ &e \vdash (a_1, b_1) = (b, b_1), \quad (a_1, b_1) \vdash e = (a_1, b),\\ &e \dashv e = (a, a), \quad e \vdash e = (b, b)\end{aligned}$$

for all $(a_1, b_1), (a_2, b_2) \in B \times B$ and $* \in \{\dashv, \vdash\}$. The algebra $((B \times B) \cup \{e\}, \dashv, \vdash)$ will be denoted by FRDop_1. An immediate verification shows that FRDop_1 is a doppelsemigroup.

Corollary 3.10. *If* $|X| = 1$, *then* $\mathrm{FRDop}_1 \cong \mathrm{FRDop}(X)$.

Proof. Let $X = \{e\}$. One can show that a map $\gamma : \mathrm{FRDop}_1 \to \mathrm{FRDop}(X)$, defined by $e\gamma = e$ and $(a_1, b_1)\gamma = (a_1, e, e, b_1)$, for all $(a_1, b_1) \in B \times B$, is an isomorphism. □

The following statement establishes a relationship between the semigroups of the free rectangular doppelsemigroup.

Corollary 3.11. *The semigroups*

$$(X \cup (B \times X \times X \times B), \dashv) \quad \text{and} \quad (X \cup (B \times X \times X \times B), \vdash)$$

of the free rectangular doppelsemigroup $\mathrm{FRDop}(X)$ *are isomorphic.*

Proof. Assume that $\widehat{a} = b$, $\widehat{b} = a$ and define a map

$$\eta : (X \cup (B \times X \times X \times B), \dashv) \to (X \cup (B \times X \times X \times B), \vdash)$$

by putting

$$\omega\eta = \begin{cases} (\widehat{a_1}, b_1, c_1, \widehat{d_1}), & \text{if } \omega = (a_1, b_1, c_1, d_1) \in B \times X \times X \times B, \\ \omega, & \text{if } \omega \in X. \end{cases}$$

It is immediate to check that η is an isomorphism. □

It is not difficult to see that the free rectangular doppelsemigroup $\mathrm{FRDop}(X)$ is determined uniquely up to isomorphism by cardinality of the set X. Hence, the automorphism group of $\mathrm{FRDop}(X)$ is isomorphic to the symmetric group on X.

At the end of this section, we present the free rectangular semigroup.

Let $\omega_1, \omega_2 \in \mathrm{FRDop}(X)$. Define a relation $\widetilde{\rho}$ on $\mathrm{FRDop}(X)$ by

$$\omega_1 \widetilde{\rho} \omega_2 \quad \text{if and only if}$$

$$\omega_1 = (a_1, b_1, c_1, b_1), \quad \omega_2 = (a_2, b_2, c_2, b_2) \in B \times X \times X \times B$$

$$\text{and} \quad (b_1, c_1) = (b_2, c_2), \quad \text{or} \quad \omega_1 = \omega_2.$$

Theorem 3.1 implies the following corollary.

Corollary 3.12. *The relation* $\widetilde{\rho}$ *is a congruence on* $\mathrm{FRDop}(X)$ *and the operations of* $\mathrm{FRDop}(X)/\widetilde{\rho}$ *coincide. Moreover,* $\mathrm{FRDop}(X)/\widetilde{\rho}$ *is the free rectangular semigroup.*

Proof. It is easy to prove that $\widetilde{\rho}$ is a congruence on $\mathrm{FRDop}(X)$. Besides, $\omega_1 \dashv \omega_2 \widetilde{\rho} \omega_1 \vdash \omega_2$ for all $\omega_1, \omega_2 \in \mathrm{FRDop}(X)$, hence $\mathrm{FRDop}(X)/\widetilde{\rho}$ is a semigroup.

Further, denote the equivalence class of $\widetilde{\rho}$ containing an element $\omega \in \mathrm{FRDop}(X)$ by $[\omega]$. It is not difficult to check that

$$[\omega_1][\omega_2][\omega_3] = [\omega_1][\omega_3]$$

for all $\omega_1, \omega_2, \omega_3 \in \mathrm{FRDop}(X)$. Consequently, the semigroup $\mathrm{FRDop}(X)/\widetilde{\rho}$ is rectangular.

Let us show that $\mathrm{FRDop}(X)/\widetilde{\rho}$ is free rectangular. Obviously, $\mathrm{FRDop}(X)/\widetilde{\rho}$ is generated by the set $[X] = \{[x] \mid x \in X\}$. Let S be an arbitrary rectangular semigroup, and let $\beta : [X] \to S$ be an arbitrary map. Define a map $\tau : \mathrm{FRDop}(X)/\widetilde{\rho} \to S$ by the rule

$$\omega\tau = \begin{cases} ([b_1]\beta)([c_1]\beta), & \text{if } \omega = [(a_1, b_1, c_1, d_1)], (a_1, b_1, c_1, d_1) \in B \times X \times X \times B, \\ \omega\beta, & \text{if } \omega \in [X]. \end{cases}$$

The reader will have no difficulty in showing that τ is the unique homomorphism extending β. Thus, $\mathrm{FRDop}(X)/\widetilde{\rho}$ is free in the variety of rectangular semigroups. $\square$

Now we give a more elegant version for free rectangular semigroups.

Define the operation $\star$ on $X \cup (X \times X)$ by

$$(x_1, y_1) \star (x_2, y_2) = (x_1, y_2), \quad x \star y = (x, y),$$

$$x \star (x_1, y_1) = (x, y_1), \quad (x_1, y_1) \star x = (x_1, x)$$

for all $(x_1, y_1), (x_2, y_2) \in X \times X$ and $x, y \in X$. The algebra $(X \cup (X \times X), \star)$ will be denoted by $\mathrm{FRSem}(X)$. A simple verification shows that $\mathrm{FRSem}(X)$ is a rectangular semigroup.

Proposition 3.13. $\mathrm{FRSem}(X)$ *is the free rectangular semigroup.*

Proof. By Corollary 3.12, $\mathrm{FRDop}(X)/\widetilde{\rho}$ is the free rectangular semigroup. One can check that the map

$$\mu : \mathrm{FRSem}(X) \to \mathrm{FRDop}(X)/\widetilde{\rho},$$

defined by $x\mu = [x]$ and $(b_1, c_1)\mu = [(a, b_1, c_1, a)]$, for all $x \in X$, $(b_1, c_1) \in X \times X$, is an isomorphism. $\square$

Proposition 3.13 implies the following statement which describes singly generated free rectangular semigroups.

Corollary 3.14. *The two-element zero semigroup is the free rectangular semigroup of rank* 1.

4. The Least Rectangular Congruence on the Free Doppelsemigroup

In this section, we characterize the least rectangular congruence on the free doppelsemigroup and the least rectangular semigroup congruence on the free semigroup.

For every nonempty word w over an alphabet X, denote the first (respectively, last) letter of w by $w^{(0)}$ (respectively, $w^{(1)}$).

Theorem 4.1. *Let* $\mathrm{FDS}(X)$ *be the free doppelsemigroup,* $(w_1, u_1), (w_2, u_2) \in \mathrm{FDS}(X)$, *and let* $\mathrm{FRDop}(X)$ *be the free rectangular doppelsemigroup. Define a relation* $\widetilde{\pi}$ *on* $\mathrm{FDS}(X)$ *by*

$$(w_1, u_1)\widetilde{\pi}(w_2, u_2)$$

if and only if

$$u_1 \neq \theta, \quad u_2 \neq \theta \quad \textit{and} \quad (u_1^{(0)}, w_1^{(0)}, w_1^{(1)}, u_1^{(1)}) = (u_2^{(0)}, w_2^{(0)}, w_2^{(1)}, u_2^{(1)}),$$

$$\textit{or} \quad (w_1, u_1) = (w_2, u_2).$$

Then $\widetilde{\pi}$ *is the least rectangular congruence on* $\mathrm{FDS}(X)$.

Proof. Clearly, if $(w,\theta)\in \mathrm{FDS}(X)$, then

$$1=\ell_w-\ell_\theta=\ell_w-0=\ell_w,$$

i.e. $w\in X$. Define a map $\pi:\mathrm{FDS}(X)\to \mathrm{FRDop}(X)$ by

$$(w,u)\mapsto (w,u)\pi=\begin{cases}(u^{(0)},w^{(0)},w^{(1)},u^{(1)}), & \text{if } u\neq\theta,\\ w, & \text{if } u=\theta.\end{cases}$$

Let us show that π is an epimorphism. We have

$$\begin{aligned}&((w_1,u_1)\dashv(w_2,u_2))\pi\\ &\quad=(w_1w_2,u_1au_2)\pi=((u_1au_2)^{(0)},(w_1w_2)^{(0)},(w_1w_2)^{(1)},(u_1au_2)^{(1)}),\end{aligned}$$

$$(w_1,u_1)\pi=\begin{cases}(u_1^{(0)},w_1^{(0)},w_1^{(1)},u_1^{(1)}), & \text{if } u_1\neq\theta,\\ w_1, & \text{if } u_1=\theta,\end{cases}$$

$$(w_2,u_2)\pi=\begin{cases}(u_2^{(0)},w_2^{(0)},w_2^{(1)},u_2^{(1)}), & \text{if } u_2\neq\theta,\\ w_2, & \text{if } u_2=\theta.\end{cases}$$

Assume that $u_1\neq\theta, u_2\neq\theta$. Then

$$\begin{aligned}&((u_1au_2)^{(0)},(w_1w_2)^{(0)},(w_1w_2)^{(1)},(u_1au_2)^{(1)})\\ &\quad=(u_1^{(0)},w_1^{(0)},w_2^{(1)},u_2^{(1)})=(u_1^{(0)},w_1^{(0)},w_1^{(1)},u_1^{(1)})\dashv(u_2^{(0)},w_2^{(0)},w_2^{(1)},u_2^{(1)}).\end{aligned}$$

Let $u_1\neq\theta, u_2=\theta$. We get

$$\begin{aligned}&((u_1au_2)^{(0)},(w_1w_2)^{(0)},(w_1w_2)^{(1)},(u_1au_2)^{(1)})\\ &\quad=(u_1^{(0)},w_1^{(0)},w_2^{(1)},a)=(u_1^{(0)},w_1^{(0)},w_1^{(1)},u_1^{(1)})\dashv w_2^{(1)}\\ &\quad=(u_1^{(0)},w_1^{(0)},w_1^{(1)},u_1^{(1)})\dashv w_2.\end{aligned}$$

Consider the case $u_1=\theta, u_2\neq\theta$:

$$\begin{aligned}&((u_1au_2)^{(0)},(w_1w_2)^{(0)},(w_1w_2)^{(1)},(u_1au_2)^{(1)})\\ &\quad=(a,w_1^{(0)},w_2^{(1)},u_2^{(1)})=w_1^{(0)}\dashv(u_2^{(0)},w_2^{(0)},w_2^{(1)},u_2^{(1)})\\ &\quad=w_1\dashv(u_2^{(0)},w_2^{(0)},w_2^{(1)},u_2^{(1)}).\end{aligned}$$

If $u_1=u_2=\theta$, then

$$\begin{aligned}&((u_1au_2)^{(0)},(w_1w_2)^{(0)},(w_1w_2)^{(1)},(u_1au_2)^{(1)})\\ &\quad=(a,w_1^{(0)},w_2^{(1)},a)=w_1^{(0)}\dashv w_2^{(1)}=w_1\dashv w_2.\end{aligned}$$

Thus,

$$((w_1,u_1)\dashv(w_2,u_2))\pi=(w_1,u_1)\pi\dashv(w_2,u_2)\pi$$

for all $(w_1,u_1),(w_2,u_2)\in\mathrm{FDS}(X)$.

The case for $\vdash$ is considered in a similar way.

So, π is a homomorphism. Evidently, π is a surjection. Since by Theorem 3.1 $\mathrm{FRDop}(X)$ is the free rectangular doppelsemigroup, Δ_π is the least rectangular congruence on $\mathrm{FDS}(X)$. From the definition of π it follows that $\Delta_\pi = \widetilde{\pi}$. □

Similarly to Theorem 4.1, one can prove the following statement.

Proposition 4.2. *Let $F[X]$ be the free semigroup. Define a relation $\widetilde{\kappa}$ on $F[X]$ by*

$$w_1 \widetilde{\kappa} w_2 \quad \text{if and only if}$$

$$\ell_{w_1} \neq 1, \quad \ell_{w_2} \neq 1 \quad \text{and} \quad (w_1^{(0)}, w_1^{(1)}) = (w_2^{(0)}, w_2^{(1)}), \quad \text{or} \quad w_1 = w_2.$$

Then $\widetilde{\kappa}$ is the least rectangular semigroup congruence on $F[X]$.

5. Some Properties

In this section, we describe some properties of free rectangular doppelsemigroups. More specially, we characterize all (maximal) subdoppelsemigroups, all idempotents and all endomorphisms of $\mathrm{FRDop}(X)$, and show that $\mathrm{End}(\mathrm{FRDop}(X))$ is isomorphic to $\mathrm{End}(\mathrm{FRDop}(Y))$ if and only if $|X| = |Y|$. We also discuss separately $\mathrm{End}(\mathrm{FRDop}(X))$ for a one-element set X, and consider the question of regularity in $\mathrm{End}(\mathrm{FRDop}(X))$.

Now we describe all (maximal) subdoppelsemigroups of the free rectangular doppelsemigroup $\mathrm{FRDop}(X)$.

Proposition 5.1. *The full list of subdoppelsemigroups of the free rectangular doppelsemigroup* $\mathrm{FRDop}(X)$ *is the following*:

(i) *subdoppelsemigroups $B_1 \times X_1 \times X_2 \times B_2$, where B_1, B_2 are nonempty subsets of B and X_1, X_2 are nonempty subsets of X, which have coinciding operations and are semigroups*;
(ii) *subdoppelsemigroups $\langle Y \rangle = Y \cup (B \times Y \times Y \times B)$ for all nonempty subsets $Y \subseteq X$*;
(iii) *subdoppelsemigroups*

$$(B_1 \times X_1 \times X_2 \times B_2) \cup Y \cup (B \times Y \times Y \times B)$$
$$\cup\, (B_1 \times X_1 \times Y \times B) \cup (B \times Y \times X_2 \times B_2),$$

where B_1, B_2 are nonempty subsets of B and Y, X_1, X_2 are nonempty subsets of X.

Proof. It is not difficult to see that the full list of subdoppelsemigroups of $\mathrm{FRDop}(X)$ consists of

(a) subdoppelsemigroups of $B \times X \times X \times B$;
(b) subdoppelsemigroups generated by nonempty subsets $Y \subseteq X$;
(c) subdoppelsemigroups generated by doppelsemigroups from (a) and (b).

By definitions of $\dashv, \vdash$, the operations of $B \times X \times X \times B$ coincide and it is a rectangular band. Obviously, each of its subsemigroups has the form $B_1 \times X_1 \times X_2 \times B_2$, where B_1, B_2 are nonempty subsets of B and X_1, X_2 are nonempty subsets of X.

Let Y be a nonempty subset of X. We have

$$(Y \dashv Y) \cup (Y \vdash Y) = (\{a\} \times Y \times Y \times \{a\}) \cup (\{b\} \times Y \times Y \times \{b\})$$

and then

$$\langle Y \rangle = Y \cup (B \times Y \times Y \times B).$$

Finally, consider subdoppelsemigroups P generated by doppelsemigroups from (a) and (b). We get

$$\begin{aligned}
&(B_1 \times X_1 \times X_2 \times B_2) \dashv (Y \cup (B \times Y \times Y \times B))\\
&\quad = ((B_1 \times X_1 \times X_2 \times B_2) \dashv Y) \cup ((B_1 \times X_1 \times X_2 \times B_2) \dashv (B \times Y \times Y \times B))\\
&\quad = (B_1 \times X_1 \times Y \times \{a\}) \cup (B_1 \times X_1 \times Y \times B) = B_1 \times X_1 \times Y \times B,\\
&(B_1 \times X_1 \times X_2 \times B_2) \vdash (Y \cup (B \times Y \times Y \times B))\\
&\quad = ((B_1 \times X_1 \times X_2 \times B_2) \vdash Y) \cup ((B_1 \times X_1 \times X_2 \times B_2) \vdash (B \times Y \times Y \times B))\\
&\quad = (B_1 \times X_1 \times Y \times \{b\}) \cup (B_1 \times X_1 \times Y \times B) = B_1 \times X_1 \times Y \times B,\\
&(Y \cup (B \times Y \times Y \times B)) \dashv (B_1 \times X_1 \times X_2 \times B_2)\\
&\quad = (Y \dashv (B_1 \times X_1 \times X_2 \times B_2)) \cup ((B \times Y \times Y \times B) \dashv (B_1 \times X_1 \times X_2 \times B_2))\\
&\quad = (\{a\} \times Y \times X_2 \times B_2) \cup (B \times Y \times X_2 \times B_2) = B \times Y \times X_2 \times B_2,\\
&(Y \cup (B \times Y \times Y \times B)) \vdash (B_1 \times X_1 \times X_2 \times B_2)\\
&\quad = (Y \vdash (B_1 \times X_1 \times X_2 \times B_2)) \cup ((B \times Y \times Y \times B) \vdash (B_1 \times X_1 \times X_2 \times B_2))\\
&\quad = (\{b\} \times Y \times X_2 \times B_2) \cup (B \times Y \times X_2 \times B_2) = B \times Y \times X_2 \times B_2.
\end{aligned}$$

Thus,

$$\begin{aligned}
P = {} & (B_1 \times X_1 \times X_2 \times B_2) \cup Y \cup (B \times Y \times Y \times B)\\
& \cup (B_1 \times X_1 \times Y \times B) \cup (B \times Y \times X_2 \times B_2).
\end{aligned}$$ □

A subdoppelsemigroup of a doppelsemigroup $(D, \dashv, \vdash)$ is proper if it does not equal to $(D, \dashv, \vdash)$. A subdoppelsemigroup T of a doppelsemigroup $(D, \dashv, \vdash)$ will be called maximal provided that $T \neq (D, \dashv, \vdash)$ and for any subdoppelsemigroup $M \leq (D, \dashv, \vdash)$ the inclusion $T \subseteq M$ implies $T = M$, or $M = (D, \dashv, \vdash)$. Equivalently, a subdoppelsemigroup of a doppelsemigroup $(D, \dashv, \vdash)$ is maximal if it is a proper subdoppelsemigroup of $(D, \dashv, \vdash)$ which is not contained in any other proper subdoppelsemigroup of $(D, \dashv, \vdash)$.

Corollary 5.2. *The only maximal subdoppelsemigroups of* $\mathrm{FRDop}(X)$ *are*

$$S_{\{x\}} = \mathrm{FRDop}(X) \backslash \{x\}$$

for all $x \in X$.

Proof. Let $x \in X$. By Proposition 5.1, $S_{\{x\}}$ is a subdoppelsemigroup of $\mathrm{FRDop}(X)$. The maximality follows from the fact that $S_{\{x\}}$ miss exactly one element from $\mathrm{FRDop}(X)$.

Let T be a maximal subdoppelsemigroup of $\mathrm{FRDop}(X)$. Assume that $X \subseteq T$. By the proof of Lemma 3.8, $\mathrm{FRDop}(X)$ is generated by X. Then $X \subseteq T$ implies $T = \mathrm{FRDop}(X)$. This contradicts the maximality of T. Thus, there is $x \in X$ with $x \notin T$, i.e. $T \subseteq S_{\{x\}} \subset \mathrm{FRDop}(X)$. This implies $T = S_{\{x\}}$. $\square$

It is easy to prove the following proposition.

Proposition 5.3. *The set of all idempotents in* $\mathrm{FRDop}(X)$ *is* $B \times X \times X \times B$.

The automorphism group of $\mathrm{FRDop}(X)$ was characterized in Sec. 3. It is natural to describe endomorphisms of the free rectangular doppelsemigroup $\mathrm{FRDop}(X)$.

Let $f : X \to \mathrm{FRDop}(X)$ be an arbitrary map. Define a map

$$\Phi_f : \mathrm{FRDop}(X) \to \mathrm{FRDop}(X)$$

as follows:

$$(a_1, b_1, c_1, d_1)\Phi_f = \begin{cases} (a_1, b_1 f, c_1 f, d_1), & \text{if } b_1 f, c_1 f \in X, \\ (a_1, b_1 f, c_2, d_2), & \text{if } b_1 f \in X \quad \text{and} \quad c_1 f = (a_2, b_2, c_2, d_2), \\ (a_2, b_2, c_1 f, d_1), & \text{if } c_1 f \in X \quad \text{and} \quad b_1 f = (a_2, b_2, c_2, d_2), \\ (a_2, b_2, c_3, d_3), & \text{if } b_1 f = (a_2, b_2, c_2, d_2) \quad \text{and} \quad c_1 f = (a_3, b_3, c_3, d_3), \end{cases}$$

$$x\Phi_f = xf$$

for $(a_1, b_1, c_1, d_1) \in B \times X \times X \times B$ and $x \in X$.

Proposition 5.4.

(i) *For any* f *as above the mapping* Φ_f *is an endomorphism of* $\mathrm{FRDop}(X)$.
(ii) *Every endomorphism of* $\mathrm{FRDop}(X)$ *has the form* Φ_f *for some* f *as above.*

Proof. (i) Since X is a generating set for $\mathrm{FRDop}(X)$ (see the proof of Lemma 3.8), by $f : X \to \mathrm{FRDop}(X)$ an endomorphism of the free rectangular doppelsemigroup $\mathrm{FRDop}(X)$ is uniquely defined. By the construction, Φ_f continues f, and the statement that any element $(a_1, b_1, c_1, d_1)\Phi_f$, where $(a_1, b_1, c_1, d_1) \in B \times X \times X \times B$, can be expressed by elements b_1f, c_1f is proved by a direct calculation. Thus, Φ_f is an endomorphism of $\mathrm{FRDop}(X)$.

(ii) Let φ be an endomorphism of $\mathrm{FRDop}(X)$ and $(a_1, b_1, c_1, d_1) \in B\times X\times X\times B$. Then there are $\star_1, \star_2 \in \{\dashv, \vdash\}$ such that

$$b_1 \star_1 b_1 = (a_1, b_1, b_1, a_1) \quad \text{and} \quad c_1 \star_2 c_1 = (d_1, c_1, c_1, d_1),$$

i.e. $(a_1, b_1, c_1, d_1) = (b_1 \star_1 b_1) \star_1 (c_1 \star_2 c_1)$. Hence,

$$(a_1, b_1, c_1, d_1)\varphi = ((b_1 \star_1 b_1) \star_1 (c_1 \star_2 c_1))\varphi = b_1\varphi \star_1 b_1\varphi \star_1 c_1\varphi \star_2 c_1\varphi.$$

Now we have to consider the following four cases.

Suppose that $b_1\varphi, c_1\varphi \in X$. Then

$$b_1\varphi \star_1 b_1\varphi \star_1 c_1\varphi \star_2 c_1\varphi = (a_1, b_1\varphi, c_1\varphi, d_1).$$

Assume that $b_1\varphi \in X$ and $c_1\varphi = (a_2, b_2, c_2, d_2) \in B \times X \times X \times B$. Then

$$\begin{aligned} &b_1\varphi \star_1 b_1\varphi \star_1 c_1\varphi \star_2 c_1\varphi \\ &\quad = (a_1, b_1\varphi, b_1\varphi, a_1) \star_1 (a_2, b_2, c_2, d_2) = (a_1, b_1\varphi, c_2, d_2). \end{aligned}$$

In the case $c_1\varphi \in X$ and $b_1\varphi = (a_2, b_2, c_2, d_2) \in B \times X \times X \times B$, we obtain

$$\begin{aligned} &b_1\varphi \star_1 b_1\varphi \star_1 c_1\varphi \star_2 c_1\varphi \\ &\quad = (a_2, b_2, c_2, d_2) \star_1 (d_1, c_1\varphi, c_1\varphi, d_1) = (a_2, b_2, c_1\varphi, d_1). \end{aligned}$$

In the remaining case $b_1\varphi = (a_2, b_2, c_2, d_2)$, $c_1\varphi = (a_3, b_3, c_3, d_3) \in B\times X\times X\times B$, we get

$$\begin{aligned} &b_1\varphi \star_1 b_1\varphi \star_1 c_1\varphi \star_2 c_1\varphi \\ &\quad = (a_2, b_2, c_2, d_2) \star_1 (a_3, b_3, c_3, d_3) = (a_2, b_2, c_3, d_3). \end{aligned}$$

As the map from X into $\mathrm{FRDop}(X)$, we can take the restriction of φ to X, i.e. $\varphi|_X$. By definition, in this notation, we have $\varphi = \Phi_f$, where $f = \varphi|_X$. This completes the proof of (ii). □

Proposition 5.4 provides that an endomorphism of $\mathrm{FRDop}(X)$ is uniquely determined by its restriction to X. Hence, by Corollary 3.9, we obtain

$$|\mathrm{End}(\mathrm{FRDop}(X))| = (|X| + 4|X|^2)^{|X|}.$$

Corollary 5.5. *Let $X = \{x\}$ be a one-element set. Then $\varphi \in \mathrm{End}(\mathrm{FRDop}(\{x\}))$ if and only if φ is the identity mapping or there is $\omega \in B \times \{x\} \times \{x\} \times B$ such that $s\varphi = \omega$ for all $s \in \{x\}\cup(B\times\{x\}\times\{x\}\times B)$. In particular,* $|\mathrm{End}(\mathrm{FRDop}(\{x\}))| = 5$.

Since $|\mathrm{FRDop}(X)| = |X| + 4|X|^2$ and $|\mathrm{End}(\mathrm{FRDop}(X))| = (|X| + 4|X|^2)^{|X|}$ for an arbitrary nonempty set X, we obtain the following criterion for an isomorphism of endomorphism semigroups of free rectangular doppelsemigroups.

Proposition 5.6. *Let X and Y be arbitrary nonempty sets. Then* $\mathrm{End}(\mathrm{FRDop}(X))$ *and* $\mathrm{End}(\mathrm{FRDop}(Y))$ *are isomorphic if and only if* $|X| = |Y|$.

Recall that an element a of a semigroup S is called regular provided that there exists $b \in S$ such that $aba = a$. A semigroup S is called regular provided that every element of S is regular.

At the end of this section, let us consider the question of regularity in $\mathrm{End}(\mathrm{FRDop}(X))$.

Proposition 5.7. *Let* $\mathrm{FRDop}(X)$ *be the free rectangular doppelsemigroup.*

(i) *If* $|X| = 1$, *then* $\mathrm{End}(\mathrm{FRDop}(X))$ *is an idempotent monoid, i.e.* $\mathrm{End}(\mathrm{FRDop}(X))$ *is regular.*
(ii) *If* $|X| > 1$, *then* $\mathrm{End}(\mathrm{FRDop}(X))$ *is not regular.*

Proof. The statement (i) is proved by a direct calculation. In order to prove (ii), let $x_1, x_2 \in X$ with $x_1 \neq x_2$, and let $\varphi \in \mathrm{End}(\mathrm{FRDop}(X))$ with

$$x_1\varphi = (a, x_1, x_1, a), \quad x_2\varphi = (a, x_1, x_1, b),$$

and $x\varphi \in B \times X \times X \times B$ for $x \in X \backslash \{x_1, x_2\}$. Let further $g \in \mathrm{End}(\mathrm{FRDop}(X))$. Then there are two cases. First, suppose that $x_1 g = b_2 \in X$ with

$$b_2\varphi = (a_3, b_3, c_3, d_3) \in B \times X \times X \times B.$$

Then

$$x_1(\varphi g\varphi) = ((a, x_1, x_1, a)g)\varphi = (a, b_2, b_2, a)\varphi = (a_3, b_3, c_3, d_3).$$

Similarly, we obtain $x_2(\varphi g\varphi) = (a_3, b_3, c_3, d_3)$. Second, let

$$x_1 g = (a_2, b_2, c_2, d_2) \in B \times X \times X \times B.$$

Then

$$\begin{aligned} x_1(\varphi g\varphi) &= ((a, x_1, x_1, a)g)\varphi \\ &= (a_2, b_2, c_2, d_2)\varphi = ((a, x_1, x_1, b)g)\varphi = x_2(\varphi g\varphi). \end{aligned}$$

This shows that $\varphi g\varphi \neq \varphi$ for all $g \in \mathrm{End}(\mathrm{FRDop}(X))$, whenever $|X| > 1$. □

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