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Stable varieties of semigroups and groupoids

Slavcho Shtrakov and Jörg Koppitz

Abstract. The paper deals with Σ-composition and Σ-essential composition of terms which lead to stable and s-stable varieties of algebras. A full description of all stable varieties of semigroups, commutative and idempotent groupoids is obtained. We use an abstract reduction system which simplifies the presentations of terms of type $\tau = (2)$ to study the variety of idempotent groupoids and s-stable varieties of groupoids. S-stable varieties are a variation of stable varieties, used to highlight replacement of subterms of a term in a deductive system instead of the usual replacement of variables by terms.

1. Introduction

Let $\mathcal{F}$ be any finite set of *operation symbols*. Let $\tau\colon \mathcal{F} \to \mathbb{N}$ be a mapping into the non-negative integers; for $f \in \mathcal{F}$, the number $\tau(f)$ will denote the *arity* of the operation symbol f. The pair $(\mathcal{F}, \tau)$ is called a *type* or *signature*. If it is obvious what the set $\mathcal{F}$ is, we will write "*type* τ". The set of symbols of arity p is denoted by $\mathcal{F}_p$.

Let $X = \{x_1, x_2, \dots\}$ be a countable set of variables, and let τ be a type with the set of operation symbols $\mathcal{F}$. The set $W_\tau(X)$ of *terms of type* τ with variables from X is the smallest set such that $X \cup \mathcal{F}_0 \subseteq W_\tau(X)$ and if f is an n-ary operation symbol, and $t_1, \dots, t_n \in W_\tau(X)$ are terms, then $f(t_1, \dots, t_n) \in W_\tau(X)$.

If $f \in \mathcal{F}$, then $f^{\mathcal{A}}$ denotes a $\tau(f)$-ary operation on the set A. An *algebra* $\mathcal{A} = \langle A; \mathcal{F}^{\mathcal{A}}\rangle$ of type τ is a pair consisting of a set A and an indexed set $\mathcal{F}^{\mathcal{A}}$ of operations, defined on A. If $s, t \in W_\tau(X)$, then the pair $s \approx t$ is called an *identity* of type τ which is satisfied in the algebra $\mathcal{A}$, $\mathcal{A} \models t \approx s$ iff $t^A = s^A$.

The operators Id and Mod are defined for classes of algebras K and for sets of identities Σ as follows

$$\mathrm{Id}(K) = \{t \approx s \mid \mathcal{A} \in K \Rightarrow \mathcal{A} \models t \approx s\}, \text{ and}$$
$$\mathrm{Mod}(\Sigma) = \{\mathcal{A} \mid t \approx s \in \Sigma \Rightarrow \mathcal{A} \models t \approx s\}.$$

The fixed points with respect to the closure operators Id Mod and Mod Id are called *equational theories* and *varieties of algebras*, respectively.

In Section 2 we introduce the inductive, positional, and Σ-composition of terms.

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We apply the concept of Σ-composition of terms to study the stable varieties of semigroups (see Theorem 3.8). We prove that a variety V of semigroups is stable if and only if $\mathrm{Id}(V)$ contains an identity of the form $(x_1x_2)x_3 \approx x_ix_j$ with $1 \leqslant i < j \leqslant 3$. We present a complete list of all the stable varieties of semigroups (see Theorem 3.9).

An Abstract Reduction System (ARS) for terms, which reduces the complexity of terms by such traditional measures as depth and length is introduced in Section 4.

The varieties of commutative and idempotent groupoids are stable which is shown in Section 5.

We present stronger conditions for stability of varieties which successfully work in the variety of groupoids. These conditions allow us to define and study the s-stable varieties of groupoids in Section 6 (see Theorem 6.13).

2. Compositions of terms

If t is a term, then the set $\mathrm{var}(t)$ consisting of those elements of X which occur in t is called the set of *input variables* (*or variables*) in t. If $t = f(t_1, \dots, t_n)$ is a non-variable term, then f is the *root symbol* (*root*) of t.

For a term $t \in W_\tau(X)$ the set $\mathrm{Sub}(t)$ of its subterms is defined as follows: if $t \in X \cup \mathcal{F}_0$, then $\mathrm{Sub}(t) = \{t\}$ and if $t = f(t_1, \dots, t_n)$, then $\mathrm{Sub}(t) = \{t\} \cup \mathrm{Sub}(t_1) \cup \cdots \cup \mathrm{Sub}(t_n)$.

Let $r, s, t \in W_\tau(X)$ be terms of type τ. By $t(r \leftarrow s)$ we denote the term, obtained by simultaneous replacement of every occurrence of r as a subterm of t by s. This term is called the *inductive composition* [8] of the terms t and r by s. If $r_i \notin \mathrm{Sub}(r_j)$ when $i \neq j$, then $t(r_1 \leftarrow s_1, \dots, r_m \leftarrow s_m)$ means the inductive composition of $t, r_1, \dots, r_m$ by $s_1, \dots, s_m$, respectively. In the particular case when $r_j = x_j$ for $j = 1, \dots, m$ and $\mathrm{var}(t) = \{x_1, \dots, x_m\}$ we will briefly write $t(s_1, \dots, s_m)$ instead of $t(x_1 \leftarrow s_1, \dots, x_m \leftarrow s_m)$.

Any term can be regarded as a tree with nodes labeled as the operation symbols and leaves labeled as variables or nullary operation symbols (see Figure 1, below).

Let τ be a type and let $\mathcal{F}$ be its set of operation symbols. Denote by

$$\mathbb{N}_\tau = \{m \in \mathbb{N} \mid m \leqslant \max_{f \in \mathcal{F}} \tau(f)\}.$$

Let $\mathbb{N}_\tau^*$ be the set of all finite strings over $\mathbb{N}_\tau$. The set $\mathbb{N}_\tau^*$ is naturally ordered by $p \preceq q \iff p$ is a prefix of q. The Greek letter ε, as usual denotes the empty word (string) over $\mathbb{N}_\tau$.

To distinguish between different occurrences of the same operation symbol in a term t we assign to each occurrence of an operation symbol a position. Usually positions are finite sequences (strings) over $\mathbb{N}_\tau$. Each position is assigned to a node of the tree diagram of t, starting with the empty sequence ε for the root and using the integer j, $1 \leqslant j \leqslant n$ for the j-th branch of an n-ary operational symbol f. Inductively, let the position $p = a_1a_2 \cdots a_s \in \mathbb{N}_\tau^*$

be assigned to a node of t labeled by the n-ary operational symbol f. Then the position assigned to the j-th child of this node is $a_1a_2\cdots a_sj$. The set of positions of a term t is denoted by $\mathrm{Pos}(t)$.

Let $t \in W_\tau(X)$ be a term of type τ and let $\mathrm{sub}_t\colon \mathrm{Pos}(t) \to \mathrm{Sub}(t)$ be the function which maps each position in a term t to the subterm of t, whose root node occurs at that position.

Let $t, r \in W_\tau(X)$ be two terms of type τ and let $p \in \mathrm{Pos}(t)$ be a position in t. The *positional composition* [8] of t and r on p is the term $s = t(p;r)$ obtained from t by replacing the term $\mathrm{sub}_t(p)$ by r on the position p, only. We will use the notation $t(p,q;r)$ for the composition $t(p;r)(q;r)$ when $p \not\preceq q \ \& \ q \not\preceq p$, and if $S = \langle p_1,\dots,p_m\rangle \in \mathrm{Pos}(t)^m$ with $(\forall i,j \leqslant m)(i \neq j \Rightarrow p_i \not\preceq p_j \ \& \ p_j \not\preceq p_i)$, then $t(S;r) = t(p_1,\dots,p_m;r) = t(p_1;r)\cdots(p_m;r)$. If $T = \langle t_1,\dots,t_m\rangle \in W_\tau(X)^m$, then $t(S;T) = t(p_1;t_1)\cdots(p_m;t_m)$.

Let $X_n = \{x_1,\dots,x_n\}$ be a finite set of variables in X. Then we denote by $W_\tau(X_n)$ the set $W_\tau(X_n) = \{t \in W_\tau(X) \mid \mathrm{var}(t) \subseteq X_n\}$ of terms.

Let $\Sigma \subseteq \mathrm{Id}(\tau)$, let $t \in W_\tau(X_n)$ be an n-ary term of type τ, let $\mathcal{A} = \langle A, \mathcal{F}\rangle$ be an algebra of type τ, and let $x_i \in \mathrm{var}(t)$ be a variable which occurs in t. The variable x_i is called *essential* [7] in t with respect to the algebra $\mathcal{A}$ if there are $n+1$ elements $a_1,\dots,a_{i-1},a,b,a_{i+1},\dots,a_n \in A$ such that

$$t^{\mathcal{A}}(a_1,\dots,a_{i-1},a,a_{i+1},\dots,a_n) \neq t^{\mathcal{A}}(a_1,\dots,a_{i-1},b,a_{i+1},\dots,a_n).$$

The set of all essential variables in t with respect to $\mathcal{A}$ is denoted by $\mathrm{Ess}(t,\mathcal{A})$. $\mathrm{Fic}(t,\mathcal{A})$ denotes the set of all variables in $\mathrm{var}(t)$, which are not essential with respect to $\mathcal{A}$, called *fictive variables*.

Let Σ be a set of identities of type τ. Then $\mathcal{A} \models \Sigma$ means that $\mathcal{A} \models t \approx s$ for all $t \approx s \in \Sigma$. For $t, s \in W_\tau(X)$, we say Σ *yields* $t \approx s$ (write: $\Sigma \models t \approx s$) if, given any algebra $\mathcal{A}$, we have $\mathcal{A} \models \Sigma \Rightarrow \mathcal{A} \models t \approx s$.

Let Σ be a set of identities of type τ. Two terms t and s are called Σ-*equivalent* (or Σ-*equal*) if $\Sigma \models t \approx s$.

A variable x_i is said to be Σ-*essential* [8] in a term t if there is an algebra $\mathcal{A}$, such that $\mathcal{A} \models \Sigma$ and $x_i \in \mathrm{Ess}(t,\mathcal{A})$. The set of all Σ-essential variables in t is denoted by $\mathrm{Ess}(t,\Sigma)$. If a variable is not Σ-essential in t, then it is called Σ-*fictive* in t. $\mathrm{Fic}(t,\Sigma)$ denotes the set of all Σ-fictive variables in t.

The concept of Σ-essential positions is a natural extension of Σ-essential variables. Let $\mathcal{A} = \langle A,\mathcal{F}\rangle$ be an algebra of type τ, let $t \in W_\tau(X_n)$, and let $p \in \mathrm{Pos}(t)$. If $x_{n+1} \in \mathrm{Ess}(t(p;x_{n+1}),\mathcal{A})$, then the position $p \in \mathrm{Pos}(t)$ is called *essential* in t with respect to $\mathcal{A}$. The set of all essential positions in t with respect to $\mathcal{A}$ is denoted by $\mathrm{PEss}(t,\mathcal{A})$ (see Example 2.3 below or Example 2.1 of [8]).

When a position $p \in \mathrm{Pos}(t)$ is not essential in t with respect to $\mathcal{A}$, it is called *fictive* in t with respect to $\mathcal{A}$. The set of all fictive positions with respect to $\mathcal{A}$ is denoted by $\mathrm{PFic}(t,\mathcal{A})$.

If $x_{n+1} \in \mathrm{Ess}(t(p;x_{n+1}),\Sigma)$, the position $p \in \mathrm{Pos}(t)$ is called Σ-*essential* in t [8]. The set of Σ-essential positions in t is denoted by $\mathrm{PEss}(t,\Sigma)$. When a position is not Σ-essential in t it is called Σ-*fictive*. $\mathrm{PFic}(t, \Sigma)$ denotes the set of all Σ-fictive positions in t.

The set of Σ-essential subterms in t is defined as follows:

$$\mathrm{SEss}(t, \Sigma) = \{r \in W_\tau(X) \mid \Sigma \models r \approx \mathrm{sub}_t(p) \text{ for } p \in \mathrm{PEss}(t, \Sigma)\}.$$

So, a term is a Σ-essential subterm of a term t if it is Σ-equivalent to a subterm of t whose root is located at a Σ-essential position in t.

Let $t, r \in W_\tau(X)$ and let $\Sigma S_r^t = \{v \in \mathrm{Sub}(t) \mid \Sigma \models r \approx v\}$ be the set of all subterms of t which are Σ-equal to r.

Let $\Sigma P_r^t = \{p \in \mathrm{Pos}(t) \mid \mathrm{sub}_t(p) \in \Sigma S_r^t\}$ be the set of all positions of subterms of t which are Σ-equivalent to r. Let $P_r^t = \{p_1, \ldots, p_m\}$ be the set of all the minimal elements in ΣP_r^t with respect to the ordering $\preceq$ in the set of positions, i.e., $p \in P_r^t$ if for each $q \in \Sigma P_r^t$ we have $q \not\preceq p$.

Definition 2.1. [8] *Term Σ-composition* $t^\Sigma(r \leftarrow s)$ of t and r by s is defined as follows

(i) $t^\Sigma(r \leftarrow s) = t$ if $P_r^t = \emptyset$;
(ii) $t^\Sigma(r \leftarrow s) = t(P_r^t; s)$ if $P_r^t \neq \emptyset$.

Lemma 2.2. *If $\Sigma \models r \approx v$, then $t^\Sigma(r \leftarrow u) = t^\Sigma(v \leftarrow u)$.*

Proof. The lemma follows from the obvious equation $P_r^t = P_v^t$ for each term $v \in W_\tau(X)$ with $\Sigma \models r \approx v$. □

Example 2.3. Let $\tau = (2)$ and let us consider the variety $RB = \mathrm{Mod}(\Sigma)$ of rectangular bands, where

$$\Sigma = \{x_1(x_2x_3) \approx (x_1x_2)x_3 \approx x_1x_3,\ x_1x_1 \approx x_1\}.$$

Let $t = ((x_1x_2)x_2)((x_1x_2)x_3)$, $r = x_1x_2$ and $s = x_4$.

The sets of Σ-essential positions and subterms in t are:

$$\mathrm{PEss}(t, \Sigma) = \{\varepsilon, 1, 11, 111, 2, 22\},$$
$$\mathrm{SEss}(t, \Sigma) = \{t, (x_1x_2)x_2, x_1x_2, x_1, (x_1x_2)x_3, x_3\}.$$

The Σ-essential and Σ-fictive positions in t are represented by large and small black circles, respectively in Figure 1. Next, we have

$$\Sigma S_r^t = \{x_1x_2, (x_1x_2)x_2\}, \quad \Sigma P_r^t = \{1, 11, 21\} \quad \text{and} \quad P_r^t = \{1, 21\}.$$

Thus, we have $t^\Sigma(r \leftarrow s) = x_4(x_4x_3)$ (see Figure 2).

Terms are important tools in various areas, such as abstract data type specifications, implementation of programming languages, automated deduction etc. They can be used as models for different structures in logic programming, term rewriting systems and other computational procedures.

A *valuation* of a term is a function $\mathrm{Val}\colon W_\tau(X) \to \mathbb{N}$ such that for some $c \in \mathbb{N}$, $\mathrm{Val}(x_i) = c$ for all $i \geqslant 1$, and $\mathrm{Val}(t) \geqslant c$ for all $t \in W_\tau(X)$. The natural number c is called initial value of the valuation Val. It is often important for

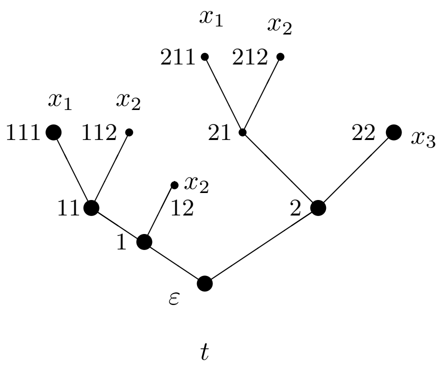

FIGURE 1. Σ-essential positions in t.

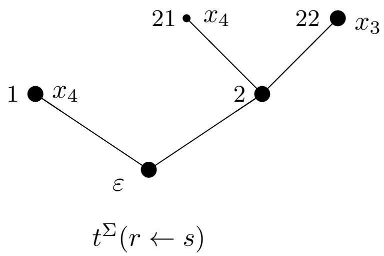

FIGURE 2. Σ-composition of terms t and r by s.

applications that terms be represented in forms with low complexity, including sometimes in normal forms.

Some common valuations are based on a linguistic point of view which counts the number of variables or the number of operation symbols occurring in the term.

If l_i denotes the number of occurrences of the variable x_i in the n-ary term t, then the valuation Len, defined by $\mathrm{Len}(t) = \sum_{x_i \in \mathrm{var}(t)} l_i$, is called the *length* of t. Its initial value is 1.

The *depth* of a term t is defined by $\mathrm{Depth}(x_i) = 0$ for $i = 1, 2, \ldots$ and otherwise, $\mathrm{Depth}(f(t_1, \ldots, t_n)) = \max\{\mathrm{Depth}(t_1), \ldots, \mathrm{Depth}(t_n)\} + 1$.

Let $t \in W_\tau(X)$ be a term and let $\mathrm{Wv}(t) = x_{i_1} \cdots x_{i_s}$ be the word of variables in t which are written from left to right, and let $\mathrm{St}(t) = i_1 \cdots i_s \in \mathbb{N}^*$ be the string of the indexes in $\mathrm{Wv}(t)$. The term t is called Σ-*minimal* if for each $s \in W_\tau(X)$ with $\Sigma \models t \approx s$, we have $\mathrm{Len}(t) < \mathrm{Len}(s)$ or $\mathrm{St}(t) \prec_{lex} \mathrm{St}(s)$ when $\mathrm{Len}(t) = \mathrm{Len}(s)$, where $\prec_{lex}$ is the lexicographical order in $\mathbb{N}^*$.

Clearly, Σ-minimal terms are unique. For instance, let t be the term defined in Example 2.3. Then $\mathrm{Len}(t) = 6$, $\mathrm{Depth}(t) = 3$, $\mathrm{Wv}(t) = x_1x_2x_2x_1x_2x_3$, and $\mathrm{St}(t) = 122123$. The Σ-minimal term corresponding to t is x_1x_3. It is clear that Σ-minimal terms minimize the valuations Depth and Len in the sets of Σ-equal terms.

We need some basic definitions from universal algebra. More detailed background about these notions can be found in the classic text [1].

Definition 2.4. [1] A set Σ of identities of type τ is D-*deductively closed* if it satisfies the following axioms (some authors call them "deductive rules", "derivation rules", "productions", etc.):

D_1 (*reflexivity*) $t \approx t \in \Sigma$ for each term $t \in W_\tau(X)$;

D_2 (*symmetry*) $(t \approx s \in \Sigma) \Rightarrow s \approx t \in \Sigma$;

D_3 (*transitivity*) $(t \approx s \in \Sigma)\ \&\ (s \approx r \in \Sigma) \Rightarrow t \approx r \in \Sigma$;

D_4 (*term positional replacement*)

$$(t \approx s \in \Sigma)\ \&\ (r \in W_\tau(X))\ \&\ (\mathrm{sub}_r(p) = t) \Rightarrow r(p; s) \approx r \in \Sigma;$$

D_5 (*variable inductive substitution*)

$$(t \approx s \in \Sigma)\ \&\ (r \in W_\tau(X)) \Rightarrow t(x \leftarrow r) \approx s(x \leftarrow r) \in \Sigma.$$

For any set Σ of identities, the smallest D-deductively closed set containing Σ is called the D-closure of Σ and it is denoted by $D(\Sigma)$.

The first three deductive rules make $D(\Sigma)$ into an equivalence relation, the fourth makes it a congruence, and the last rule says $D(\Sigma)$ is a *fully invariant congruence.*

Let Σ be a set of identities of type τ. For $t \approx s \in \mathrm{Id}(\tau)$, we say Σ *proves* $t \approx s$ and write $\Sigma \vdash t \approx s$ if there is a sequence of identities (D-*deductions*) $t_1 \approx s_1, \ldots, t_n \approx s_n$, such that each identity belongs to Σ or is a result of applying any of the derivation rules D_1–D_5 to previous identities in the sequence and the last identity $t_n \approx s_n$ is $t \approx s$. It is well-known that $\Sigma \vdash t \approx s \iff \Sigma \models t \approx s$.

In [8], a variation of the derivation rules D_1–D_5 is given which is used to define a *globally invariant congruence.*

Definition 2.5. [8] A set Σ of identities is $\Sigma\,$R-*deductively closed* if it satisfies the rules D_1, D_2, D_3, D_5 and

$\Sigma\,R_1$ (Σ *replacement*)

$$\left(\begin{array}{c} r, t, s, u \in W_\tau(X)\ \&\ (t \approx s \in \Sigma) \\ \&\ r \in \mathrm{SEss}(t, \Sigma) \cap \mathrm{SEss}(s, \Sigma) \end{array}\right) \Rightarrow t^{\Sigma}(r \leftarrow u) \approx s^{\Sigma}(r \leftarrow u) \in \Sigma.$$

For any set Σ of identities, the smallest $\Sigma\,$R-deductively closed set containing Σ is called the $\Sigma\,$R-*closure* of Σ and it is denoted by $\Sigma\,R(\Sigma)$.

$\Sigma\,$R is a closure operator which implies that:

(1) $\Sigma\,R(\Sigma\,R(\Sigma)) = \Sigma\,R(\Sigma)$, and

(2) for each $\Delta \subseteq \Sigma$, if $\Delta \vdash t \approx s$, then $t \approx s \in \Sigma\,R(\Sigma)$.

A set $\Sigma \subseteq \mathrm{Id}(\tau)$ is called a *globally invariant congruence* if it is $\Sigma\,$R-deductively closed. In [8], it is proved that each globally invariant congruence is a fully invariant congruence.

A variety V of type τ is called *stable* if $\Sigma = \mathrm{Id}(V)$ is $\Sigma\,$R-deductively closed.

3. Stable varieties of semigroups

We are going to describe all stable varieties of semigroups in analogy to the solid varieties [5, 6], using some fundamental results in semigroup theory as presented in [2, 3].

Throughout the rest of the paper, we write $f(x_1, x_2)$ as (x_1x_2) or x_1x_2. The following identities of type (2) are important for the achievement of our aim:

$$x_1x_2x_3 \approx x_ix_j \quad \text{for} \quad i, j \in \{1, 2, 3\}. \tag{3.1}$$

They allow us to define a special class of varieties of semigroups. Let i and j be two natural numbers from the set $\{1, 2, 3\}$. Then we consider the following variety of semigroups:

$$V_{ij} = \mathrm{Mod}(\{(x_1x_2)x_3 \approx x_1(x_2x_3),\ x_1x_2x_3 \approx x_ix_j\}).$$

Let $t \in W_\tau(X)$ be a term and let $\mathrm{Wv}(t) = x_{i_1} \cdots x_{i_s}$ be the word of the variables in t. We use this notation: $\mathrm{first}(t) = x_{i_1}$, $\mathrm{second}(t) = x_{i_2}$, …, s - th$(t) = x_{i_s}$. Often, the last variable x_{i_s} is denoted by $\mathrm{last}(t)$ or $\mathrm{rightmost}(t)$. Also, some authors write $\mathrm{first}(t)$ as $\mathrm{leftmost}(t)$. For instance, if $t = x_3x_1x_2x_2$ then we have $\mathrm{first}(t) = \mathrm{leftmost}(t) = x_3$, $\mathrm{second}(t) = x_1$, $\mathrm{third}(t) = x_2$ and $\mathrm{fourth}(t) = \mathrm{last}(t) = \mathrm{rightmost}(t) = x_2$.

Lemma 3.1. *The varieties LZ (of Left-Zero-semigroups), RZ (of Right-Zero-semigroups), Z (of Zero-semigroups), and the varieties V_{ij} with $1 \leqslant i < j \leqslant 3$ are stable.*

Proof. Let $\mathcal{V} \in \{LZ, RZ, Z, V_{12}, V_{13}, V_{23}\}$ and $\Sigma = \mathrm{Id}(\mathcal{V})$. Since $\mathrm{Id}(\mathcal{V})$ is a fully invariant congruence, it satisfies the derivation rules D_1, D_2, D_3, D_4, and D_5. We have to prove that $\Sigma\,\mathrm{R}_1$ is also satisfied in $\mathcal{V}$, i.e.,

$$\Sigma \models t^\Sigma(r \leftarrow u) \approx s^\Sigma(r \leftarrow u), \tag{3.2}$$

when $\Sigma \models t \approx s$, $r \in \mathrm{SEss}(t, \Sigma) \cap \mathrm{SEss}(s, \Sigma)$, and $u \in W_\tau(X)$.

Let $t, s, r \in W_\tau(X)$ be terms with $\Sigma \models t \approx s$ and $r \in \mathrm{SEss}(t, \Sigma) \cap \mathrm{SEss}(s, \Sigma)$, and let $u \in W_\tau(X)$ be a term. To prove (3.2), let us observe the following two common facts:

First, if $P_r^t = \{\varepsilon\}$ or $P_r^s = \{\varepsilon\}$, then $r \in \mathrm{SEss}(t, \Sigma) \cap \mathrm{SEss}(s, \Sigma)$ implies $\Sigma \models t \approx r$ and $\Sigma \models s \approx r$. Thus, we have $t^\Sigma(r \leftarrow u) = s^\Sigma(r \leftarrow u) = u$, which shows that (3.2) is satisfied.

Second, if $\mathrm{Depth}(t) = 0$, then we have $t = x_i$ for some variable $x_i \in X$, and hence $r \in \mathrm{SEss}(t, \Sigma) \cap \mathrm{SEss}(s, \Sigma)$ implies $\Sigma \models r \approx x_i$. Now, (3.2) is satisfied, according to D_5.

Next, let $\Sigma \models t \approx s$, $r \in \mathrm{SEss}(t, \Sigma) \cap \mathrm{SEss}(s, \Sigma)$, $1 \leqslant \mathrm{Depth}(t) \leqslant \mathrm{Depth}(s)$, and $\Sigma \not\models t \approx r$.

Claim 1: LZ and RZ are stable varieties.

Consider the variety $\mathcal{V} = LZ$. Then $\Sigma \models x_1x_2 \approx x_1$ and $\Sigma \models w \approx \mathrm{first}(w)$ for all terms w. Consequently, $\Sigma \models t \approx s$ and $r \in \mathrm{SEss}(t, \Sigma) \cap \mathrm{SEss}(s, \Sigma)$ imply

$\mathrm{first}(t) = \mathrm{first}(s) = \mathrm{first}(r)$. Thus, we have $\Sigma \models t \approx s \approx r$, and hence (3.2) is satisfied, which shows that LZ is stable. The variety RZ is stable by dual arguments.

Claim 2: Z is a stable variety.

We have $\Sigma \models x_1x_2 \approx x_3x_4$. Clearly, $P_r^t = P_r^s = \{\varepsilon\}$ for all terms t, s, r with $\Sigma \models t \approx s$, and $r \in \mathrm{SEss}(t, \Sigma) \cap \mathrm{SEss}(s, \Sigma)$, which proves the stability of Z.

Claim 3: V_{12}, V_{13} and V_{23} are stable varieties.

We shall show that the variety $\mathcal{V} = V_{12}$ is stable. Hence, we have

$$\Sigma \models (x_1x_2)x_3 \approx x_1(x_2x_3) \quad \text{and} \quad \Sigma \models x_1x_2x_3 \approx x_1x_2.$$

Since $\mathrm{Depth}(t) \geqslant 1$, we have $\Sigma \models t \approx \mathrm{first}(t)\,\mathrm{second}(t)$. Let us assume, with no loss of generality, that $x_1 = \mathrm{first}(t)$ and $x_2 = \mathrm{second}(t)$.

Next, $\Sigma \not\models t \approx r$ and $r \in \mathrm{SEss}(t, \Sigma) \cap \mathrm{SEss}(s, \Sigma)$ implies $\Sigma \models r \approx x_1$ or $\Sigma \models r \approx x_2$. Without loss of generality, let us assume that $\Sigma \models r \approx x_1$. According to Lemma 2.2, we have $\Sigma \models t^\Sigma(r \leftarrow u) \approx t^\Sigma(x_1 \leftarrow u)$. Since x_1 is a variable, it is easy to see that $\Sigma \models t^\Sigma(x_1 \leftarrow u) \approx t(x_1 \leftarrow u)$. Hence, for satisfaction of (3.2), we need $\Sigma \models t(x_1 \leftarrow u) \approx s(x_1 \leftarrow u)$, which follows from D_5. Consequently, V_{12} is a stable variety. The proof that V_{13} and V_{23} are stable varieties is left to the reader. □

Remark 3.2. Let us consider the variety $V_{21} = \mathrm{Mod}(\Sigma)$, where

$$\Sigma = \{(x_1x_2)x_3 \approx x_1(x_2x_3),\ x_1x_2x_3 \approx x_2x_1\}.$$

Then we have

$$\begin{aligned}\Sigma \models x_1x_2 &\approx (x_2x_1)x_3 \approx ((x_1x_2)x_4)x_3 \approx x_1(x_2x_4)x_3\\ &\approx x_1((x_4x_2)x_5)x_3 \approx ((x_1x_4)x_2)x_5x_3 \approx x_4((x_1x_5)x_3)\\ &\approx x_4(x_5x_1) \approx ((x_4x_5)x_1) \approx x_5x_4.\end{aligned}$$

Hence, $V_{21} = Z$. Using similar or dual arguments one can show that $V_{31} = V_{32} = Z$.

Proposition 3.3. *The varieties of semigroups V_{ii}, for $i \in \{1, 2, 3\}$ are not stable.*

Proof. We prove that $V_{11} = \mathrm{Mod}(\{(x_1x_2)x_3 \approx x_1(x_2x_3),\ x_1x_2x_3 \approx x_1x_1\})$ is not a stable variety.

Consider the following terms: $t = (x_1x_2)x_3$, $s = (x_1x_2)x_4$, and $r = x_1x_2$. Clearly, $\Sigma \models t \approx s$. Then we have $t(1; x_1) = x_1x_3$, $t(1; x_3) = x_3x_3$, and $\Sigma \not\models x_3x_3 \approx x_1x_3$. Since $\Sigma \not\models x_1x_2 \approx (x_1x_2)x_3$, we have $1 \in P_r^t$. In an analogous way, we obtain $1 \in P_r^s$. Next, we have $t^\Sigma(r \leftarrow x_3) = x_3x_3$ and $s^\Sigma(r \leftarrow x_3) = x_3x_4$, which shows that V_{11} is not stable. In a similar way, one can prove that V_{22} and V_{33} are not stable varieties. □

Lemma 3.4. *The following varieties of semigroups are stable:*

$$V_1 = \mathrm{Mod}(\{(x_1x_2)x_3 \approx x_1(x_2x_3),\ x_1x_2 \approx x_1x_3\}),$$
$$V_3 = \mathrm{Mod}(\{(x_1x_2)x_3 \approx x_1(x_2x_3),\ x_1x_3 \approx x_2x_3\}).$$

Proof. We shall prove that V_1 is stable. Let t, s and r be terms for which $\Sigma \models t \approx s$ and $r \in \mathrm{SEss}(t, \Sigma) \cap \mathrm{SEss}(s, \Sigma)$.

If $\mathrm{Depth}(t) = 0$, then $\mathrm{Depth}(s) = 0$ and (3.2) is clearly satisfied.

If $\mathrm{Depth}(t) = 1$ and $\mathrm{Depth}(s) = 1$, then $\mathrm{first}(t) = \mathrm{first}(s)$ and $r = \mathrm{first}(t)$ or $\Sigma \models t \approx r$, and (3.2) is obvious, again.

If $\mathrm{Depth}(t) \geqslant 2$, then $\Sigma \models t \approx s \approx \mathrm{first}(t)x_2$ for an arbitrary variable $x_2 \in X$. This implies that $\Sigma \models r \approx \mathrm{first}(t)$ or $\Sigma \models t \approx r$. In both cases, (3.2) is satisfied.

By dual arguments it follows that V_3 is stable. $\square$

Remark 3.5. Since $\{x_1x_2 \approx x_1x_3\} \models x_1x_2x_3 \approx x_1x_1$, it follows that $V_1 \subseteq V_{11}$ and by dual arguments, we have $V_3 \subseteq V_{33}$.

Lemma 3.6. *Let $\mathcal{V} = \mathrm{Mod}(\Sigma)$ be a stable variety of semigroups. If we have $\Sigma \models x_1x_2x_3 \approx x_1x_2x_4$, then Σ proves at least one identity among* (3.1) *with $1 \leqslant i < j \leqslant 3$.*

Proof. Consider the following terms: $t = (x_1x_2)x_3$, $s = (x_1x_2)x_4$, $r = x_1x_2$, and $u = x_1$. Clearly, $\Sigma \models t \approx s$.

If $r \notin \mathrm{SEss}(t, \Sigma) \cap \mathrm{SEss}(s, \Sigma)$, then we are done because $\Sigma \models (x_1x_2)x_3 \approx x_5x_3$ and from D_5, we have $\Sigma \models (x_1x_2)x_3 \approx x_1x_3$.

Let $r \in \mathrm{SEss}(t, \Sigma) \cap \mathrm{SEss}(s, \Sigma)$. If $\Sigma \models t \approx r$, then $\Sigma \models s \approx r$, and we are done again because of $\Sigma \models (x_1x_2)x_3 \approx x_1x_2$.

Next, assume that $\Sigma \not\models t \approx r$. Then $\Sigma \not\models s \approx r$, $P^t_r = \{1\}$, and $P^s_r = \{1\}$. From $\Sigma \models t \approx s$ and $\Sigma\, \mathrm{R}_1$, we obtain

$$\Sigma \models t^\Sigma(r \leftarrow x_1) \approx s^\Sigma(r \leftarrow x_1), \quad \text{and} \quad \Sigma \models x_1x_3 \approx x_1x_4.$$

According to D_5, we can replace x_4 by x_2x_3 in the last identity, and hence $\Sigma \models x_1x_3 \approx x_1(x_2x_3)$. $\square$

In a similar way, one can show that if we have $\Sigma \models x_1x_2x_3 \approx x_4x_2x_3$ or $\Sigma \models x_1x_2x_3 \approx x_1x_4x_3$, then Σ proves at least one identity among (3.1) with $1 \leqslant i < j \leqslant 3$.

Lemma 3.7. *If $\mathcal{V} = \mathrm{Mod}(\Sigma)$ is a stable variety of semigroups, then we have $\Sigma \models x_1x_1x_1 \approx x_1x_1$.*

Proof. If Σ proves at least one identity among (3.1) with $1 \leqslant i < j \leqslant 3$, then $\Sigma \models x_1x_1x_1 \approx x_1x_1$ is clear.

Assume that Σ does not prove any identity among (3.1) with $1 \leqslant i < j \leqslant 3$. We shall prove the lemma by considering cases:

Case A: $\Sigma \not\models (x_1x_2)(x_1x_2) \approx x_1x_2$.

Let us put $t = ((x_1x_2)x_1)x_2$, $s = (x_1x_2)(x_1x_2)$, $r = x_1x_2$, and $u = x_3$. Clearly, $\Sigma \models t \approx s$. Suppose $r \notin \mathrm{SEss}(t, \Sigma)$; then $\Sigma \models (x_3x_1)x_2 \approx (x_4x_1)x_2$, which contradicts Lemma 3.6, and so $r \in \mathrm{SEss}(t, \Sigma)$. Suppose $r \notin \mathrm{SEss}(s, \Sigma)$; then $\Sigma \models x_3x_1 \approx x_3x_4 \approx x_3(x_1x_2)$, a contradiction. Hence, $r \in \mathrm{SEss}(s, \Sigma)$.

Then we have $t^{\Sigma}(r \leftarrow u) = (x_3x_1)x_2$ and $s^{\Sigma}(r \leftarrow u) = x_3x_3$. Hence, $\Sigma \models (x_3x_1)x_2 \approx x_3x_3$ and after replacing x_3 and x_2 by x_1, we obtain that $\Sigma \models x_1x_1x_1 \approx x_1x_1$.

Case B: $\Sigma \models (x_1x_2)(x_1x_2) \approx x_1x_2$.

The associative law and D_5 imply

$$\Sigma \models ((x_1x_1)x_1)x_1 \approx (x_1x_1)(x_1x_1) \approx x_1x_1. \tag{3.3}$$

Let us put $t = ((x_1x_2)x_3)(x_1x_2)$, $s = (((x_1x_2)x_3)x_1)x_2$, $r = x_1x_2$, and $u = x_4$. Clearly, $\Sigma \models t \approx s$.

If $\Sigma \models t \approx r$, then we are done after replacing x_2 and x_3 by x_1 in $\Sigma \models t \approx s$.

If we suppose that $r \notin \mathrm{SEss}(t, \Sigma)$, then $\Sigma \models (x_4x_3)x_5 \approx (x_6x_3)x_7$, which contradicts Lemma 3.6. Hence, $r \in \mathrm{SEss}(t, \Sigma)$.

Let us assume that $r \in \mathrm{SEss}(t, \Sigma) \cap \mathrm{SEss}(s, \Sigma)$. Then from (3.2), we obtain $\Sigma \models (x_4x_3)x_4 \approx ((x_4x_3)x_1)x_2$. Replacing x_2, x_3, x_4 by x_1, and using (3.3), we obtain $\Sigma \models x_1x_1 \approx (x_1x_1)x_1$.

Assume that $r \in \mathrm{SEss}(t, \Sigma) \setminus \mathrm{SEss}(s, \Sigma)$; then

$$\Sigma \models (((x_1x_2)x_3)x_1)x_2 \approx ((x_4x_3)x_1)x_2.$$

Replacing x_2, x_3, x_4 by x_1, and using (3.3), we obtain $\Sigma \models x_1x_1 \approx x_1x_1x_1$. $\square$

Theorem 3.8. *If a variety $\mathcal{V}$ of semigroups is stable, then $\mathcal{V} \subseteq V_{12}$ or $\mathcal{V} \subseteq V_{13}$, or $\mathcal{V} \subseteq V_{23}$.*

Proof. Let $\mathcal{V} = \mathrm{Mod}(\Sigma)$ be a stable variety of semigroups.

First, let $\Sigma \not\models ((x_1x_2)x_2)x_3 \approx ((x_1x_2)x_2)x_4$ and let us put $t = ((x_1x_2)x_2)x_3$, $s = (((x_1x_2)x_2)x_2)x_3$, $r = (x_1x_2)x_2$, and $u = x_4$. Lemma 3.7 implies that $\Sigma \models t \approx s$.

If $r \notin \mathrm{SEss}(t, \Sigma)$, then $\Sigma \models x_1x_3 \approx x_2x_3$ and according to D_5, we have $\Sigma \models x_1x_3 \approx (x_1x_2)x_3$.

If $r \notin \mathrm{SEss}(s, \Sigma)$, then $\Sigma \models (x_1x_2)x_3 \approx (x_4x_2)x_3$, and hence we are done because of Lemma 3.6.

If we suppose that $\Sigma \models r \approx t$, i.e., $\Sigma \models (x_1x_2)x_2 \approx ((x_1x_2)x_2)x_3$, then D_3 implies $\Sigma \models (x_1x_2)x_2 \approx ((x_1x_2)x_2)x_4$, which contradicts our assumption. Hence, $\Sigma \not\models r \approx t$, $P_r^t = \{1\}$, $P_r^s = \{11\}$, and (3.2) implies $\Sigma \models x_4x_2 \approx (x_4x_2)x_3$.

Second, assume that $\Sigma \models ((x_1x_2)x_2)x_3 \approx ((x_1x_2)x_2)x_4$. Lemma 3.7 then implies $\Sigma \models ((x_1x_2)x_2)x_3 \approx ((x_1x_2)x_2)x_2 \approx (x_1x_2)x_2$. Consider the terms $t = (x_1x_2)x_2$, $s = ((x_1x_2)x_2)x_3$, $r = x_1x_2$, and $u = x_4$. Clearly, $\Sigma \models t \approx s$.

If $r \notin \mathrm{SEss}(t, \Sigma)$, then $\Sigma \models x_1x_2 \approx x_3x_2$, and hence $\Sigma \models x_1x_2 \approx (x_1x_3)x_2$.

If $r \notin \mathrm{SEss}(s, \Sigma)$, then $\Sigma \models (x_1x_2)x_3 \approx (x_4x_2)x_3$, and hence we are done because of Lemma 3.6.

Let $r \in \mathrm{SEss}(t,\Sigma) \cap \mathrm{SEss}(s,\Sigma)$. If $\Sigma \models t \approx r$, then $\Sigma \models s \approx r$, i.e., $\Sigma \models ((x_1x_2)x_2)x_3 \approx x_1x_2 \approx (x_1x_2)x_3$.

If $\Sigma \not\models t \approx r$, then $P_r^t = \{1\}$ and $P_r^s = \{11\}$. Thus, from (3.2), we obtain $\Sigma \models x_4x_2 \approx (x_4x_2)x_3$, which completes the proof. □

Theorem 3.9. *Let $\mathcal{V}$ be a variety of semigroups. Then $\mathcal{V}$ is stable if and only if $\mathcal{V}$ is one of the following ten varieties:*

$\mathcal{V}_1 = \mathrm{Mod}(\{(x_1x_2)x_3 \approx x_1(x_2x_3),\ x_1x_2x_3 \approx x_1x_3\})$,
$\mathcal{V}_2 = \mathrm{Mod}(\{(x_1x_2)x_3 \approx x_1(x_2x_3),\ x_1x_2x_3 \approx x_1x_2\})$,
$\mathcal{V}_3 = \mathrm{Mod}(\{(x_1x_2)x_3 \approx x_1(x_2x_3),\ x_1x_2x_3 \approx x_2x_3\})$,
$\mathcal{V}_4 = \mathrm{Mod}(\{(x_1x_2)x_3 \approx x_1(x_2x_3),\ x_1x_2 \approx x_1x_3\})$,
$\mathcal{V}_5 = \mathrm{Mod}(\{(x_1x_2)x_3 \approx x_1(x_2x_3),\ x_1x_2 \approx x_3x_2\})$,
$\mathcal{V}_6 = RB$ *the variety of rectangular bands,*
$\mathcal{V}_7 = LZ$ *the variety of Left-Zero-semigroups,*
$\mathcal{V}_8 = RZ$ *the variety of Right-Zero-semigroups,*
$\mathcal{V}_9 = Z$ *the variety of Zero-semigroups,*
$\mathcal{V}_{10} = TR$ *the trivial variety.*

All these varieties are pairwise distinct.

Proof. First, assume that $\mathcal{V}$ is a stable variety of semigroups. Then from Theorem 3.8, we have $\mathcal{V} = \mathrm{Mod}(\Sigma)$ for some set of identities Σ, which proves at least one identity among (3.1) with $1 \leqslant i < j \leqslant 3$. Then

$$\mathcal{V} \in \{\mathcal{V}_1, \mathcal{V}_2, \mathcal{V}_3, \mathcal{V}_4, \mathcal{V}_5, RB, LZ, RZ, Z, TR\}$$

follows by the following well-known facts (see [2]):

Fact 1: The non-trivial proper subvarieties of $\mathcal{V}_2$ are $\mathcal{V}_4$, LZ, and Z.
Fact 2: The non-trivial proper subvarieties of $\mathcal{V}_3$ are $\mathcal{V}_5$, RZ, and Z.
Fact 3: The non-trivial proper subvarieties of $\mathcal{V}_1$ are RB, $\mathcal{V}_4$, $\mathcal{V}_5$, LZ, RZ, and Z.

Second, the varieties $\mathcal{V}_1, \mathcal{V}_2, \mathcal{V}_3, LZ, RZ$, and Z are stable, according to Lemma 3.1; the varieties $\mathcal{V}_4$ and $\mathcal{V}_5$ are stable, according to Lemma 3.4. The stability of RB is proved in [8]. The variety TR is obviously stable.

Finally, it is a well-known fact that all these varieties are pairwise distinct [2].

All stable varieties of semigroups are shown in Figure 3. □

4. Abstract reduction systems and deduction of identities

A *Term Rewriting System* (TRS) for deductions of identities is a pair $(\tau, \mathcal{R})$ consisting of a type and a set of reduction (rewrite) rules which are binary relations on $W_\tau(X)$ written as $t \to r$.

Our aim is to use a TRS and apply its well-developed tools to investigate the stability of several varieties of groupoids. For this purpose, we consider a TRS as an Abstract Reduction System (ARS).

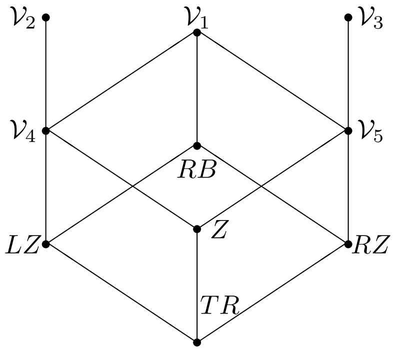

FIGURE 3. Stable varieties of semigroups.

An ARS is a structure $\mathcal{W} = \langle W_\tau(X), (\rightarrow_i)_{i\in I}, \Sigma\rangle$, where $(\rightarrow_i)_{i\in I}$ is a family of binary relations on $W_\tau(X)$, called *reductions or rewrite relations*. For a reduction $\rightarrow_i$, the transitive and reflexive closure is denoted by $\twoheadrightarrow_i$. A term $r \in W_\tau(X)$ is a *normal form* if there is no $v \in W_\tau(X)$ such that $r \rightarrow_i v$.

TRSs, and in particular ARSs, play an important role in various areas such as abstract data type specification, functional programming, automated deductions, etc. For more detailed information about TRSs, we refer to J. W. Klop and Roel de Vrijer [4]. The concepts and properties of ARSs also apply to other rewrite systems such as string rewrite systems (Thue systems), tree rewrite systems, graph grammars, etc.

Many computations, constructions, processes, translations, mappings and so on, can be modeled as stepwise transformations of objects known as rewriting systems. In all different branches of rewriting, two basic concepts occur, known as termination (guaranteeing the existence of normal forms) and confluence (securing the uniqueness of normal forms).

Let us consider the ARS $\mathcal{W} = \langle W_{(2)}(X), \{\rightarrow_R\}, \Sigma\rangle$ determined by the following reduction:

$$t \rightarrow_R r \overset{\text{def}}{\iff} r = t(p; u)$$

where $s = \mathrm{sub}_t(p)$, $\Sigma \models s \approx u$, and u is Σ-minimal. According to D_4, we have that if $t \rightarrow_R r$, then $\Sigma \models t \approx r$.

Our intention is to reduce the terms in an identity to normal forms and then implement the deductive rules on these normal forms, preferably with low complexity terms. First, we are interested in existence and uniqueness of normal forms for the reduction $\rightarrow_R$.

A reduction $\rightarrow$ has the *unique normal form property* (UN) if whenever $t, r \in W_\tau(X)$ are normal forms and $\Sigma \models t \approx r$ then $t = r$. We are going to prove that $\rightarrow_R$ is UN when Σ determines the variety of idempotent groupoids or consists of identities as from (3.1). This we shall do using Newman's Lemma (Theorem 1.2.1. [4]).

A reduction $\to$ is *terminating* (or *strongly normalizing* SN) if every reduction sequence $t \to t_1 \to t_2 \cdots$ eventually must terminate. A reduction $\to$ is *weakly confluent* (or *has weakly Church-Rosser property* WCR) if $t \to r$ and $t \to v$ imply that there is $w \in W_\tau(X)$ such that $r \twoheadrightarrow w$ and $v \twoheadrightarrow w$.

Theorem 4.1. *The reduction $\to_R$ is SN and WCR.*

Proof. (SN): Clearly, if $t \to_R r$, then $\mathrm{Len}(t) \geqslant \mathrm{Len}(r)$ or $\mathrm{St}(t) \prec_{lex} \mathrm{St}(r)$ when $\mathrm{Len}(t) = \mathrm{Len}(r)$. Since the lengths $\mathrm{Len}(z)$ of the terms z in any reduction sequence decrease and strings $\mathrm{St}(z)$ strongly decrease, it follows that the sequence eventually must terminate, i.e., the reduction is terminating.

(WCR): Let t be a term, $p, q \in \mathrm{Pos}(t)$, $s = \mathrm{sub}_t(p)$, $w = \mathrm{sub}_t(q)$, $r = t(p; u)$, and $v = t(q; z)$, where u and z are Σ-minimal. If $p \prec q$, then we have $t \to_R r \to_R w$. If $q \prec p$, then $t \to_R w \to_R r$, which shows that reduction $\to_R$ is WCR in these two cases.

Let $p \not\prec q$ and $q \not\prec p$, and let y be the Σ-minimal term with $\Sigma \models t \approx y$. Then we have

$$t \to_R w \to_R r \twoheadrightarrow_R y \quad \text{and} \quad t \to_R r \to_R w \twoheadrightarrow_R y. \qquad \square$$

Corollary 4.2. *The reduction $\to_R$ is UN.*

Proof. This follows from Newman's Lemma, which states that WCR & SN $\Rightarrow$ UN (see Theorem 1.2.1. [4]). $\square$

For each term $t \in W_\tau(X)$, we denote by $\mathrm{Red}(t)$ the normal form obtained from t under the reduction $\to_R$.

Corollary 4.3. $\Sigma \models t \approx \mathrm{Red}(t)$ *for any term* $t \in W_\tau(X)$.

It is easy to see that the normal form operator Red minimizes the valuations Len and Depth.

5. Stable varieties of groupoids

We are going to study stable varieties of groupoids. Let us note that if $\Sigma = \emptyset$, then $\mathrm{Mod}(\Sigma)$ is a stable variety.

First, we consider the variety of idempotent groupoids. Note that if $t \in W_\tau(X)$ and $s \in \mathrm{Sub}(\mathrm{Red}(t))$, then there is $r \in \mathrm{Sub}(t)$ such that $\Sigma \models r \approx s$, and if $t = t_1 t_2$, then $\Sigma \models \mathrm{Red}(t) \approx \mathrm{Red}(t_1)\,\mathrm{Red}(t_2)$.

Lemma 5.1. *If* $\Sigma = \{x_1 x_1 \approx x_1\}$, *then*

$$\Sigma \models \mathrm{Red}(t^\Sigma(r \leftarrow u)) \approx \mathrm{Red}(t)^\Sigma(r \leftarrow u) \tag{5.1}$$

for every $r, t, u \in W_\tau(X)$.

Proof. Let $t, r, u \in W_\tau(X)$ be three terms. We shall proceed by induction on $\mathrm{Depth}(t)$. If $\mathrm{Depth}(t) = 0$, then $t = x_i$ for some natural number i. Then $\mathrm{Red}(t) = x_i$ and it is obvious that (5.1) is satisfied.

Let us assume that for some natural number $k \geqslant 2$, if $\mathrm{Depth}(t) < k$, then (5.1) is satisfied for t. Let $\mathrm{Depth}(t) = k$ and $t = t_1t_2$. Let $r \in \mathrm{Sub}(t)$. If $\Sigma \models r \approx t$, then $\Sigma \models r \approx \mathrm{Red}(t)$ and we have $\Sigma \models r \approx \mathrm{Red}(t)$. Hence, $\mathrm{Red}(t^{\Sigma}(r \leftarrow u)) = u$ and $\mathrm{Red}(t)^{\Sigma}(r \leftarrow u) = u$, which proves (5.1).

Let $r \in \mathrm{Sub}(t)$ and $\Sigma \not\models r \approx t$. If $\Sigma S_r^t = \emptyset$, then clearly $\Sigma S_r^{\mathrm{Red}(t)} = \emptyset$ and (5.1) is obviously satisfied in this case.

Assume that $\Sigma S_r^t \neq \emptyset$. By the inductive assumption we have

$$\Sigma \models \mathrm{Red}(t_i^{\Sigma}(r \leftarrow u)) \approx \mathrm{Red}(t_i)^{\Sigma}(r \leftarrow u)$$

for $i = 1, 2$ and $r, u \in W_\tau(X)$. Hence,

$$\Sigma \models \mathrm{Red}(t_1^{\Sigma}(r \leftarrow u))\,\mathrm{Red}(t_2^{\Sigma}(r \leftarrow u)) \approx \mathrm{Red}(t_1)^{\Sigma}(r \leftarrow u)\,\mathrm{Red}(t_2)^{\Sigma}(r \leftarrow u).$$

Thus, we have

$$\begin{aligned}\Sigma &\models \mathrm{Red}(t_1^{\Sigma}(r \leftarrow u))\,\mathrm{Red}(t_2^{\Sigma}(r \leftarrow u)) \\ &\approx \mathrm{Red}((t_1^{\Sigma}(r \leftarrow u)t_2^{\Sigma}(r \leftarrow u))) \approx \mathrm{Red}(t^{\Sigma}(r \leftarrow u)).\end{aligned}$$

Let us assume that $\Sigma \not\models t_1 \approx t$ and $\Sigma \not\models t_2 \approx t$. Then

$$\begin{aligned}\Sigma &\models \mathrm{Red}(t_1)^{\Sigma}(r \leftarrow u)\,\mathrm{Red}(t_2)^{\Sigma}(r \leftarrow u) \\ &\approx (\mathrm{Red}(t_1)\,\mathrm{Red}(t_2))^{\Sigma}(r \leftarrow u) = \mathrm{Red}(t)^{\Sigma}(r \leftarrow u),\end{aligned}$$

which proves (5.1) in this case.

Let $\Sigma \models t_1 \approx t$. Then $\{1, 2\} \subseteq \mathrm{PEss}(t, \Sigma)$ implies $\Sigma \models t_1 \approx t_2$. Then $\Sigma \models \mathrm{Red}(t_1) \approx \mathrm{Red}(t_2)$ and $\Sigma \models \mathrm{Red}(t_1t_2) \approx \mathrm{Red}(t_1)\,\mathrm{Red}(t_2) \approx \mathrm{Red}(t_1)$. Hence,

$$\begin{aligned}\Sigma &\models \mathrm{Red}(t_1)^{\Sigma}(r \leftarrow u)\,\mathrm{Red}(t_2)^{\Sigma}(r \leftarrow u) \\ &\approx \mathrm{Red}(t_1)^{\Sigma}(r \leftarrow u) \approx \mathrm{Red}(t)^{\Sigma}(r \leftarrow u).\end{aligned}$$

□

Theorem 5.2. *The variety* $IG = \mathrm{Mod}(\{x_1x_1 \approx x_1\})$ *of idempotent groupoids is stable.*

Proof. We put $\Sigma = \mathrm{Id}(\mathrm{Mod}(\{x_1x_1 \approx x_1\}))$. We have to prove (3.2) when $\Sigma \models t \approx s$ and $r \in \mathrm{SEss}(t, \Sigma) \cap \mathrm{SEss}(s, \Sigma)$. Without loss of generality, let us assume that $\mathrm{Depth}(t) \leqslant \mathrm{Depth}(s)$. We shall proceed by induction on $\mathrm{Depth}(t)$. Our inductive basis is $\mathrm{Depth}(t) \leqslant 1$. Then clearly $t = s$ or $s = x_1x_2$, and (3.2) is satisfied.

Assume that (3.2) is satisfied when $\mathrm{Depth}(t) < k$ for some natural number $k \geqslant 2$. Let $\mathrm{Depth}(t) = k$. Then $t = t_1t_2$ and $s = s_1s_2$. Lemma 5.1 allows us to think that terms t and s are presented in their normal forms under $\to_R$, i.e., $t = \mathrm{Red}(t)$ and $s = \mathrm{Red}(s)$. Hence, $1, 2 \in \mathrm{PEss}(t, \Sigma) \cap \mathrm{PEss}(s, \Sigma)$ and $\Sigma \not\models t_i \approx t$, and $\Sigma \not\models s_i \approx s$ for $i = 1, 2$. This shows that $\Sigma \not\models t_1 \approx t_2$ and $\Sigma \not\models s_1 \approx s_2$. Hence, $\Sigma \models t_i \approx s_i$ for $i = 1, 2$. Now $\mathrm{Depth}(t_i) < k$, so our inductive assumption and Lemma 5.1 prove (3.2). □

Theorem 5.3. *The variety* $CG = \mathrm{Mod}(\{x_1x_2 \approx x_2x_1\})$ *of all commutative groupoids is stable.*

Proof. Let $\Sigma = \{x_1x_2 \approx x_2x_1\}$. Let us note that $\Sigma \models u \approx v$ implies $\mathrm{Len}(u) = \mathrm{Len}(v)$, $\mathrm{Depth}(u) = \mathrm{Depth}(v)$, and $\mid \mathrm{Pos}(u) \mid = \mid \mathrm{Pos}(v) \mid$, for all $v, u \in W_\tau(X)$.

We shall prove (3.2) by induction on the depth of terms t and s. Let $\mathrm{Depth}(t) = \mathrm{Depth}(s) = 0$. Then $t = s = x_1$ for some variable $x_1 \in X$ and (3.2) is obvious.

Assume that (3.2) is satisfied when $\mathrm{Depth}(t) = \mathrm{Depth}(s) < k$ for some natural number k, $k > 1$. Let $\mathrm{Depth}(t) = \mathrm{Depth}(s) = k$, $\Sigma \models t \approx s$, and $r \in \mathrm{SEss}(t,\Sigma) \cap \mathrm{SEss}(s,\Sigma)$. Let n be a natural number such that $t, s, r, u \in W_\tau(X_n)$ and let us denote by $z_{n+1}, \ldots, z_{n+p} \in W_\tau(X_n)$ all subterms of t, s or r with depths equal to 1 which are distinguished by Σ, i.e., $\Sigma \not\models z_{n+i} \approx z_{n+j}$ when $i \neq j$. Using inductive composition, we obtain three new terms, namely:

$$\begin{aligned} t' &= t(z_{n+1} \leftarrow x_{n+1}, \ldots, z_{n+p} \leftarrow x_{n+p}),\\ s' &= s(z_{n+1} \leftarrow x_{n+1}, \ldots, z_{n+p} \leftarrow x_{n+p}), \text{ and}\\ r' &= r(z_{n+1} \leftarrow x_{n+1}, \ldots, z_{n+p} \leftarrow x_{n+p}). \end{aligned}$$

Thus, we have $t', s', r' \in W_\tau(X_{n+p})$ and $\mathrm{Depth}(t') = \mathrm{Depth}(s') = k-1 < k$. It is easy to see that $\Sigma \models t' \approx s'$ and $r' \in \mathrm{SEss}(t',\Sigma) \cap \mathrm{SEss}(s',\Sigma)$. Our inductive assumption implies $\Sigma \models t'^{\Sigma}(r' \leftarrow u) \approx s'^{\Sigma}(r' \leftarrow u)$. Let us put $t'' = t'^{\Sigma}(r' \leftarrow u)$ and $s'' = s'^{\Sigma}(r' \leftarrow u)$. Then from D_5 it follows that

$$t''(x_{n+1} \leftarrow z_{n+1}, \ldots, x_{n+p} \leftarrow z_{n+p}) \approx s''(x_{n+1} \leftarrow z_{n+1}, \ldots, x_{n+p} \leftarrow z_{n+p}).$$

Now, the equations

$$\begin{aligned} t^{\Sigma}(r \leftarrow u) &= t''(x_{n+1} \leftarrow z_{n+1}, \ldots, x_{n+p} \leftarrow z_{n+p}), \text{ and}\\ s^{\Sigma}(r \leftarrow u) &= s''(x_{n+1} \leftarrow z_{n+1}, \ldots, x_{n+p} \leftarrow z_{n+p}) \end{aligned}$$

complete the proof. □

Remark 5.4. (i) It is surprising that the variety CG of all commutative groupoids is stable, but the analogous variety of commutative semigroups is not stable, as shown by Theorem 3.8. Hence, stability is not inherited by subvarieties of groupoids.

(ii) Theorem 3.9 and the description of the lattice of the varieties of semigroups given in [2] show that if a variety $\mathcal{V}$ of semigroups is stable, then all subvarieties of $\mathcal{V}$ are stable.

Next, we consider the following varieties of groupoids:

$$V_{lm}^{ijk} = \mathrm{Mod}(\{(x_ix_j)x_k \approx x_lx_m\}) \quad \text{and} \quad W_{lm}^{ijk} = \mathrm{Mod}(\{x_i(x_jx_k) \approx x_lx_m\}),$$

where $i, j, k, l, m \in \{1, 2, 3\}$.

Theorem 5.5. *The varieties of groupoids V_{lm}^{ijk} and W_{lm}^{ijk} for $i, j, k, l, m \in \{1, 2\}$ are stable.*

Proof. Since $\mathrm{Id}(V_{lm}^{ijk})$ and $\mathrm{Id}(W_{lm}^{ijk})$ are fully invariant congruences, they satisfy rules $\mathrm{D}_1 - \mathrm{D}_5$. Thus, we have to prove that $\Sigma\,\mathrm{R}_1$ is satisfied in $\mathrm{Id}(V_{lm}^{ijk})$ and $\mathrm{Id}(W_{lm}^{ijk})$, i.e., that (3.2) is satisfied in V_{lm}^{ijk} and W_{lm}^{ijk}.

Let $t, s, r \in W_\tau(X)$ be three terms for which $r \in \mathrm{SEss}(t, \Sigma) \cap \mathrm{SEss}(s, \Sigma)$ and $\Sigma \models t \approx s$. Suppose, with no loss of generality, that $\mathrm{Depth}(t) \leqslant \mathrm{Depth}(s)$. If $\Sigma \models t \approx r$ then (3.2) is obvious. Thus, we assume that $\Sigma \not\models t \approx r$.

Claim 1: The varieties V_{11}^{ijk} and W_{11}^{ijk} for $i, j, k \in \{1, 2\}$ are stable.

In [8], it is proved that V_{11}^{121} is stable (see Proposition 3.1 of [8]). In a similar way, one can prove that V_{11}^{ijk} and W_{11}^{ijk} for $i, j, k \in \{1, 2\}$ are stable.

Claim 2: The varieties V_{12}^{ijk} and W_{12}^{ijk} for $i, j, k \in \{1, 2\}$ are stable.

We shall show that V_{12}^{121} is stable by induction on $\mathrm{Depth}(t)$. If $\mathrm{Depth}(t) = 0$, then (3.2) is clearly satisfied. Let $\mathrm{Depth}(t) = 1$. Then, with no loss of generality, we can assume that $t = x_1x_1$ or $t = x_1x_2$. Hence, $\Sigma \models r \approx x_1$ or $\Sigma \models r \approx x_2$. Then (3.2) follows from D_5.

Assume that (3.2) is satisfied when $\mathrm{Depth}(t) < k$ for some natural number k, $k > 2$. Let $\mathrm{Depth}(t) = k$. Then we have $t = t_1t_2$ with $1 \leqslant \mathrm{Depth}(t_i) < k$ for $i = 1, 2$. Then $\Sigma \models t \approx s$ implies that $s = s_1s_2$ or $s = (s_1s_2)s_1$ with $\Sigma \models t_1 \approx s_1$ and $\Sigma \models t_2 \approx s_2$. Since $\Sigma \not\models t \approx r$, we then have that $r \in \mathrm{SEss}(t_1, \Sigma) \cap \mathrm{SEss}(s_1, \Sigma)$ or $r \in \mathrm{SEss}(t_2, \Sigma) \cap \mathrm{SEss}(s_2, \Sigma)$. Thus, we have

$$\begin{aligned}
&t^\Sigma(r \leftarrow u) = t_1^\Sigma(r \leftarrow u)t_2^\Sigma(r \leftarrow u),\\
&s^\Sigma(r \leftarrow u) = s_1^\Sigma(r \leftarrow u)s_2^\Sigma(r \leftarrow u), \text{ and}\\
&s^\Sigma(r \leftarrow u) = (s_1^\Sigma(r \leftarrow u)s_2^\Sigma(r \leftarrow u))s_1^\Sigma(r \leftarrow u),
\end{aligned}$$

which proves (3.2), according to our inductive assumption.

In a similar way, one can show that V_{12}^{211} and V_{12}^{112} are stable varieties. By dual arguments, we obtain that W_{12}^{121}, W_{12}^{211} and W_{12}^{112} are stable varieties.

Claim 3: The varieties V_{ii}^{111} and W_{ii}^{111} for $i \in \{1, 2\}$ are stable.

We shall prove that V_{11}^{111} is stable by induction on $\mathrm{Depth}(t)$. If $\mathrm{Depth}(t) = 0$ or $\mathrm{Depth}(t) = 1$, then (3.2) can be proved as in the previous case.

Assume that (3.2) is satisfied when $\mathrm{Depth}(t) < k$ for some natural number k, $k > 2$. Let $\mathrm{Depth}(t) = k$. Then we have $t = t_1t_1$ or $t = t_1t_2$ with $1 \leqslant \mathrm{Depth}(t_i) < k$ for $i = 1, 2$.

If $t = t_1t_1$, then $\Sigma \models t \approx s$ implies that $s = s_1s_1$ or $s = (s_1s_1)s_1$ with $\Sigma \models t_1 \approx s_1$. Since $\Sigma \not\models t \approx r$, it follows that $r \in \mathrm{SEss}(t_1, \Sigma) \cap \mathrm{SEss}(s_1, \Sigma)$. Thus, we have

$$\begin{aligned}
&t^\Sigma(r \leftarrow u) = t_1^\Sigma(r \leftarrow u)t_1^\Sigma(r \leftarrow u),\\
&s^\Sigma(r \leftarrow u) = s_1^\Sigma(r \leftarrow u)s_1^\Sigma(r \leftarrow u), \text{ and}\\
&s^\Sigma(r \leftarrow u) = (s_1^\Sigma(r \leftarrow u)s_1^\Sigma(r \leftarrow u))s_1^\Sigma(r \leftarrow u),
\end{aligned}$$

which proves (3.2), according to our inductive assumption.

If $t = t_1t_2$, then $\Sigma \models t \approx s$ implies that $s = s_1s_2$ with $\Sigma \models t_1 \approx s_1$ and $\Sigma \models t_2 \approx s_2$. Since $\Sigma \not\models t \approx r$, it follows that $r \in \mathrm{SEss}(t_1,\Sigma) \cap \mathrm{SEss}(s_1,\Sigma)$ or $r \in \mathrm{SEss}(t_2,\Sigma) \cap \mathrm{SEss}(s_2,\Sigma)$. Thus, we have

$$t^\Sigma(r \leftarrow u) = t_1^\Sigma(r \leftarrow u)t_2^\Sigma(r \leftarrow u), \text{ and}$$
$$s^\Sigma(r \leftarrow u) = s_1^\Sigma(r \leftarrow u)s_2^\Sigma(r \leftarrow u)$$

which proves (3.2) again, according to our inductive assumption.

The varieties V_{22}^{111} and W_{22}^{111} are clearly stable. □

6. S-stable varieties

Let us go back to the identities (3.1). These identities guarantee stability of a variety of semigroups that satisfies one of them. It is natural to expect that the identities (3.1) will provide for stability of a variety of groupoids. The next proposition is a counterexample to that expectation.

Proposition 6.1. *The varieties V_{lm}^{ijk} and W_{lm}^{ijk} are not stable when $\{i,j,k\} = \{1,2,3\}$ and $l,m \in \{1,2,3\}$ with $l \neq m$.*

Proof. Without loss of generality, we shall prove that $V_{23}^{123} = \mathrm{Mod}(\Sigma)$ is not stable, where $\Sigma = \{(x_1x_2)x_3 \approx x_2x_3\}$. Let us put $t = (x_3(x_1x_2))(x_2(x_1x_2))$, $s = x_2(x_2(x_1x_2))$, $r = x_1x_2$, and $u = x_4$. Clearly, $\Sigma \models t \approx s$. Since $\mathrm{PFic}(t,\Sigma) = \{11, 121\}$ and $\mathrm{PFic}(s,\Sigma) = \emptyset$, it follows that $P_r^t = \{12, 22\}$ and $P_r^s = \{22\}$. Thus, we have $t^\Sigma(r \leftarrow u) = (x_3x_4)(x_2x_3)$ and $s^\Sigma(r \leftarrow u) = x_2(x_2x_4)$. Clearly, $\Sigma \not\models t^\Sigma(r \leftarrow u) \approx s^\Sigma(r \leftarrow u)$. Hence, the variety $\mathrm{Mod}(\Sigma)$ is not stable. □

Our aim in this section is to define additional sufficient conditions for stability such that if a variety of groupoids satisfies an identity among (3.1), then it is stable under these conditions. Also, we expect the varieties V_{11}, V_{22} and V_{33} of semigroups to be included in this new concept of stability. We are going to define an s-stable variety for an arbitrary type τ.

For two terms t, r, let $EP_r^t = \{p \in P_r^t \mid p \preceq q \in \mathrm{Pos}(t) \Rightarrow q \in \mathrm{PEss}(t,\Sigma)\}$ be the set of all the minimal elements in ΣP_r^t whose successors are Σ-essential in t.

Definition 6.2. Let $r, s, t \in W_\tau(X)$ be terms of type τ. The Σ-*essential composition* of the terms t and r by s is defined as follows:

(i) $t(r * s) = t$ if $EP_r^t = \emptyset$;
(ii) $t(r * s) = t(EP_r^t; s)$ if $EP_r^t \neq \emptyset$.

Example 6.3. Let us consider the terms $t = (x_3(x_1x_2))(x_2(x_1x_2))$ and $r = x_1x_2$ from the Proposition 6.1 and let $s = x_4$. Then we have $EP_r^t = \{22\}$ and $t(r * s) = (x_3(x_1x_2))(x_2x_4)$. On the other hand, $P_r^t = \{12, 22\}$ implies $t^\Sigma(r \leftarrow s) = (x_3x_4)(x_2x_4)$. Clearly, $\Sigma \not\models t(r * s) \approx t^\Sigma(r \leftarrow s)$.

Definition 6.4. A set Σ of identities is SR-deductively closed if it satisfies the rules $\mathrm{D}_1, \mathrm{D}_2, \mathrm{D}_3, \mathrm{D}_5$ and

SR_1 (*Star Replacement*)

$$\left(\begin{array}{c} r, t, s, u \in W_\tau(X) \;\;\&\;\; (t \approx s \in \Sigma) \\ \&\;\; (EP_r^t \neq \emptyset) \;\;\&\;\; (EP_r^s \neq \emptyset) \end{array} \right) \Rightarrow t(r * u) \approx s(r * u) \in \Sigma.$$

For any set of identities Σ, the smallest SR-deductively closed set containing Σ is called the SR-*closure* of Σ, and is denoted by $\mathrm{SR}(\Sigma)$. For $t \approx s \in \mathrm{Id}(\tau)$, we say $\Sigma \vdash_{\mathrm{SR}} t \approx s$ ("Σ SR-*proves* $t \approx s$") if there is a sequence of identities $t_1 \approx s_1, \ldots, t_n \approx s_n$, such that each identity belongs to Σ or is a result of applying any of the derivation rules $\mathrm{D}_1, \mathrm{D}_2, \mathrm{D}_3, \mathrm{D}_5$, or SR_1 to previous identities in the sequence and the last identity $t_n \approx s_n$ is $t \approx s$.

Let $t \approx s$ be an identity and let $\mathcal{A}$ be an algebra of type τ. $\mathcal{A} \models_{\mathrm{SR}} t \approx s$ means that $\mathcal{A} \models t(r * v) \approx s(r * v)$ for every $r \in \mathrm{SEss}(t, \Sigma) \cap \mathrm{SEss}(s, \Sigma)$ and $v \in W_\tau(X)$. For $t, s \in W_\tau(X)$, we say $\Sigma \models_{\mathrm{SR}} t \approx s$ (read: "Σ SR-*yields* $t \approx s$") if, given any algebra $\mathcal{A}$, $\mathcal{A} \models_{\mathrm{SR}} \Sigma \;\Rightarrow\; \mathcal{A} \models_{\mathrm{SR}} t \approx s$.

As in [8] (see Theorem 3.4 and Theorem 3.6), one can prove that SR is a closure operator, and prove a completeness theorem that $\Sigma \models_{\mathrm{SR}} t \approx s \iff \Sigma \vdash_{\mathrm{SR}} t \approx s$.

Theorem 6.5. *For each set of identities Σ, the closure* $\mathrm{SR}(\Sigma)$ *is a fully invariant congruence.*

Proof. It is enough to prove that $\mathrm{SR}(\Sigma)$ satisfies the rule D_4. Let $r \in W_\tau(X)$, $t \approx s \in \Sigma$, and $p \in \mathrm{Pos}(r)$. If $p \notin \mathrm{PEss}(r, \Sigma)$, then we have $r(p; v) \approx r(p; w) \in \mathrm{SR}(\Sigma)$ for all terms $v, w \in W_\tau(X)$. Let $p \in \mathrm{PEss}(r, \Sigma)$ and let n be a natural number such that $r, t, s \in W_\tau(X_n)$. Write $v = r(p; x_{n+1})$ and $u = x_{n+1}$. Clearly, $u \in \mathrm{Sub}(v)$ and $EP_u^v = \{p\}$. We have $v(u * t) = r(p; t)$ and $v(u * s) = r(p; s)$. Now from SR_1, we obtain $v(u * t) \approx v(u * s) \in \mathrm{SR}(\Sigma)$, i.e., $r(p; t) \approx r(p; s) \in \mathrm{SR}(\Sigma)$. □

As $EP_r^t \subseteq P_r^t$ for all $t, s \in W_\tau(X)$, so $t \approx s \in \mathrm{SR}(\Sigma) \Rightarrow t \approx s \in \Sigma\mathrm{R}(\Sigma)$, for each identity $t \approx s \in \mathrm{Id}(\tau)$. Thus, we obtain the following inclusions, $\mathrm{D}(\Sigma) \subseteq \mathrm{SR}(\Sigma) \subseteq \Sigma\mathrm{R}(\Sigma)$ for each $\Sigma \subseteq \mathrm{Id}(\tau)$. Hence, each stable variety is an s-stable one.

Definition 6.6. A set of identities Σ is called an *s-globally invariant congruence* if it is SR-deductively closed. A variety V of type τ is called *s-stable* if $\mathrm{Id}(V)$ is an s-globally invariant congruence.

Proposition 6.7. *There exist sets Σ_1 and Σ_2 of identities such that* $\mathrm{D}(\Sigma_1) \subsetneqq \mathrm{SR}(\Sigma_1)$ *and* $\mathrm{SR}(\Sigma_2) \subsetneqq \Sigma\mathrm{R}(\Sigma_2)$.

Proof. First, let $\Sigma_1 = \{x_1(x_2x_3) \approx (x_1x_2)x_3\}$ be the set of identities which define the variety $SG = \mathrm{Mod}(\Sigma_1)$ of semigroups. Clearly, $\mathrm{Id}(SG) = \mathrm{D}(\Sigma_1)$. Let us set $t = ((x_1x_2)x_1)x_2$, $s = (x_1x_2)(x_1x_2)$, $r = x_1x_2$, and $u = x_3$. Clearly, $\Sigma_1 \models t \approx s$. Since $EP_r^t = \{11\}$ and $EP_r^s = \{1, 2\}$, we obtain $t(r * u) = (x_3x_1)x_2$

and $s(r*u) = x_3x_3$. Hence, $\Sigma_1 \not\models t(r*u) \approx s(r*u)$. Consequently, $\mathrm{D}(\Sigma_1)$ is a proper subset of $\mathrm{SR}(\Sigma_1)$ and $\mathrm{Mod}(\mathrm{SR}(\Sigma_1))$ is a proper subvariety of SG.

Second, let $\Sigma_2 = \{(x_1x_2)x_3 \approx x_2x_3\}$. Recall the terms t, s, and r considered in Proposition 6.1. It is easy to see that $EP_r^t = EP_r^s = \{22\}$. Thus, we have $t(r*u) = (x_3(x_1x_2))(x_2x_3)$, $s(r*u) = x_2(x_2x_3)$, so $\Sigma_2 \models t(r*u) \approx s(r*u)$, but $\Sigma_2 \not\models t^{\Sigma_2}(r \leftarrow u) \approx s^{\Sigma_2}(r \leftarrow u)$. □

Lemma 6.8. *Let $x_i \in X$ be a Σ-essential variable which occurs once in the term $t \in W_\tau(X)$. Then the variable x_i is Σ-essential in* $\mathrm{Red}(t)$ *with a unique occurrence.*

Proof. According to Theorem 4.1, it is enough to prove that $x_i \in X$ is Σ-essential in r with unique occurrence when $t \to_R r$. Corollary 3.8 of [7] and Corollary 4.3 imply $x_i \in \mathrm{Ess}(r, \Sigma)$.

Let $t \to_R r$, $r = t(p; u)$, $s = \mathrm{sub}_t(p)$, and $\Sigma \models s \approx u$, where u is Σ-minimal. Let q be the unique position on which x_i occurs in t. Since q is a position of a variable, it follows that $q \not\prec p$.

If $p \prec q$, then the unique occurrence of x_i in r follows by the Σ-minimality of u. If $p \not\prec q$, then x_i occurs once on the position $q \in \mathrm{Pos}(r)$ in r. □

Lemma 6.9. *If $\Sigma = \{x_1(x_2x_3) \approx x_ix_j\}$ with $1 \leqslant i \leqslant j \leqslant 3$, then*

$$\Sigma \models t(r*u) \approx \mathrm{Red}(t)^\Sigma(r \leftarrow u)$$

for all $t, r, u \in W_\tau(X)$.

Proof. Let n be a natural number such that $r, t, u \in W_\tau(X_n)$. If $EP_r^t = \emptyset$, then $P_r^t = \emptyset$ and we are done. Let $EP_r^t = \{p_1, \dots, p_m\}$ and let us put $s = t(p_1, \dots, p_m; x_{n+1} \cdots x_{n+m})$. Clearly, $x_{n+1} \cdots x_{n+m} \in \mathrm{Ess}(s, \Sigma)$ and x_{n+i} occurs only once in s for $i = 1, \dots, m$. From Lemma 6.8, it follows that $x_{n+1} \cdots x_{n+m} \in \mathrm{Ess}(\mathrm{Red}(s), \Sigma)$ and x_{n+i} occurs only once in $\mathrm{Red}(s)$ for $i = 1, \dots, m$. If we suppose that there is a term v such that $\Sigma \models r \approx v$ and $v \in \mathrm{Sub}(\mathrm{Red}(s))$, then there is $w \in \mathrm{Sub}(s)$ such that $\Sigma \models v \approx w$. Since $\mathrm{Ess}(v, \Sigma) \subseteq \mathrm{Ess}(\mathrm{Red}(s), \Sigma)$, it follows that $\mathrm{Ess}(v, \Sigma) \subseteq \mathrm{Ess}(s, \Sigma)$. Then from Theorem 2.13 of [8], it follows that $v \in EP_r^s \subseteq EP_r^t$, which is a contradiction. Hence, $\Sigma \not\models r \approx v$ for all $v \in \mathrm{Sub}(\mathrm{Red}(s))$. Consequently, $EP_r^s = P_r^s = \emptyset$ and we obtain $t(r*u) = s(x_{n+1} \leftarrow u, \dots, x_{n+m} \leftarrow u)$ and

$$\mathrm{Red}(s)^\Sigma(r \leftarrow u) = \mathrm{Red}(s)(x_{n+1} \leftarrow u, \dots, x_{n+m} \leftarrow u).$$

From Corollary 4.3, we have

$$\Sigma \models s(x_{n+1} \leftarrow u, \dots, x_{n+m} \leftarrow u) \approx \mathrm{Red}(s)(x_{n+1} \leftarrow u, \dots, x_{n+m} \leftarrow u),$$

which completes the proof. □

Lemma 6.10. *If $\Sigma = \{(x_1x_2)x_3 \approx x_ix_j\}$ with $1 \leqslant i \leqslant j \leqslant 3$, then the normal form under the reduction $\to_R$ of a term $t \in W_\tau(X)$ is presented in the following form:*

$$\mathrm{Red}(t) = x_{i_1}(x_{i_2}(\cdots(x_{i_{n-1}}x_{i_n})\cdots)), \tag{6.1}$$

where $x_{i_m} \in \mathrm{var}(t)$ for $m = 1, \dots, n$.

Proof. Let $\mathcal{V}$ be the variety defined by Σ, i.e., $\mathcal{V} = \mathrm{Mod}(\Sigma)$. We shall prove the lemma when $\Sigma \models (x_1x_2)x_3 \approx x_1x_2$. The other cases follow by similar arguments.

So, let us consider the term $t = (x_1x_2)x_3$. Then we have $2 \notin \mathrm{PEss}(t, \Sigma)$. Hence, $\mathrm{Red}(t) = x_1x_2$ and we are done.

Assume that if $\mathrm{Depth}(t) < k$, for some natural number k with $k > 2$, then $\mathrm{Red}(t)$ is presented in the form of (6.1).

Let $\mathrm{Depth}(t) = k$. Then we have $t = t_1t_2$ with $t_1, t_2 \in W_\tau(X)$ and with $0 \leqslant \mathrm{Depth}(t_i) < k$ for $i = 1, 2$. Clearly, $\Sigma \models \mathrm{Red}(t) \approx \mathrm{Red}(t_1)\,\mathrm{Red}(t_2)$. From the inductive assumption, we know that $\mathrm{Red}(t_1)$ and $\mathrm{Red}(t_2)$ are presented in the form of (6.1). If $\mathrm{Red}(t_1) = x_{i_1}$, then we are done. Let $\mathrm{Depth}(\mathrm{Red}(t_1)) \geqslant 1$ and $\mathrm{Red}(t_1) = x_{i_1}t_{12}$ for some $t_{12} \in W_\tau(X)$. Then

$$\Sigma \models \mathrm{Red}(t) \approx (x_{i_1}t_{12})\,\mathrm{Red}(t_2) \approx x_{i_1}t_{12} = \mathrm{Red}(t_1),$$

which completes the proof. □

By dual arguments one can prove the following lemma.

Lemma 6.11. *If $\Sigma = \{x_1(x_2x_3) \approx x_ix_j\}$ with $1 \leqslant i \leqslant j \leqslant 3$, then the normal form under the reduction $\to_R$ of a term $t \in W_\tau(X)$ is presented in the following form:*

$$\mathrm{Red}(t) = (\cdots((x_{i_1}x_{i_2})x_{i_3})\cdots)x_{i_n},$$

where $x_{i_m} \in \mathrm{var}(t)$ for $m = 1, \ldots, n$.

Theorem 6.12. *The varieties of semigroups V_{11}, V_{22}, and V_{33} are s-stable (see Proposition 3.3).*

Proof. We shall prove that V_{11} is an s-stable variety. To show that $\Sigma = \mathrm{Id}(V_{11})$ is SR-deductively closed, i.e., $\mathrm{SR}(\Sigma) = \Sigma$, we let r, s, t be three terms such that $t \approx s \in \Sigma$, $EP_r^t \neq \emptyset$, and $EP_r^s \neq \emptyset$. We have to prove

$$\Sigma \models t(r * u) \approx s(r * u). \tag{6.2}$$

If $\mathrm{Depth}(t) \leqslant 1$, then we have

$$\Sigma \models t \approx s \Longrightarrow t = s$$

and (6.2) is obviously satisfied.

Let $\mathrm{Depth}(t) \geqslant 2$ and $\mathrm{Depth}(s) \geqslant 2$. Since $x_1x_2x_3 \approx x_1x_1 \in \Sigma$, the set of Σ-essential positions in each term w consists of all strings over $\{1\}$ which belong to $\mathrm{Pos}(w)$, including the empty string ε. Consequently, for each term r, we have $EP_r^w = \emptyset$ or $EP_r^w = \{p_w\}$, where p_w is the longest string over $\{1\}$ in $\mathrm{Pos}(w)$.

Next, $EP_r^t \neq \emptyset$ and $EP_r^s \neq \emptyset$ imply $EP_r^t = \{p_t\}$ and $EP_r^s = \{p_s\}$. Since p_t and p_s are the longest strings in $\mathrm{Pos}(t)$ and $\mathrm{Pos}(s)$, respectively, it follows that r is a variable and $r = \mathrm{first}(t) = \mathrm{first}(s)$. Thus, (6.2) follows by D_5.

In a similar way, one can prove that V_{33} is an s-stable variety. The proof that V_{22} is an s-stable variety is left to the reader. □

Theorem 6.13. *The varieties of groupoids V_{lm}^{ijk} and W_{lm}^{ijk} for $i,j,k,l,m \in \{1,2,3\}$ are s-stable.*

Proof. If $i,j,k,l,m \in \{1,2\}$, we are done because of Theorem 5.5.

Claim 1: V_{lm}^{123} and W_{lm}^{123} with $1 \leqslant l \leqslant m \leqslant 3$ are s-stable varieties.

We are going to prove that $V_{12}^{123} = \mathrm{Mod}(\Sigma)$ is an s-stable variety, where $\Sigma = \{(x_1x_2)x_3 \approx x_1x_2\}$. Lemma 6.9 implies $t(r * u) = \mathrm{Red}(t)^{\Sigma}(\mathrm{Red}(r) \leftarrow u)$ and it is enough to prove

$$\Sigma \models \mathrm{Red}(t)^{\Sigma}(\mathrm{Red}(r) \leftarrow u) \approx \mathrm{Red}(s)^{\Sigma}(\mathrm{Red}(r) \leftarrow u) \tag{6.3}$$

when $\Sigma \models t \approx s$, $r \in \mathrm{SEss}(t,\Sigma) \cap \mathrm{SEss}(s,\Sigma)$ and $u \in W_\tau(X)$.

Let $t,s,r \in W_\tau(X)$ be terms with $\Sigma \models t \approx s$ and $r \in \mathrm{SEss}(t,\Sigma) \cap \mathrm{SEss}(s,\Sigma)$. Suppose with no loss of generality that $\mathrm{Depth}(t) \leqslant \mathrm{Depth}(s)$. We argue by induction on $\mathrm{Depth}(t)$. If $\Sigma \models t \approx r$ then (6.3) is obvious.

Assume that $\Sigma \not\models t \approx r$. Let $\mathrm{Depth}(t) = 1$. Then, without loss of generality, we can assume that $t = x_1x_2$. Hence, $\Sigma \models r \approx x_1$ or $\Sigma \models r \approx x_2$, and (6.3) follows from D_5.

Assume for some natural number $k \geqslant 2$ that if $\mathrm{Depth}(t) < k$, then (6.3) is satisfied. Let $\mathrm{Depth}(t) = k$. From Lemma 6.10, it follows that

$$\begin{aligned}\mathrm{Red}(t) &= x_{i_1}(x_{i_2}(\cdots(x_{i_{n-1}}x_{i_n})\cdots)), \text{ and}\\ \mathrm{Red}(s) &= x_{j_1}(x_{j_2}(\cdots(x_{j_{m-1}}x_{j_m})\cdots)),\end{aligned}$$

where $x_{i_l} \in \mathrm{var}(t)$ and $x_{j_k} \in \mathrm{var}(s)$ for $l = 1,\ldots,n$ and $k = 1,\ldots,m$. Clearly, $x_{i_1} = x_{j_1}$ because $\Sigma \models t \approx s$ and $1 \in \mathrm{PEss}(t,\Sigma) \cap \mathrm{PEss}(s,\Sigma)$.

If $\mathrm{Red}(r) = x_{i_1}$, then we are done because of D_5. If $\mathrm{Red}(r) \neq x_{i_1}$, then $r \in \mathrm{SEss}(t_2,\Sigma) \cap \mathrm{SEss}(s_2,\Sigma)$ where

$$t_2 = x_{i_2}(\cdots(x_{i_{n-1}}x_{i_n})\cdots) \quad \text{and} \quad s_2 = x_{j_2}(\cdots(x_{j_{m-1}}x_{j_m})\cdots).$$

Clearly, $\Sigma \models t_2 \approx s_2$ and we have

$$\begin{aligned}\mathrm{Red}(t)^{\Sigma}(\mathrm{Red}(r) \leftarrow u) &= x_{i_1}\,\mathrm{Red}(t_2)^{\Sigma}(\mathrm{Red}(r) \leftarrow u), \text{ and}\\ \mathrm{Red}(s)^{\Sigma}(\mathrm{Red}(r) \leftarrow u) &= x_{i_1}\,\mathrm{Red}(s_2)^{\Sigma}(\mathrm{Red}(r) \leftarrow u)\end{aligned}$$

for each $u \in W_\tau(X)$, which together with our inductive assumption proves (6.3).

Claim 2: V_{lm}^{123} and W_{lm}^{123} with $1 \leqslant m < l \leqslant 3$ are s-stable varieties.

We shall show that V_{31}^{123} is s-stable. Thus, we have

$$\begin{aligned}\Sigma \models x_1x_3 &\approx (x_3(x_4x_5))x_1 \approx (x_1x_2)(x_3(x_4x_5))\\ &\approx ((x_2x_6)x_1)(x_3(x_4x_5)) \approx (x_3(x_4x_5))(x_2x_6)\\ &\approx (x_2x_6)x_3 \approx x_3x_2.\end{aligned}$$

Hence, $\Sigma \models x_1x_3 \approx x_3x_2 \approx x_2x_4$. So, if $\mathrm{Depth}(t) \geqslant 1$, then without loss of generality, we can assume that $\Sigma \models t \approx \mathrm{Red}(t) = x_1x_2$ with $\mathrm{PEss}(\mathrm{Red}(t)) = \varepsilon$. Consequently, for each term r, we have $EP_r^t = \emptyset$.

This completes the proof of the theorem. $\square$

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Slavcho Shtrakov
Department of Computer Science, South-West University, 2700 Blagoevgrad, Bulgaria
e-mail: shtrakov@swu.bg
URL: http://shtrakov.swu.bg/

Jörg Koppitz
Institute of Mathematics, University of Potsdam, 14415 Potsdam
e-mail: koppitz@uni-potsdam.de