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RESEARCH ARTICLE

Partial automorphisms and injective partial endomorphisms of a finite undirected path

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Abstract
In this paper, we study partial automorphisms and, more generally, injective partial endomorphisms of a finite undirected path from Semigroup Theory perspective. Our main objective is to give formulas for the ranks of the monoids $\mathrm{IEnd}(P_n)$ and $\mathrm{PAut}(P_n)$ of all injective partial endomorphisms and of all partial automorphisms of the undirected path P_n with n vertices. We also describe Green's relations of $\mathrm{PAut}(P_n)$ and $\mathrm{IEnd}(P_n)$ and calculate their cardinals.

Keywords Injective partial endomorphisms · Partial automorphisms · Paths · Generators · Rank

Mathematics Subject Classification 05C38 · 20M10 · 20M20 · 05C25

Introduction and preliminaries

As well as automorphisms of graphs allow one to establish natural connections between Graph Theory and Group Theory, endomorphisms of graphs allow similar connections between Graph Theory and Semigroup Theory. Likewise, in particular, partial automorphisms of graphs relate Graph Theory with Inverse Semigroup Theory. This has led, over the last decades, many authors to become interested in the study of combinatorial and algebraic properties of monoids of endomorphisms of graphs. One of the most studied algebraic notions is *regularity*, in the sense of Semigroup Theory. A general solution to the problem, posed in 1987 by Knauer and Wilkeit, see [29], of

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which graphs have a regular monoid of endomorphisms has proved to be very difficult to obtain. Despite that, various authors studied and solved this question for some special classes of graphs (see, for instance, [7–9,16–18,20,21,25–28,31,32]).

The *rank* of a monoid S, denoted by rank S, is the least number of generators of S. In this paper, we focus our attention on this important notion of Semigroup Theory, which has been, in recent years, the subject of intensive research.

Let Ω be a finite set with at least 3 elements. It is well known that the symmetric group $\mathscr{S}(\Omega)$ of Ω has rank 2 (as a semigroup, a monoid or a group). Furthermore, the monoid of all transformations $\mathscr{T}(\Omega)$ of Ω, the monoid of all partial transformations $\mathscr{PT}(\Omega)$ of Ω and the symmetric inverse monoid $\mathscr{I}(\Omega)$ of Ω have ranks 3, 4, and 3, respectively. The survey [10] presents these results and similar ones for other classes of transformation monoids, in particular, for monoids of order-preserving transformations and for some of their extensions. More recently, for instance, the papers [1,2,5,11–15,23,33,34] are dedicated to the computation of the ranks of certain (classes of transformation) semigroups or monoids.

Now, let $G = (V, E)$ be a simple graph (i.e. an undirected graph without loops and without multiple edges). Let α be a partial transformation of V. Denote by $\operatorname{Dom}\alpha$ the domain of α and by $\operatorname{Im}\alpha$ the image of α. We say that α is:

– A *partial endomorphism* of G if $\{u, v\} \in E$ implies $\{u\alpha, v\alpha\} \in E$, for all $u, v \in \operatorname{Dom}\alpha$;
– A *partial automorphism* of G if α is an injective mapping (i.e. a partial permutation) and α and α^{-1} are both partial endomorphisms.

If α is a full mapping (i.e. $\alpha \in \mathscr{T}(V)$) then a partial endomorphism (respectively, partial automorphism) is just called *endomorphism* (respectively, *automorphism*).

Notice that, for finite graphs, any bijective endomorphism is an automorphism.

Denote by:

– $\operatorname{PEnd}(G)$ the set of all partial endomorphisms of G;
– $\operatorname{End}(G)$ the set of all endomorphisms of G;
– $\operatorname{IEnd}(G)$ the set of all injective partial endomorphisms of G;
– $\operatorname{PAut}(G)$ the set of all partial automorphisms of G;
– $\operatorname{Aut}(G)$ the set of all automorphisms of G.

Clearly, $\operatorname{PEnd}(G)$, $\operatorname{End}(G)$, $\operatorname{IEnd}(G)$, $\operatorname{PAut}(G)$, and $\operatorname{Aut}(G)$ are monoids under composition of maps with the identity mapping id as the identity element. Moreover, $\operatorname{Aut}(G)$ is also a group and $\operatorname{PAut}(G)$ is an inverse semigroup: $\operatorname{Aut}(G) \subseteq \mathscr{S}(V)$ and $\operatorname{PAut}(G) \subseteq \mathscr{I}(V)$. It is also clear that

$$\operatorname{Aut}(G) \subseteq \operatorname{End}(G) \subseteq \operatorname{PEnd}(G)$$

and

$$\operatorname{Aut}(G) \subseteq \operatorname{PAut}(G) \subseteq \operatorname{IEnd}(G) \subseteq \operatorname{PEnd}(G)$$

(these inclusions may not be strict).

Let $\mathbb{N}$ be the set of all positive integers and let $n \in \mathbb{N}$. Let P_n be the undirected path with n vertices. Notice that we may take

$$P_n = (\{1, \ldots, n\}, \{\{i, i+1\} \mid i = 1, \ldots, n-1\}).$$

The number of endomorphisms of P_n has been determined by Arworn [3] (see also the paper [30] by Michels and Knauer). In addition, several other combinatorial and algebraic properties of P_n were also studied in these two papers and also, for instance, in [4,19]. The authors in [6] studied several properties of the monoid $\mathrm{End}(P_n)$. In particular, they characterized regular elements and determined the rank of $\mathrm{End}(P_n)$.

The main objective of the present paper is to determine the ranks of the monoids $\mathrm{PAut}(P_n)$ and $\mathrm{IEnd}(P_n)$. We will show that

$$\operatorname{rank}\mathrm{PAut}(P_n) = \begin{cases} 2 & \text{for } n = 1 \\ 2 & \text{for } n = 2 \\ 3 & \text{for } n = 3 \\ n-1 & \text{for } n \geq 4 \end{cases} \quad \text{and} \quad \operatorname{rank}\mathrm{IEnd}(P_n) = \begin{cases} 2 & \text{for } n = 1 \\ 2 & \text{for } n = 2 \\ 4 & \text{for } n = 3 \\ n + \lceil \frac{n}{2} \rceil - 2 & \text{for } n \geq 4. \end{cases}$$

We also aim to describe Green's relations of $\mathrm{PAut}(P_n)$ and $\mathrm{IEnd}(P_n)$ and to calculate the cardinals of both monoids.

Observe that $\mathrm{PAut}(P_n)$ and $\mathrm{IEnd}(P_n)$ are submonoids of the symmetric inverse monoid $\mathscr{I}_n = \mathscr{I}(\{1, \ldots, n\})$.

Recall that the Green's relations $\mathscr{L}$, $\mathscr{R}$, and $\mathscr{J}$ of a monoid S are defined as following: for $\alpha, \beta \in S$,

- $\alpha \mathscr{L} \beta$ if and only if there exist $\gamma, \delta \in S$ such that $\alpha = \gamma\beta$ and $\beta = \delta\alpha$;
- $\alpha \mathscr{R} \beta$ if and only if there exist $\gamma', \delta' \in S$ such that $\alpha = \beta\gamma'$ and $\beta = \alpha\delta'$;
- $\alpha \mathscr{J} \beta$ if and only if there exist $\gamma, \gamma', \delta, \delta' \in S$ such that $\alpha = \gamma\beta\gamma'$ and $\beta = \delta\alpha\delta'$.

The relations $\mathscr{L}$ and $\mathscr{R}$ commute (i.e. $\mathscr{L} \circ \mathscr{R} = \mathscr{R} \circ \mathscr{L}$) and Green's relation $\mathscr{D}$ is defined by $\mathscr{D} = \mathscr{L} \circ \mathscr{R} = \mathscr{R} \circ \mathscr{L}$ (i.e. $\alpha \mathscr{D} \beta$ if and only if there exists $\sigma \in S$ such that $\alpha \mathscr{L} \sigma \mathscr{R} \beta$, for $\alpha, \beta \in S$). Notice that for a finite monoid the relations $\mathscr{J}$ and $\mathscr{D}$ coincide. Finally, we have Green's relation $\mathscr{H}$ defined by $\mathscr{H} = \mathscr{L} \cap \mathscr{R}$.

If S is an inverse semigroup of injective partial transformations on a given set, then the relations $\mathscr{L}$, $\mathscr{R}$, and $\mathscr{H}$ can be described as following: for $\alpha, \beta \in S$,

- $\alpha \mathscr{L} \beta$ if and only if $\mathrm{Im}\,\alpha = \mathrm{Im}\,\beta$;
- $\alpha \mathscr{R} \beta$ if and only if $\mathrm{Dom}\,\alpha = \mathrm{Dom}\,\beta$;
- $\alpha \mathscr{H} \beta$ if and only if $\mathrm{Im}\,\alpha = \mathrm{Im}\,\beta$ and $\mathrm{Dom}\,\alpha = \mathrm{Dom}\,\beta$.

Since $\mathrm{PAut}(P_n)$ is an inverse semigroup, it remains to obtain a description of its Green's relation $\mathscr{J}$. On the other hand, that is not the situation of $\mathrm{IEnd}(P_n)$, for $n \geq 3$, since $\mathrm{IEnd}(P_n)$ is not an inverse semigroup (for instance, $\begin{pmatrix} 1 & 3 \\ 1 & 2 \end{pmatrix} \in \mathrm{IEnd}(P_n)$ is not a regular element of $\mathrm{IEnd}(P_n)$). Notice that $\mathrm{IEnd}(P_n) = \mathrm{PAut}(P_n)$, for $n = 1, 2$.

For general background on Semigroup Theory and standard notation, we refer the reader to Howie's book [22]. Regarding Algebraic Graph Theory, our main reference is Knauer's book [24].

1 Green's relations

Let $n \in \mathbb{N}$. We now describe Green's relations $\mathscr{L}$, $\mathscr{R}$, $\mathscr{H}$, and $\mathscr{J}$ of the monoid $\mathrm{IEnd}(P_n)$ as well as Green's relation $\mathscr{J}$ of the inverse monoid $\mathrm{PAut}(P_n)$.

In this section, for a set $X \subseteq \mathbb{N}$, we need the following concept. A set $I \subseteq X$ is called a *maximal interval* of X if I satisfies the following properties:

- I is an interval of X (i.e. $x, y \in I$ and $z \in \mathbb{N}$ with $x < z < y$ implies $z \in I$);
- If $J \subseteq X$ is an interval of X then $I \subseteq J$ implies $I = J$.

Recall that a partial transformation α of $\{1, \ldots, n\}$ is said to be *order-preserving* (respectively, *order-reversing*) if $x < y$ implies $x\alpha \leq y\alpha$ (respectively, if $x < y$ implies $x\alpha \geq y\alpha$), for all $x, y \in \mathrm{Dom}\,\alpha$.

Let $\alpha \in \mathscr{I}_n$. The following observations are easy to show:

- $\alpha \in \mathrm{IEnd}(P_n)$ if and only if for each interval I of $\mathrm{Dom}\,\alpha$ the image $I\alpha$ is an interval of $\mathrm{Im}\,\alpha$;
- If $\alpha \in \mathrm{IEnd}(P_n)$ then α is order-preserving or order-reversing in I (i.e. the restriction $\alpha|_I$ of α to I is an order-preserving or order-reversing transformation), for each interval I of $\mathrm{Dom}\,\alpha$;
- If for each maximal interval I of $\mathrm{Dom}\,\alpha$ the image $I\alpha$ is an interval of $\mathrm{Im}\,\alpha$ and α is order-preserving or order-reversing in I then $\alpha \in \mathrm{IEnd}(P_n)$;
- If $\alpha \in \mathrm{PAut}(P_n)$ and I is a maximal interval of $\mathrm{Dom}\,\alpha$ then the image $I\alpha$ is a maximal interval of $\mathrm{Im}\,\alpha$;
- If for each maximal interval I of $\mathrm{Dom}\,\alpha$ the image $I\alpha$ is a maximal interval of $\mathrm{Im}\,\alpha$ and α is order-preserving or order-reversing in I then $\alpha \in \mathrm{PAut}(P_n)$.

Let $\alpha \in \mathscr{I}_n$. Let $\{X_1, \ldots, X_k\}$ be a partition of $\mathrm{Dom}\,\alpha$. We will use the notation $\alpha = \begin{pmatrix} X_1 & \cdots & X_k \\ Y_1 & \cdots & Y_k \end{pmatrix}$ to express that $Y_i = (X_i)\alpha$, for $i \in \{1, \ldots, k\}$.

Let $\alpha, \beta \in \mathrm{IEnd}(P_n)$. Since $\mathrm{IEnd}(P_n)$ is a submonoid of the inverse monoid $\mathscr{I}_n$, if $\alpha \mathscr{L} \beta$ (respectively, $\alpha \mathscr{R} \beta$) in $\mathrm{IEnd}(P_n)$ then $\alpha \mathscr{L} \beta$ (respectively, $\alpha \mathscr{R} \beta$) in $\mathscr{I}_n$, whence $\mathrm{Im}\,\alpha = \mathrm{Im}\,\beta$ (respectively, $\mathrm{Dom}\,\alpha = \mathrm{Dom}\,\beta$). Moreover, we have the following descriptions of the relations $\mathscr{L}$ and $\mathscr{R}$ in $\mathrm{IEnd}(P_n)$:

Proposition 1 *Let* $\alpha, \beta \in \mathrm{IEnd}(P_n)$ *and let* $\{I_1, I_2, \ldots, I_k\}$ *and* $\{I'_1, I'_2, \ldots, I'_l\}$ *be the (partitions into) maximal intervals of* $\mathrm{Dom}\,\alpha$ *and in* $\mathrm{Dom}\,\beta$*, respectively. Then, the following three conditions are equivalent:*

1. $\alpha \mathscr{L} \beta$;
2. $\{I_1\alpha, I_2\alpha, \ldots, I_k\alpha\} = \{I'_1\beta, I'_2\beta, \ldots, I'_l\beta\}$;
3. $\mathrm{Im}\,\alpha = \mathrm{Im}\,\beta$ *and* $\alpha\beta^{-1} \in \mathrm{PAut}(P_n)$.

Proof [$1 \Rightarrow 2$] Suppose that $\alpha \mathscr{L} \beta$. Then, by the definition of Green's relation $\mathscr{L}$, there exist $\gamma, \delta \in \mathrm{IEnd}(P_n)$ such that $\alpha = \gamma\beta$ and $\beta = \delta\alpha$. Let $i \in \{1, \ldots, k\}$. Since $\mathrm{Dom}\,\alpha \subseteq \mathrm{Dom}\,\gamma$, I_i is also an interval of $\mathrm{Dom}\,\gamma$, whence $I_i\gamma$ is an interval of $\mathrm{Dom}\,\beta$ and so $I_i\gamma \subseteq I'_j$, for some $j \in \{1, \ldots, l\}$. It follows that $I_i\alpha = I_i\gamma\beta \subseteq I'_j\beta$, for some $j \in \{1, \ldots, l\}$. Similarly, we may show that, for all $j \in \{1, \ldots, l\}$, there exists $i \in \{1, \ldots, k\}$ such that $I'_j\beta \subseteq I_i\alpha$. Now, since $\mathrm{Im}\,\alpha = \mathrm{Im}\,\beta$, we may deduce that $\{I_1\alpha, I_2\alpha, \ldots, I_k\alpha\} = \{I'_1\beta, I'_2\beta, \ldots, I'_l\beta\}$.

$[2 \Rightarrow 3]$ From $\{I_1\alpha, I_2\alpha, \dots, I_k\alpha\} = \{I'_1\beta, I'_2\beta, \dots, I'_l\beta\}$ it follows immediately that $k = l$ and $\operatorname{Im}\alpha = \operatorname{Im}\beta$. Let σ be the permutation of $\{1, \dots, k\}$ such that $I_i\alpha = I'_{i\sigma}\beta$, for all $i \in \{1, \dots, k\}$. Then $\alpha\beta^{-1} = \begin{pmatrix} I_1 & I_2 & \cdots & I_k \\ I'_{1\sigma} & I'_{2\sigma} & \cdots & I'_{k\sigma} \end{pmatrix}$ and so $\alpha\beta^{-1}$ maps maximal intervals of its domain into maximal intervals of its image. Hence, in order to prove that $\alpha\beta^{-1} \in \mathrm{PAut}(P_n)$, it suffices to show that $\alpha\beta^{-1}$ is order-preserving or order-reversing in I_i, for $i \in \{1, \dots, k\}$. Let $i \in \{1, \dots, k\}$. Then, we have $\alpha\beta^{-1}|_{I_i} = \alpha|_{I_i}\beta^{-1}|_{I_i\alpha} = \alpha|_{I_i}\beta^{-1}|_{I'_{i\sigma}\beta}$ and $\beta^{-1}|_{I'_{i\sigma}\beta} = \begin{pmatrix} I'_{i\sigma}\beta \\ I'_{i\sigma} \end{pmatrix}$. As I_i is an interval, $\alpha|_{I_i}$ is order-preserving or order-reversing. On the other hand, as $I'_{i\sigma}$ is an interval, $\beta|_{I'_{i\sigma}}$ is order-preserving or order-reversing and so its inverse mapping $\beta^{-1}|_{I'_{i\sigma}\beta}$ is also order-preserving or order-reversing. Thus, $\alpha\beta^{-1}$ is order-preserving or order-reversing in I_i, as required.

$[3 \Rightarrow 1]$ From $\operatorname{Im}\alpha = \operatorname{Im}\beta$ and $\alpha\beta^{-1} \in \mathrm{PAut}(P_n)$, it follows that $\alpha\beta^{-1}$ and $\beta\alpha^{-1} = (\alpha\beta^{-1})^{-1}$ lie in $\mathrm{IEnd}(P_n)$, $(\alpha\beta^{-1})\beta = \alpha(\beta^{-1}\beta) = \alpha\,\mathrm{id}|_{\operatorname{Im}\beta} = \alpha\,\mathrm{id}|_{\operatorname{Im}\alpha} = \alpha$ and $(\alpha\beta^{-1})^{-1}\alpha = (\beta\alpha^{-1})\alpha = \beta(\alpha^{-1}\alpha) = \beta\,\mathrm{id}|_{\operatorname{Im}\alpha} = \beta\,\mathrm{id}|_{\operatorname{Im}\beta} = \beta$, whence $\alpha\mathscr{L}\beta$. $\square$

Proposition 2 *Let* $\alpha, \beta \in \mathrm{IEnd}(P_n)$. *Then* $\alpha\mathscr{R}\beta$ *if and only if* $\operatorname{Dom}\alpha = \operatorname{Dom}\beta$ *and* $\alpha^{-1}\beta \in \mathrm{PAut}(P_n)$.

Proof Suppose that $\alpha\mathscr{R}\beta$. Then $\operatorname{Dom}\alpha = \operatorname{Dom}\beta$. Moreover, there exist transformations $\gamma, \delta \in \mathrm{IEnd}(P_n)$ such that $\beta = \alpha\gamma$ and $\alpha = \beta\delta$. Then, we have $\alpha^{-1}\beta = \alpha^{-1}\alpha\gamma = \mathrm{id}|_{\operatorname{Im}\alpha}\gamma = \gamma|_{\operatorname{Im}\alpha}$ and $(\alpha^{-1}\beta)^{-1} = \beta^{-1}\alpha = \beta^{-1}\beta\delta = \mathrm{id}|_{\operatorname{Im}\beta}\delta = \delta|_{\operatorname{Im}\beta}$. Since, clearly, any restriction of a transformation of $\mathrm{IEnd}(P_n)$ is still a transformation of $\mathrm{IEnd}(P_n)$, we have $\alpha^{-1}\beta, \beta^{-1}\alpha \in \mathrm{IEnd}(P_n)$ and so $\alpha^{-1}\beta \in \mathrm{PAut}(P_n)$.

Conversely, admit that $\operatorname{Dom}\alpha = \operatorname{Dom}\beta$ and $\alpha^{-1}\beta \in \mathrm{PAut}(P_n)$. Then $\alpha^{-1}\beta$, $\beta^{-1}\alpha = (\alpha^{-1}\beta)^{-1} \in \mathrm{IEnd}(P_n)$, $\beta = \mathrm{id}|_{\operatorname{Dom}\beta}\beta = \mathrm{id}|_{\operatorname{Dom}\alpha}\beta = (\alpha\alpha^{-1})\beta = \alpha(\alpha^{-1}\beta)$ and $\alpha = \mathrm{id}|_{\operatorname{Dom}\alpha}\alpha = \mathrm{id}|_{\operatorname{Dom}\beta}\alpha = (\beta\beta^{-1})\alpha = \beta(\beta^{-1}\alpha)$, whence $\alpha\mathscr{R}\beta$, as required. $\square$

Since $\mathscr{H} = \mathscr{R} \cap \mathscr{L}$, it follows immediately that:

Corollary 1 *Let* $\alpha, \beta \in \mathrm{IEnd}(P_n)$. *Then* $\alpha\mathscr{H}\beta$ *if and only if* $\operatorname{Dom}\alpha = \operatorname{Dom}\beta$, $\operatorname{Im}\alpha = \operatorname{Im}\beta$ *and* $\alpha^{-1}\beta, \alpha\beta^{-1} \in \mathrm{PAut}(P_n)$.

Before presenting our descriptions of Green's relation $\mathscr{J}$ on $\mathrm{IEnd}(P_n)$ and on $\mathrm{PAut}(P_n)$, we need to introduce some notions and notations.

For $A, B \subseteq \mathbb{N}$, denote by $A < B$ if $a < b$ for all $a \in A$ and $b \in B$.

Let $a = (a_1, \dots, a_p)$ be a sequence of elements of $\mathbb{N}$. We define the *reverse of* a as being the sequence $a^R = (a_p, \dots, a_1)$.

Let $\alpha \in \mathrm{IEnd}(P_n)$ and let J be a maximal interval of $\operatorname{Im}\alpha$. Define the *type of* J to be the sequence $\tau_\alpha(J) = (|I_1|, |I_2|, \dots, |I_p|)$, where $\{I_1, I_2, \dots, I_p\}$ are the maximal intervals of $J\alpha^{-1}$ such that $I_i\alpha < I_{i+1}\alpha$, for $1 \le i < p$.

Now, let $\alpha, \beta \in \mathrm{IEnd}(P_n)$. We say that α and β have *similar type* if there exists a bijection σ from the set of maximal intervals of $\operatorname{Im}\alpha$ into the set of maximal intervals of $\operatorname{Im}\beta$ such that $\tau_\alpha(J) \in \{\tau_\beta(J\sigma), \tau_\beta(J\sigma)^R\}$, for any maximal interval J of $\operatorname{Im}\alpha$.

Observe that two elements α and β of $\mathrm{IEnd}(P_n)$ have similar type if and only if they have maximal intervals of their images with the same type up to reversion and the same number of occurrences.

Lemma 1 *Let $\alpha, \beta \in \mathrm{IEnd}(P_n)$ be such that α and β have similar type. Then, there exist $\gamma, \delta \in \mathrm{PAut}(P_n)$ such that $\beta = \gamma\alpha\delta$ and $\alpha = \gamma^{-1}\beta\delta^{-1}$.*

Proof Let $\{J_1, J_2, \ldots, J_k\}$ and $\{J'_1, J'_2, \ldots, J'_k\}$ be the maximal intervals of $\mathrm{Im}\,\alpha$ and $\mathrm{Im}\,\beta$, respectively. Then there exist a permutation σ of $\{1, \ldots, k\}$ such that $\tau_\beta(J'_r) \in \{\tau_\alpha(J_{r\sigma}), \tau_\alpha(J_{r\sigma})^R\}$, for $r = 1, \ldots, k$.

For $1 \le r \le k$, let $\{I'_{r,1}, I'_{r,2}, \ldots, I'_{r,p_r}\}$ and $\{I_{r\sigma,1}, I_{r\sigma,2}, \ldots, I_{r\sigma,p_r}\}$ be the maximal intervals of $J'_r\beta^{-1}$ and $J_{r\sigma}\alpha^{-1}$, respectively, such that $I'_{r,i}\beta < I'_{r,i+1}\beta$ and $I_{r\sigma,i}\alpha < I_{r\sigma,i+1}\alpha$, for all $1 \le i < p_r$. Moreover, let $J'_{r,i} = I'_{r,i}\beta$ and $J_{r\sigma,i} = I_{r\sigma,i}\alpha$, for $r = 1, \ldots, k$ and $i = 1, \ldots, p_r$. Clearly, $J'_r = J'_{r,1} \cup J'_{r,2} \cup \cdots \cup J'_{r,p_r}$ and $J_{r\sigma} = J_{r\sigma,1} \cup J_{r\sigma,2} \cup \cdots \cup J_{r\sigma,p_r}$.

Let $r = 1, \ldots, k$. We define partial transformations γ_r and δ_r as following:

- $\mathrm{Dom}\,\gamma_r = \cup\{I'_{r,1}, I'_{r,2}, \ldots, I'_{r,p_r}\} = J'_r\beta^{-1}$;
- $\mathrm{Dom}\,\delta_r = \cup\{J_{r\sigma,1}, J_{r\sigma,2}, \ldots, J_{r\sigma,p_r}\} = J_{r\sigma}$;
- $I'_{r,i}\gamma_r = \begin{cases} I_{r\sigma,i} & \text{if } \tau_\beta(J'_r) = \tau_\alpha(J_{r\sigma}) \\ I_{r\sigma,p_r-i+1} & \text{if } \tau_\beta(J'_r) = \tau_\alpha(J_{r\sigma})^R, \end{cases}$
  for $i = 1, \ldots, p_r$;
- $J_{r\sigma,i}\delta_r = \begin{cases} J'_{r,i} & \text{if } \tau_\beta(J'_r) = \tau_\alpha(J_{r\sigma}) \\ J'_{r,p_r-i+1} & \text{if } \tau_\beta(J'_r) = \tau_\alpha(J_{r\sigma})^R, \end{cases}$
  for $i = 1, \ldots, p_r$;
- $\gamma_r|_{I'_{r,i}}$ is $\begin{cases} \text{order-preserving if (a) or (b) is satisfied} \\ \text{order-reversing} \quad \text{otherwise,} \end{cases}$
  where

  (a) $\tau_\beta(J'_r) = \tau_\alpha(J_{r\sigma})$ and $\alpha|_{I_{r\sigma,i}}$ and $\beta|_{I'_{r,i}}$ are both order-preserving or both order-reversing, and
  (b) $\tau_\beta(J'_r) = \tau_\alpha(J_{r\sigma})^R$ and $\alpha|_{I_{r\sigma,p_r-i+1}}$ is order-preserving and $\beta|_{I'_{r,i}}$ is order-reversing or vice versa,

  for $i = 1, \ldots, p_r$;
- $\delta_r|_{J_{r\sigma}}$ is $\begin{cases} \text{order-preserving if } \tau_\beta(J'_r) = \tau_\alpha(J_{r\sigma}) \\ \text{order-reversing} \quad \text{if } \tau_\beta(J'_r) = \tau_\alpha(J_{r\sigma})^R. \end{cases}$

It is easy to verify that both γ_r and δ_r are well defined. Then, we define partial transformations γ and δ as follows:

- $\mathrm{Dom}\,\gamma = \cup\{I'_{1,1}, \ldots, I'_{1,p_1}, \ldots, I'_{k,1}, \ldots, I'_{k,p_k}\} = \mathrm{Dom}\,\beta$;
- $\mathrm{Dom}\,\delta = \cup\{J_{1\sigma,1}, \ldots, J_{1\sigma,p_1}, \ldots, J_{k\sigma,1}, \ldots, J_{k\sigma,p_k}\} = \mathrm{Im}\,\alpha$;
- $\gamma|_{I'_{r,s}} = \gamma_r|_{I'_{r,s}}$ for $r = 1, \ldots, k$ and $s = 1, \ldots, p_r$;
- $\delta|_{J_{r\sigma,s}} = \delta_r|_{J_{r\sigma,s}}$ for $r = 1, \ldots, k$ and $s = 1, \ldots, p_r$.

Clearly, both transformations γ and δ are partial automorphisms. Let $r = 1, \ldots, k$ and $s = 1, \ldots, p_r$. Then

$$I'_{r,s}\gamma\alpha\delta = \begin{cases} I_{r\sigma,s}\alpha\delta = & J_{r\sigma,s}\delta = J'_{r,s} = I'_{r,s}\beta \text{if } \tau_\beta\left(J'_r\right) = \tau_\alpha\left(J_{r\sigma}\right) \\ I_{r\sigma,p_r-s+1}\alpha\delta = & J_{r\sigma,p_r-s+1}\delta = J'_{r,s} = I'_{r,s}\beta \text{if } \tau_\beta\left(J'_r\right) = \tau_\alpha\left(J_{r\sigma}\right)^R. \end{cases}$$

Taking into account properties (a) and (b), we can deduce that $\gamma\alpha\delta|_{I'_{r,s}}$ is order-preserving if $\beta|_{I'_{r,s}}$ is order-preserving and $\gamma\alpha\delta|_{I'_{r,s}}$ is order-reversing if $\beta|_{I'_{r,s}}$ is order-reversing, which allows us to conclude that $\beta = \gamma\alpha\delta$. On the other hand, since $\operatorname{Im}\gamma = \operatorname{Dom}\alpha$ and $\operatorname{Im}\alpha = \operatorname{Dom}\delta$, we obtain $\gamma^{-1}\gamma\alpha\delta\delta^{-1} = \mathrm{id}\,|_{\operatorname{Dom}\alpha}\alpha\,\mathrm{id}\,|_{\operatorname{Im}\alpha} = \alpha$ and so we also have $\alpha = \gamma^{-1}\beta\delta^{-1}$, as required. □

Now, we can describe Green's relation $\mathscr{J}$ for the monoid $\mathrm{IEnd}(P_n)$.

Proposition 3 *Let $\alpha, \beta \in \mathrm{IEnd}(P_n)$. Then $\alpha \mathscr{J} \beta$ if and only if α and β have similar type.*

Proof Let $\alpha, \beta \in \mathrm{IEnd}(P_n)$ be such that $\alpha \mathscr{J} \beta$. Then, there exists $\gamma \in \mathrm{IEnd}(P_n)$ such that $\alpha \mathscr{L} \gamma \mathscr{R} \beta$ and so, by Propositions 1 and 2 , we have $\operatorname{Dom}\gamma = \operatorname{Dom}\beta$ and $\operatorname{Im}\gamma = \operatorname{Im}\alpha$ and $\alpha\gamma^{-1}, \gamma^{-1}\beta \in \mathrm{PAut}(P_n)$. In addition, $\alpha^{-1}\alpha\gamma^{-1} = \gamma^{-1} = \gamma^{-1}\beta\beta^{-1}$. Moreover, $\operatorname{Dom}(\gamma^{-1}\beta) = \operatorname{Im}\alpha$ and $\operatorname{Im}(\gamma^{-1}\beta) = \operatorname{Im}\beta$. Hence, $\gamma^{-1}\beta \in \mathrm{PAut}(P_n)$ maps each maximal interval J of $\operatorname{Im}\alpha$ into a maximal interval $J\gamma^{-1}\beta$ of $\operatorname{Im}\beta$, thus defining a bijection σ ($J \mapsto J\sigma = J\gamma^{-1}\beta$) from the set of maximal intervals of $\operatorname{Im}\alpha$ into the set of maximal intervals of $\operatorname{Im}\beta$. Let J be a maximal interval of $\operatorname{Im}\alpha$. Then $(J\sigma)\beta^{-1} = J\gamma^{-1}\beta\beta^{-1} = J\gamma^{-1} = J\alpha^{-1}\alpha\gamma^{-1} = (J\alpha^{-1})\alpha\gamma^{-1}$. Since $\alpha\gamma^{-1} \in \mathrm{PAut}(P_n)$, we may deduce that $\tau_\alpha(J) \in \{\tau_\beta(J\sigma), \tau_\beta(J\sigma)^R\}$. Therefore α and β have similar type.

Conversely, let $\alpha, \beta \in \mathrm{IEnd}(P_n)$ be such that α and β have similar type. Then, by Lemma 1, we have directly $\alpha \mathscr{J} \beta$, as required. □

We finish this section with the description of Green's relation $\mathscr{J}$ of $\mathrm{PAut}(P_n)$, which follows immediately from Lemma 1 and Proposition 3.

Corollary 2 *Let $\alpha, \beta \in \mathrm{PAut}(P_n)$. Then $\alpha \mathscr{J} \beta$ if and only if α and β have similar type.*

Observe that the type of a maximal interval of the image of an element of $\mathrm{PAut}(P_n)$ is always a unitary sequence which we can identify with the size of the interval taken. Therefore, two elements α and β of $\mathrm{PAut}(P_n)$ have similar type if and only if they have maximal intervals of their images with the same size and with the same number of occurrences.

2 Cardinality

Let $n \in \mathbb{N}$ and $\overline{n} = \{1, \ldots, n\}$. We will determine the cardinality of $\mathrm{PAut}(P_n)$ as well as of $\mathrm{IEnd}(P_n)$. For this, we need some technical notations.

Let $A \in \{0, 1\}^n$ and let $A(p)$ denotes the element on the position p in A. Further, let $A(0) = A(n + 1) = 0$.

Let $R_A = \{p \in \overline{n} \mid A(p - 1) = 0 \text{ and } A(p) = 1\}$ and $r_A = |R_A|$.

Let $s_A = \sum\limits_{p=1}^{n} A(p)$.

Let $z(1) = 1$, $z(2) = r_A$ and

$$q_{A,i} = \begin{cases} \dbinom{n - s_A + z(i)}{r_A} & \text{if } A(p) \neq 0, \text{ for some } p \in \overline{n} \\ 1 & \text{otherwise,} \end{cases}$$

for $i \in \{1, 2\}$.

Let $t_{A,i} = (r_A!)q_{A,i}$, for $i \in \{1, 2\}$ and let $T_A = |\{p \in R_A \mid A(p+1) = 1\}|$.

Theorem 1 *One has*

$$|\mathrm{PAut}\,(P_n)| = \sum_{A\in\{0,1\}^n} 2^{T_A} t_{A,1} \quad \textit{and} \quad |\mathrm{IEnd}\,(P_n)| = \sum_{A\in\{0,1\}^n} 2^{T_A} t_{A,2}.$$

Proof The domain of an injective endomorphism on P_n is a subset of $\overline{n}$. For each $A \in \{0, 1\}^n$, let A^* be the subset of $\overline{n}$ with $x \in A^*$ if and only if $A(x) = 1$. In particular, by $A \longmapsto A^*$, a bijection between $\{0, 1\}^n$ and the powerset of $\overline{n}$, i.e. between $\{0, 1\}^n$ and the possible domains of injective endomorphisms on P_n, is given. Let $A \in \{0, 1\}^n$.

First, we suppose that $A \neq (0, 0, \ldots, 0)$. Then A^* consists of r_A maximal intervals $A_1 < A_2 < \cdots < A_{r_A}$ of A^*. For $i \in \{1, \ldots, r_A\}$, let p_i be the minimal element in the set A_i. So, we have $A(p_i - 1) = 0$ and $A(p_i) = 1$, for $i \in \{1, \ldots, r_A\}$. This provides $R_A = \{p_i \mid i \in \{1, \ldots, r_A\}\}$. Moreover, we have $s_A = |A^*|$.

An injective endomorphism on P_n with domain A^* has the form

$$\begin{pmatrix} A_1 & A_2 & \cdots & A_{r_A} \\ B_1 & B_2 & \cdots & B_{r_A} \end{pmatrix},$$

where $B_1, \ldots, B_{r_A}$ are intervals. We observe that for each permutation σ on $\{1, \ldots, r_A\}$, there is a possible image sequence $B_1, \ldots, B_{r_A}$ such that $B_{1\sigma} < B_{2\sigma} < \cdots < B_{r_A\sigma}$, i.e. there are $r_A!$ possibilities in which the intervals $B_1, \ldots, B_{r_A}$ are ordered. If the image sequence $B_1, \ldots, B_{r_A}$ is ordered by $B_{1\sigma} < B_{2\sigma} < \cdots < B_{r_A\sigma}$, for some permutation σ on $\{1, \ldots, r_A\}$, then there are still $n - s_A$ elements being not in the image of an injective endomorphism. If we restricted us to partial automorphisms then there are $b_1, \ldots, b_{r_A-1} \in \overline{n}$ such that $B_{1\sigma} < b_1 < B_{2\sigma} < b_2 < \cdots < b_{r_A-1} < B_{r_A\sigma}$ and so there are still $n - s_A - r_A + 1$ elements being not in the image of a partial automorphism. These remaining elements can be distributed before or after the B_i's, i.e. at $r_A + 1$ places. The number of all these possibilities is

$$\binom{(r_A+1) + (n - s_A) - 1}{n - s_A} = \binom{r_A + n - s_A}{n - s_A} = \binom{r_A + n - s_A}{r_A} = q_{A,2}$$

for injective endomorphism and

$$\binom{(r_A+1) + (n - s_A - r_A + 1) - 1}{n - s_A - r_A + 1} = \binom{n - s_A + 1}{n - s_A - r_A + 1} = \binom{n - s_A + 1}{r_A} = q_{A,1}$$

if we only consider partial automorphisms. In other words, we have $q_{A,2}(r_A!) = t_{A,2}$ and $q_{A,1}(r_A!) = t_{A,1}$ possibilities for the intervals $B_1, \ldots, B_{r_A}$, whenever A^* (with the partition $A_1 < \cdots < A_{r_A}$) is the domain of an injective endomorphism and of a partial automorphism, respectively. For $i \in \{1, \ldots, r_A\}$, if $|A_i| \geq 2$ then we have to consider two cases, namely $\begin{pmatrix} A_i \\ B_i \end{pmatrix}$ is order-preserving or order-reversing. In order to realize it, we consider the cardinality T_A of the set $D_A = \{i \in \{1, \ldots, r_A\} \mid |A_i| \geq 2\}$, i.e. $T_A = |D_A|$. So, we have still to consider 2^{T_A} possibilities, whenever the intervals $B_1, \ldots, B_{r_A}$ are already fixed. Observe that $D_A = \{p \in R_A \mid A(p+1) = 1\}$.

Thus, there are $2^{T_A} t_{A,2}$ injective endomorphisms and $2^{T_A} t_{A,1}$ partial automorphisms on P_n with domain A^*.

Next, suppose that $A = (0, 0, \ldots, 0)$. Then, there exists exactly one injective endomorphism on P_n with the domain $A^* = \emptyset$, namely the empty transformation. In this case, we have $q_{A,1} = q_{A,2} = 1$ and $r_A = T_A = 0$. Hence, $t_{A,1} = t_{A,2} = q_{A,1}(r_A!) = 1(0!) = 1$, $2^{T_A} = 2^0 = 1$ and $2^{T_A} t_{A,1} = 2^{T_A} t_{A,2} = 1$.

We conclude that $|\mathrm{PAut}(P_n)| = \sum_{A \in \{0,1\}^n} 2^{T_A} t_{A,1}$ and $|\mathrm{IEnd}(P_n)| = \sum_{A \in \{0,1\}^n} 2^{T_A} t_{A,2}$, as required. $\square$

3 Generators and rank

In this section we present the main results of this paper. We are referring to the calculation of the ranks of $\mathrm{PAut}(P_n)$ and $\mathrm{IEnd}(P_n)$. In both cases, we proceed by determining a generating set of minimal size.

It is clear that $\mathrm{PAut}(P_1) = \{\mathrm{id}, \emptyset\}$ is a generating set of minimal size of $\mathrm{PAut}(P_1) = \mathrm{IEnd}(P_1)$, where $\emptyset$ is the empty transformation. Moreover, it is easy to verify that

$$\mathscr{G} = \left\{ \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}$$

is a generating set of minimal size of

$$\mathrm{PAut}(P_2) = \mathrm{IEnd}(P_2) = \left\{ \mathrm{id}, \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 2 \end{pmatrix}, \emptyset \right\}.$$

This shows that

$$\operatorname{rank} \mathrm{PAut}(P_1) = \operatorname{rank} \mathrm{IEnd}(P_1) = \operatorname{rank} \mathrm{PAut}(P_2) = \operatorname{rank} \mathrm{IEnd}(P_2) = 2.$$

Next, let $n \geq 3$ and define

$$\tau = \begin{pmatrix} 1 & 2 & \cdots & n-1 & n \\ n & n-1 & \cdots & 2 & 1 \end{pmatrix}$$

and

$$\alpha_i = \begin{pmatrix} 1 & 2 & \cdots & i-1 & i+1 & i+2 & \cdots & n-1 & n \\ 1 & 2 & \cdots & i-1 & n & n-1 & \cdots & i+2 & i+1 \end{pmatrix},$$

for $i = 1, 2, \ldots, n$.

Let

$$\mathscr{A} = \begin{cases} \{\tau, \alpha_1, \alpha_2\} & \text{if } n = 3 \\ \{\tau\} \cup \{\alpha_i \mid i = 1, 2, \ldots, n-2\} & \text{if } n \geq 4. \end{cases}$$

First, we will show that $\mathscr{A}$ is a generating set of $\mathrm{PAut}(P_n)$. To accomplish this aim we start by proving a series of lemmas.

Lemma 2 *One has* $\{\alpha_i \mid i = n-1, n\} \subseteq \langle \mathscr{A} \rangle$.

Proof The proof follows immediately from the relations $\alpha_i = \tau \alpha^2_{n-i+1} \tau$, for $i = n-1, n$. □

Let

$$\alpha_i^* = \begin{pmatrix} 1 & 2 & \cdots & i-2 & i-1 & i+1 & \cdots & n-1 & n \\ i-1 & i-2 & \cdots & 2 & 1 & i+1 & \cdots & n-1 & n \end{pmatrix},$$

for $i = 1, 2, \ldots, n$.

Lemma 3 *One has* $\alpha_i^* \in \langle \mathscr{A} \rangle$, *for* $i = 1, 2, \ldots, n$.

Proof We have $\alpha_i^* = \alpha_i \tau \alpha_{n-i+1} \tau \alpha_i$, whence $\alpha_i^* \in \langle \mathscr{A} \rangle$, for $i = 1, 2, \ldots, n$. □

Let

$$\varepsilon_{i,j} = \begin{pmatrix} 1 & 2 & \cdots & i-1 & i+1 & \cdots & j-1 & j+1 & \cdots & n \\ 1 & 2 & \cdots & i-1 & i+1 & \cdots & j-1 & j+1 & \cdots & n \end{pmatrix},$$

for $1 \leq i < i+1 < j \leq n$.

Lemma 4 *One has* $\varepsilon_{i,j} \in \langle \mathscr{A} \rangle$, *for* $1 \leq i < i+1 < j \leq n$.

Proof We have $\varepsilon_{i,j} = \alpha_i^2 \alpha_j^2$, whence $\varepsilon_{i,j} \in \langle \mathscr{A} \rangle$, for $1 \leq i < i+1 < j \leq n$. □

Let

$$\varepsilon_{i,j}^* = \begin{pmatrix} 1 & 2 & \cdots & i-1 & i+1 & i+2 & \cdots & j-2 & j-1 & j+1 & \cdots & n \\ 1 & 2 & \cdots & i-1 & j-1 & j-2 & \cdots & i+2 & i+1 & j+1 & \cdots & n \end{pmatrix},$$

for $1 \leq i < i+1 < j \leq n$.

Lemma 5 *One has* $\varepsilon_{i,j}^* \in \langle \mathscr{A} \rangle$, *for* $1 \leq i < i+1 < j \leq n$.

Proof We have $\varepsilon_{i,j}^* = \varepsilon_{i,j} \alpha_j^* \alpha_{j-i}^* \alpha_j^*$, which implies $\varepsilon_{i,j}^* \in \langle \mathscr{A} \rangle$, for $1 \leq i < i+1 < j \leq n$. □

Define $\alpha_0 = \tau$, $\alpha_{n+1} = \mathrm{id}$, $\varepsilon^*_{0,n+1} = \tau$, $\varepsilon^*_{0,j} = \alpha^*_j$, for $j = 2, \ldots, n$, and $\varepsilon^*_{i,n+1} = \alpha_i$, for $i = 1, \ldots, n-1$.

Let

$$\rho^+_{i,j} = \begin{pmatrix} 1\ 2 \cdots i-1 & i+2 \cdots & j & j+2 \cdots n \\ 1\ 2 \cdots i-1 & i+1 \cdots & j-1 & j+2 \cdots n \end{pmatrix},$$

for $0 \le i < i+2 < j \le n$.

Lemma 6 *One has* $\rho^+_{i,j} \in \langle \mathscr{A} \rangle$, *for* $0 \le i < i+2 < j \le n$.

Proof We have $\rho^+_{i,j} = \alpha_i^2 \alpha_{i+1}^2 \alpha_{j+1}^2 \varepsilon^*_{i,j+1} \varepsilon^*_{i,j}$, whence $\rho^+_{i,j} \in \langle \mathscr{A} \rangle$, for $0 \le i < i+2 < j \le n$. □

Let

$$\rho^-_{i,j} = \begin{pmatrix} 1\ 2 \cdots i-2 & i & \cdots j-2 & j+1 \cdots n \\ 1\ 2 \cdots i-2 & i+1 & \cdots j-1 & j+1 \cdots n \end{pmatrix},$$

for $1 \le i < i+2 < j \le n+1$.

Lemma 7 *One has* $\rho^-_{i,j} \in \langle \mathscr{A} \rangle$, *for* $1 \le i < i+2 < j \le n+1$.

Proof We have $\rho^-_{i,j} = \alpha_{i-1}^2 \alpha_{j-1}^2 \alpha_j^2 \varepsilon^*_{i-1,j} \varepsilon^*_{i,j}$, which implies $\rho^-_{i,j} \in \langle \mathscr{A} \rangle$, for $1 \le i < i+2 < j \le n+1$. □

Now, we are prepared to prove that $\mathscr{A}$ is a generating set of the monoid $\mathrm{PAut}(P_n)$.

Proposition 4 *One has* $\mathrm{PAut}(P_n) = \langle \mathscr{A} \rangle$.

Proof We will perform this proof by using a recurring construction. First, for an arbitrary element α of $\mathrm{PAut}(P_n)$, we set some notations. Denote by $I_1^\alpha, I_2^\alpha, \ldots, I_k^\alpha$ the maximal intervals of $\mathrm{Dom}\,\alpha$ such that

$$I_1^\alpha < I_2^\alpha < \cdots < I_k^\alpha.$$

Let $J_r^\alpha = I_r^\alpha \alpha$, for $r = 1, \ldots, k$. Then $J_1^\alpha, J_2^\alpha, \ldots, J_k^\alpha$ are the maximal intervals of $\mathrm{Im}\,\alpha$. Denote by σ_α the permutation of $\{1, 2, \ldots, k\}$ such that

$$J^\alpha_{1\sigma_\alpha} < J^\alpha_{2\sigma_\alpha} < \cdots < J^\alpha_{k\sigma_\alpha}.$$

Now, fix $\alpha \in \mathrm{PAut}(P_n)$. Let $I = \overline{n} \setminus \mathrm{Dom}\,\alpha$ and define $\beta = \prod_{i \in I} \alpha_i^2$ (observe that α_i^2, $i \in \overline{n}$, is an idempotent and idempotents commute). Clearly, $\mathrm{Dom}\,\beta = \mathrm{Dom}\,\alpha$.

Let s be the least number $r \in \{1, \ldots, k\}$ such that $r\sigma_\alpha \neq r\sigma_\beta$. Let t be the minimal element in the set $J^\beta_{s\sigma_\beta}$ and q be the maximal element of $J^\beta_{s\sigma_\alpha\sigma_\beta}$. Then, we put

$$\beta = \beta \varepsilon^*_{t-1,q+1}$$

(i.e. we define a *new* β as being $\beta\varepsilon^*_{t-1,q+1}$; below we will made similar *variables*'s redefinitions). Then either $r\sigma_\alpha = r\sigma_\beta$, for all $r \in \{1,\dots,k\}$, or the least number $r \in \{1,\dots,k\}$ such that $r\sigma_\alpha \neq r\sigma_\beta$ is greater than s.

We repeat the procedure until $r\sigma_\alpha = r\sigma_\beta$ for all $r \in \{1,\dots,k\}$.

Further, we put $\gamma = \beta$ and let u be the least number $p \in \{1,\dots,k\}$ such that $\gamma|_{I^\alpha_{p\sigma_\gamma}} \neq \alpha|_{I^\alpha_{p\sigma_\gamma}}$.

If $\operatorname{Im}\gamma|_{I^\alpha_{u\sigma_\gamma}} = \operatorname{Im}\alpha|_{I^\alpha_{u\sigma_\gamma}}$ then we put

$$\gamma = \gamma\varepsilon^*_{a,b},$$

where a and b are the greatest and respectively the least number with $a < J^\gamma_{u\sigma_\gamma} < b$.

If $\operatorname{Im}\gamma|_{I^\alpha_{u\sigma_\gamma}} \neq \operatorname{Im}\alpha|_{I^\alpha_{u\sigma_\gamma}}$ then there exist $x, y \in \overline{n}$ such that $\operatorname{Im}\alpha|_{I^\alpha_{u\sigma_\gamma}} = \{x,\dots,z\}$ and either $\operatorname{Im}\gamma|_{I^\alpha_{u\sigma_\gamma}} = \{x-y,\dots,z-y\}$ or $\operatorname{Im}\gamma|_{I^\alpha_{u\sigma_\gamma}} = \{x+y,\dots,z+y\}$, where $z = x + |I^\alpha_{u\sigma_\gamma}| - 1$.

First, suppose that $\operatorname{Im}\gamma|_{I^\alpha_{u\sigma_\gamma}} = \{x-y,\dots,z-y\}$. Then, there exists $j \in \overline{n}$ with $j > J^\gamma_{u\sigma_\gamma}$ such that $j, j+1 \notin \operatorname{Im}\gamma$. In this case, we put

$$\gamma = \gamma\rho^-_{x-y,j+1}.$$

On the other hand, admit that $\operatorname{Im}\gamma|_{I^\alpha_{u\sigma_\gamma}} = \{x+y,\dots,z+y\}$. Then, there exists $j < J^\gamma_{u\sigma_\gamma}$, with $j > J^\gamma_{p\sigma_\gamma}$, for all $p < u$ such that $j-1, j \notin \operatorname{Im}\gamma$. In this case, we put

$$\gamma = \gamma\rho^+_{j-1,z+y}.$$

After y such steps, we obtain a transformation γ such that $\operatorname{Im}\gamma|_{I^\alpha_{u\sigma_\gamma}} = \operatorname{Im}\alpha|_{I^\alpha_{u\sigma_\gamma}}$. If $\gamma|_{I^\alpha_{u\sigma_\gamma}} \neq \alpha|_{I^\alpha_{u\sigma_\gamma}}$ then we put

$$\gamma = \gamma\varepsilon^*_{a,b},$$

where a and b are the greatest and the least number, respectively, such that $a < J^\gamma_{u\sigma_\gamma} < b$.

We repeat the procedure until $\gamma = \alpha$. Therefore, by Lemmas 2-7, we may deduce that $\alpha \in \langle\mathscr{A}\rangle$ and so $\mathscr{A}$ is a generating set of $\operatorname{PAut}(P_n)$, as required. □

Next, we will show that $\mathscr{A}$ is a generating set of $\operatorname{PAut}(P_n)$ of minimal size.

Let G be a generating set of $\operatorname{PAut}(P_n)$.

First, notice that $\operatorname{Dom}\tau = \overline{n}$. Moreover, for $\alpha \in \operatorname{PAut}(P_n)$, clearly, we have $\operatorname{Dom}\alpha = \overline{n}$ if and only if $\alpha = \tau$ or $\alpha = \mathrm{id} = \tau^2$. Thus, it follows immediately that:

Lemma 8 *One has $\tau \in G$.*

Let

$$A_i = \{\alpha \in \operatorname{PAut}(P_n) \mid \operatorname{Dom}\alpha = \overline{n}\backslash\{i\} \text{ or } \operatorname{Dom}\alpha = \overline{n}\backslash\{n-i+1\}\},$$

for $i = 1, \dots, \lceil \frac{n}{2} \rceil$.

Lemma 9 *One has* $|G \cap A_i| \geq 1$, *for all* $i \in \{1, \dots, \lceil \frac{n}{2} \rceil\}$.

Proof Let $i \in \{1, \dots, \lceil \frac{n}{2} \rceil\}$ and consider the transformation α_i defined previously. Notice that $\operatorname{Dom} \alpha_i = \overline{n} \backslash \{i\}$ and so $\alpha_i \in A_i$. Let $\beta_1, \dots, \beta_k \in G \backslash \{\mathrm{id}\}$ be such that $\alpha_i = \beta_1 \cdots \beta_k$ and $\{\beta_j, \beta_{j+1}\} \neq \{\tau\}$, for $j = 1, \dots, k-1$. Since $\operatorname{Dom} \alpha_i = \operatorname{Dom}(\beta_1 \cdots \beta_k) \subseteq \operatorname{Dom} \beta_1$, $\operatorname{rank} \alpha_i = n-1$ and $\beta_1 \neq \mathrm{id}$, we have $\operatorname{Dom} \beta_1 = \operatorname{Dom} \alpha_i$ or $\beta_1 = \tau$.

If $\operatorname{Dom} \beta_1 = \operatorname{Dom} \alpha_i$ then $\beta_1 \in A_i$ and so $\beta_1 \in G \cap A_i$.

On the other hand, suppose that $\beta_1 = \tau$. In that case, since $\operatorname{Dom} \alpha_i = \operatorname{Dom}(\beta_1 \cdots \beta_k) \subseteq \operatorname{Dom}(\tau\beta_2)$, $\operatorname{rank} \alpha_i = n-1$ and $\beta_2 \in G \backslash \{\mathrm{id}, \tau\}$, we have $\operatorname{Dom} \alpha_i = \operatorname{Dom}(\tau\beta_2)$, whence $\operatorname{Dom} \beta_2 = \overline{n} \backslash \{n-i+1\}$ and so $\beta_2 \in G \cap A_i$.

Thus, in both cases, we have shown that $|G \cap A_i| \neq \emptyset$, as required. $\square$

Lemma 10 *Let* $n \geq 6$. *Then* $|G \cap A_i| \geq 2$, *for all* $i \in \{3, \dots, \lfloor \frac{n}{2} \rfloor\}$.

Proof First, observe that it is a routine matter to check that $|A_i| = 16$, for all $i \in \{3, \dots, \lfloor \frac{n}{2} \rfloor\}$. Recall also that $\tau \in G$, by Lemma 8.

Now, assume by contradiction that $|G \cap A_i| < 2$, for some $i \in \{3, \dots, \lfloor \frac{n}{2} \rfloor\}$. Then, by Lemma 9, we have $G \cap A_i = \{\alpha\}$, for some $\alpha \in \mathrm{PAut}(P_n)$. Without loss of generality, we may suppose that $\operatorname{Dom} \alpha = \overline{n} \backslash \{i\}$. Hence, we have two cases:

Case 1. $\operatorname{Im} \alpha = \overline{n} \backslash \{i\}$. Then, as $\alpha^3 = \alpha$ and $\operatorname{rank} \alpha\tau\alpha = n-2$, we have

$$\langle G \rangle \cap A_i = \{\alpha, \alpha^2, \alpha\tau, \tau\alpha, \alpha^2\tau, \tau\alpha^2, \tau\alpha\tau, \tau\alpha^2\tau\} \neq A_i,$$

which is a contradiction (since G is a generating set of $\mathrm{PAut}(P_n)$).

Case 2. $\operatorname{Im} \alpha = \overline{n} \backslash \{n-i+1\}$. In this case, as $(\alpha\tau)^2 = \mathrm{id}|_{\operatorname{Dom} \alpha}$, $(\tau\alpha)^2 = \mathrm{id}|_{\operatorname{Im} \alpha}$ and $\operatorname{rank} \alpha^2 = n-2$, we obtain

$$\langle G \rangle \cap A_i = \{\alpha, \alpha\tau, \tau\alpha, \alpha\tau\alpha, \tau\alpha\tau, (\alpha\tau)^2, (\tau\alpha)^2, \tau(\alpha\tau)^2\} \neq A_i,$$

which again is a contradiction, as required. $\square$

Now, as a consequence of Proposition 4 and Lemmas 8–10, we may prove the first of our main results:

Theorem 2 *The rank of* $\mathrm{PAut}(P_3)$ *is equal to* 3 *and, for* $n \geq 4$, *the rank of* $\mathrm{PAut}(P_n)$ *is equal to* $n-1$.

Proof By Proposition 4, the set $\mathscr{A}$ generates $\mathrm{PAut}(P_n)$. Thus,

$$\operatorname{rank} \mathrm{PAut}(P_n) \leq |\mathscr{A}| = \begin{cases} 3 & \text{if } n = 3 \\ n-1 & \text{if } n \geq 4. \end{cases}$$

Let G be any generating set of $\mathrm{PAut}(P_n)$. By Lemmas 8 and 9 , the transformation τ and $\left\lceil \frac{n}{2} \right\rceil$ pairwise different transformations of rank $n-1$ are in G. Thus, $|G| \geq 1 + \left\lceil \frac{n}{2} \right\rceil$. In particular, we have $|G| \geq 3$, if $n = 3, 4$, and $|G| \geq 4$, if $n = 5$. If $n \geq 6$ then, by Lemma 10, there exist $\lfloor \frac{n}{2} \rfloor - 2$ additional pairwise different transformations with rank $n-1$ in G. This shows that

$$\operatorname{rank} \mathrm{PAut}(P_n) \geq \begin{cases} 3 & \text{if } n = 3 \\ n-1 & \text{if } n \geq 4, \end{cases}$$

as required. □

Next, we calculate the rank of the monoid $\mathrm{IEnd}(P_n)$.
Define

$$\beta_i = \begin{pmatrix} 1 & 2 & \cdots & i-1 & i+1 & i+2 & \cdots & n-1 & n \\ 1 & 2 & \cdots & i-1 & i & i+1 & \cdots & n-2 & n-1 \end{pmatrix},$$

for $i = 2, \ldots, n-1$, and let $\mathscr{B} = \mathscr{A} \cup \{\beta_i \mid i = 2, \ldots, \lceil \frac{n}{2} \rceil\}$.

Lemma 11 *One has* $\{\beta_i \mid i = 2, \ldots, n-1\} \subseteq \langle \mathscr{B} \rangle$.

Proof For $i = 2, \ldots, \lceil \frac{n}{2} \rceil$, we have $\beta_i \in \mathscr{B}$. Let $i = \lceil \frac{n}{2} \rceil + 1, \ldots, n-1$ then $\beta_i = \tau \beta_{n-i+1} \alpha_n^*$. □

Proposition 5 *Let* $\beta \in \mathrm{IEnd}(P_n) \backslash \mathrm{PAut}(P_n)$. *Then*

$$\beta \in \langle \mathrm{PAut}(P_n) \cup \{\beta_i \mid i = 2, \ldots, n-1\} \rangle.$$

Proof Let $\beta \in \mathrm{IEnd}(P_n) \backslash \mathrm{PAut}(P_n)$. Then, it is easy to show that there exists a transformation $\delta \in \mathrm{PAut}(P_n)$ with $\operatorname{Dom} \delta = \operatorname{Im} \beta$ and $\operatorname{Im} \delta \subseteq \{1, 2, \ldots, |\operatorname{Im} \beta| + \mathrm{m}_\beta - 1\}$, where m_β is the number of the maximal intervals of $\operatorname{Im} \beta$.

Define $\bar{\beta} = \beta\delta$. It is clear that $\operatorname{Dom} \bar{\beta} = \operatorname{Dom} \beta$.

Further, let I be the set of all $x \in \operatorname{Dom} \bar{\beta}$ such that $x\bar{\beta} + 1 \in \operatorname{Im} \bar{\beta}$ and $x\bar{\beta} + 1 \notin \{(x-1)\bar{\beta}, (x+1)\bar{\beta}\}$. Clearly, $I \neq \emptyset$ since $\beta \notin \mathrm{PAut}(P_n)$. We let $I = \{i_1, \ldots, i_k\}$ be such that $i_1\bar{\beta} < i_2\bar{\beta} < \cdots < i_k\bar{\beta}$.

Let $X_1, X_2, \ldots, X_{k+1}$ be the partition of $\operatorname{Dom} \bar{\beta}$ such that

$$X_r = \{x \in \operatorname{Dom} \bar{\beta} \mid i_{r-1}\bar{\beta} < x\bar{\beta} \leq i_r\bar{\beta}\},$$

for $r \in \{1, 2, \ldots, k+1\}$, where $i_0\bar{\beta} = 0$ and $i_{k+1}\bar{\beta} = n$.

Let β^* be the transformation defined by $x\beta^* = x\bar{\beta} + r - 1$, for all $x \in X_r$ and $r = 1, 2, \ldots, k+1$. It is clear that $\operatorname{Dom} \beta^* = \operatorname{Dom} \bar{\beta} = \operatorname{Dom} \beta$ and $\beta^* \in \mathrm{IEnd}(P_n)$.

First, we show $\beta^* \in \mathrm{PAut}(P_n)$. Let $u \in \bar{n}$ be such that $u, u+1 \in \operatorname{Im} \beta^*$. Then there exist $a \in X_{r_1}$ and $b \in X_{r_2}$ such that $a\bar{\beta} + r_1 - 1 = u$ and $b\bar{\beta} + r_2 - 1 = u + 1$, for some $r_1, r_2 \in \{1, \ldots, k+1\}$. In order to obtain a contradiction, assume that $r_1 \neq r_2$.

Suppose that $r_1 < r_2$. Then $X_{r_1}\bar{\beta} < X_{r_2}\bar{\beta}$ and so $a\bar{\beta} < b\bar{\beta}$. This implies $r_1 + 1 \leq r_2$ and $a\bar{\beta} + 1 \leq b\bar{\beta}$, whence $b\bar{\beta} + r_2 \geq a\bar{\beta} + 1 + r_1 + 1 = a\bar{\beta} + r_1 - 1 + 3 = u + 3 = b\bar{\beta} + r_2 - 1 + 2 = b\bar{\beta} + r_2 + 1$. Thus $b\bar{\beta} \geq b\bar{\beta} + 1$, which is a contradiction.

On the other hand, suppose that $r_1 > r_2$. Then $X_{r_1}\bar{\beta} > X_{r_2}\bar{\beta}$ and so $a\bar{\beta} > b\bar{\beta}$. This implies $r_1 \geq r_2 + 1$ and $a\bar{\beta} \geq b\bar{\beta} + 1$. Thus, we have $a\bar{\beta} + r_1 \geq b\bar{\beta} + 1 + r_2 + 1 =$

$b\bar{\beta} + r_2 - 1 + 3 = u + 1 + 3 = a\bar{\beta} + r_1 + 3$, whence $a\bar{\beta} \geq a\bar{\beta} + 3$, which is a contradiction.

Therefore, we have $r_1 = r_2$. Then $u = a\bar{\beta} + r_1 - 1$, $u + 1 = b\bar{\beta} + r_1 - 1$ and so $a, b \in X_{r_1}$. This implies $b\bar{\beta} + r_1 = b\bar{\beta} + r_1 - 1 + 1 = u + 1 + 1 = a\bar{\beta} + r_1 - 1 + 2 = a\bar{\beta} + r_1 + 1$, i.e. $b\bar{\beta} = a\bar{\beta} + 1$. Thus $a\bar{\beta} + 1 \in \operatorname{Im}\bar{\beta}$, since $b \in \operatorname{Dom}\bar{\beta}$. Assume $a\bar{\beta} + 1 \notin \{(a-1)\bar{\beta}, (a+1)\bar{\beta}\}$. Then $a \in I$ and so $a = i_{r_1}$ and $b\bar{\beta} \leq a\bar{\beta}$, since $a, b \in X_{r_1}$. Hence, $a\bar{\beta} + 1 = b\bar{\beta} \leq a\bar{\beta}$, which is a contradiction. Thus, $b\bar{\beta} = a\bar{\beta} + 1 \in \{(a-1)\bar{\beta}, (a+1)\bar{\beta}\}$ and so we obtain $b \in \{a-1, a+1\}$.

This shows that $\beta^* \in \operatorname{PAut}(P_n)$.

Finally, we show that $\beta = \beta^* \beta_{i_1\bar{\beta}+1} \beta_{i_2\bar{\beta}+1} \cdots \beta_{i_k\bar{\beta}+1} \delta^{-1}$, from which follows that

$$\beta \in \langle \operatorname{PAut}(P_n) \cup \{\beta_i \mid i = 2, \ldots, n-1\} \rangle.$$

Since $\bar{\beta}\delta^{-1} = \beta\delta\delta^{-1} = \beta \operatorname{id}|_{\operatorname{Dom}\delta} = \beta \operatorname{id}|_{\operatorname{Im}\beta} = \beta$, it suffices to show that $\bar{\beta} = \beta^* \beta_{i_1\bar{\beta}+1} \beta_{i_2\bar{\beta}+1} \cdots \beta_{i_k\bar{\beta}+1}$.

We proceed by showing that

$$\begin{aligned} & x\beta^* \beta_{i_1\bar{\beta}+1} \beta_{i_2\bar{\beta}+1} \cdots \beta_{i_s\bar{\beta}+1} \\ & \quad = \begin{cases} x\bar{\beta}, & \text{if } x \in X_1 \cup \cdots \cup X_s \\ x\bar{\beta} + r - 1 - s, & \text{if } x \in X_r, \text{ for some } r \in \{s+1, \ldots, k+1\}, \end{cases} \end{aligned}$$

by induction on $1 \leq s \leq k$.

Let $s = 1$. Then

$$\begin{aligned} & x\beta^* \beta_{i_1\bar{\beta}+1} = (x\bar{\beta} + r - 1)\beta_{i_1\bar{\beta}+1} \\ & \quad = \begin{cases} (x\bar{\beta})\beta_{i_1\bar{\beta}+1} = x\bar{\beta} & \text{if } x \in X_1, \\ & \text{since } x\bar{\beta} < i_1\bar{\beta} + 1 \\ x\bar{\beta} + r - 1 - 1 = x\bar{\beta} + r - 1 - s & \text{if } x \in X_r \text{ for some } r > 1, \\ & \text{since } x\bar{\beta} \geq i_1\bar{\beta} + 1. \end{cases} \end{aligned}$$

Assume that the above expression is true for some $s < k$. We will prove it for $s + 1$.

Let $x \in X_1 \cup \cdots \cup X_{s+1}$. If $x \notin X_{s+1}$ then $x\beta^* \beta_{i_1\bar{\beta}+1} \cdots \beta_{i_s\bar{\beta}+1} = x\bar{\beta}$, by the induction hypothesis and $(x\bar{\beta})\beta_{i_{s+1}\bar{\beta}+1} = x\bar{\beta}$, since $x\bar{\beta} < i_{s+1}\bar{\beta} + 1$.

If $x \in X_{s+1}$ then $x\beta^* \beta_{i_1\bar{\beta}+1} \cdots \beta_{i_s\bar{\beta}+1} = x\bar{\beta} + s + 1 - 1 - s = x\bar{\beta}$, by the induction hypothesis and $(x\bar{\beta})\beta_{i_{s+1}\bar{\beta}+1} = x\bar{\beta}$, since $x\bar{\beta} < i_{s+1}\bar{\beta} + 1$.

Now, let $x \in X_r$ for some $r \in \{s+2, \ldots, k+1\}$. Then

$$\begin{aligned} (x\beta^* \beta_{i_1\bar{\beta}+1} \cdots \beta_{i_s\bar{\beta}+1})\beta_{i_{s+1}\bar{\beta}+1} &= (x\bar{\beta} + r - 1 - s)\beta_{i_{s+1}\bar{\beta}+1} = (x\bar{\beta} + r - 1 - s) - 1 \\ &= x\bar{\beta} + r - 1 - (s+1), \end{aligned}$$

since $x\bar{\beta} > i_{s+1}\bar{\beta} + 1$, which completes the proof. □

From Proposition 4, Lemma 11 and Proposition 5, we obtain immediately:

Corollary 3 *One has* $\operatorname{IEnd}(P_n) = \langle \mathscr{B} \rangle$.

Now, we will prove that $\mathscr{B}$ is a generating set of $\mathrm{IEnd}(P_n)$ of minimal size. We start by presenting a series of five lemmas.

Let G' be a generating set of $\mathrm{IEnd}(P_n)$.

Lemma 12 *One has* $|G' \cap (\mathrm{IEnd}(P_n)\backslash\mathrm{PAut}(P_n))| \geq \lceil \frac{n}{2} \rceil - 1$.

Proof Let $2 \leq j \leq n-1$. Then $\beta_j = \gamma_1 \cdots \gamma_k$, for some $k \geq 1$ and $\gamma_1, \ldots, \gamma_k \in G'$. As $\operatorname{rank}\beta_j = n-1$ then $\operatorname{rank}\gamma_i \geq n-1$, for all $i = 1, \ldots, k$, and there exists $i \in \{1, \ldots, k\}$ such that $\gamma_i \notin \mathrm{PAut}(P_n)$. Let i be the least $r \in \{1, \ldots, k\}$ such as $\gamma_r \notin \mathrm{PAut}(P_n)$. Let $\gamma = \gamma_1 \cdots \gamma_{i-1} \in \mathrm{PAut}(P_n)$ (with $\gamma = \mathrm{id}$ if $i = 1$). Thus, $\beta_j = \gamma\gamma_i \cdots \gamma_k$ implies $\gamma^{-1}\beta_j = \gamma^{-1}\gamma\gamma_i \cdots \gamma_k = \gamma_i \cdots \gamma_k$ (since $\gamma^{-1}\gamma = \mathrm{id}|_{\mathrm{Dom}\,\gamma_i}$).

We have $\operatorname{rank}\gamma = n-1$ or $\operatorname{rank}\gamma = n$. If $\operatorname{rank}\gamma = n-1$ then $\operatorname{Im}\gamma^{-1} = \operatorname{Dom}\beta_j = \{1, \ldots, n\}\backslash\{j\}$. Thus $\operatorname{Dom}\gamma = \{1, \ldots, n\}\backslash\{j\}$, whence $\operatorname{Im}\gamma = \operatorname{Dom}\beta_j$ or $\operatorname{Im}\gamma = \operatorname{Dom}\beta_{n-j+1}$, and so $\operatorname{Dom}\gamma_i = \operatorname{Im}\gamma = \operatorname{Dom}\beta_j$ or $\operatorname{Dom}\gamma_i = \operatorname{Im}\gamma = \operatorname{Dom}\beta_{n-j+1}$ (since $\operatorname{rank}\gamma_i = n-1$). If $\operatorname{rank}\gamma = n$ then $\gamma = \mathrm{id}$ or $\gamma = \tau$. If $\gamma = \mathrm{id}$ then $\operatorname{Dom}\gamma_i = \operatorname{Dom}\beta_j$. If $\gamma = \tau$ then $\operatorname{Dom}\gamma_i = \operatorname{Dom}\beta_{n-j+1}$. Note that $n - j + 1 \geq \lceil \frac{n}{2} \rceil$.

Therefore, we must have in G' at least $\lceil \frac{n-2}{2} \rceil = \lceil \frac{n}{2} \rceil - 1$ distinct elements of $\mathrm{IEnd}(P_n)\backslash\mathrm{PAut}(P_n)$. □

Lemma 13 *For* $\alpha \in \mathrm{IEnd}(P_n)$ *such that* $\operatorname{Dom}\alpha \in \{\{1, \ldots, n-1\}, \{2, \ldots, n\}\}$, *we have* $\alpha \in \mathrm{PAut}(P_n)$.

Proof It is a routine matter to verify that

$$\alpha \in \{\alpha_1, \tau\alpha_1, \alpha_1\tau, \tau\alpha_1\tau, \alpha_n, \tau\alpha_n, \alpha_n\tau, \tau\alpha_n\tau\} \subseteq \mathrm{PAut}(P_n),$$

as required. □

Lemma 14 *If* $\alpha \in \mathrm{IEnd}(P_n)\backslash\mathrm{PAut}(P_n)$ *has rank* $n-1$, *then* $\operatorname{Im}\alpha \in \{\{1, \ldots, n-1\}, \{2, \ldots, n\}\}$.

Proof As $\operatorname{rank}\alpha = n-1$, we conclude that $\alpha \in \{\beta_i, \tau\beta_i, \beta_i\tau, \tau\beta_i\tau \mid i = 2, \ldots, n-1\}$. Let $i = 2, \ldots, n-1$. Since $\operatorname{Im}\beta_i = \{1, \ldots, n-1\}$ and $\operatorname{Dom}\tau = \operatorname{Im}\tau = \overline{n}$, we obtain

$$\operatorname{Im}\beta_i, \operatorname{Im}(\tau\beta_i), \operatorname{Im}(\beta_i\tau), \operatorname{Im}(\tau\beta_i\tau) \in \{\{1, \ldots, n-1\}, \{2, \ldots, n\}\},$$

whence $\operatorname{Im}\alpha \in \{\{1, \ldots, n-1\}, \{2, \ldots, n\}\}$, as required. □

Lemma 15 *One has* $\langle G' \cap \mathrm{PAut}(P_n)\rangle = \mathrm{PAut}(P_n)$.

Proof First, notice that it is clear that $\tau \in G'$.

On the other hand, let α be any transformation of $\mathrm{PAut}(P_n)$ with $\operatorname{rank}\alpha = n-1$.

Then, there exist $\gamma_1, \ldots, \gamma_k \in G'$ such that $\alpha = \gamma_1 \cdots \gamma_k$ ($k \geq 1$). Assume that there exists $i \in \{1, \ldots, k\}$ such that $\gamma_i \notin \mathrm{PAut}(P_n)$. Let i be the least index $r \in \{1, \ldots, k\}$ such that $\gamma_r \notin \mathrm{PAut}(P_n)$ and let $\gamma = \gamma_1 \cdots \gamma_{i-1} \in \mathrm{PAut}(P_n)$ (with $\gamma = \mathrm{id}$ if $i = 1$). Then $\alpha = \gamma\gamma_i \cdots \gamma_k$ implies $\gamma_i \cdots \gamma_k = \gamma^{-1}\gamma\gamma_i \cdots \gamma_k = \gamma^{-1}\alpha \in \mathrm{PAut}(P_n)$

(since $\gamma^{-1}\gamma = \mathrm{id}|_{\mathrm{Dom}\,\gamma_i}$). Hence, we have $i < k$. We have $\gamma_{i+1}\cdots\gamma_k \notin \{\mathrm{id}, \tau\}$ (otherwise $\gamma_i = \gamma^{-1}\alpha(\gamma_{i+1}\cdots\gamma_k)^{-1} \in \mathrm{PAut}(P_n)$, which is a contradiction). Hence, $\mathrm{rank}(\gamma_{i+1}\cdots\gamma_k) = n-1$. Let $\lambda = \gamma_{i+1}\cdots\gamma_k$. Then $\mathrm{Dom}\,\lambda = \mathrm{Im}\,\gamma_i \in \{\{1,\ldots,n-1\},\{2,\ldots,n\}\}$, by Lemma 14. Therefore, we obtain $\lambda \in \mathrm{PAut}(P_n)$, by Lemma 13. Thus, $\gamma^{-1}\alpha = \gamma_i\lambda$ implies that $\gamma_i = \gamma^{-1}\alpha\lambda^{-1} \in \mathrm{PAut}(P_n)$, which is a contradiction. Thus, $\gamma_1,\ldots,\gamma_k \in \mathrm{PAut}(P_n)$.

Therefore, in particular, we showed that $\mathscr{A} \subseteq \langle G' \cap \mathrm{PAut}(P_n)\rangle$, and therefore, $\langle G' \cap \mathrm{PAut}(P_n)\rangle = \mathrm{PAut}(P_n)$, by Proposition 4. □

Now, as an immediate consequence of Lemma 15 and Theorem 2, we have:

Lemma 16 *One has* $|G' \cap \mathrm{PAut}(P_n)| \geq \begin{cases} 3 & \text{if } n = 3 \\ n-1 & \text{if } n \geq 4. \end{cases}$

Finally, we conclude with the presentation of our second main result.

Theorem 3 *The rank of* $\mathrm{IEnd}(P_3)$ *is equal to* 4 *and, for* $n \geq 4$, *the rank of* $\mathrm{IEnd}(P_n)$ *is equal to* $n + \lceil \frac{n}{2} \rceil - 2$.

Proof By Corollary 3, we have

$$\mathrm{rank}\,\mathrm{IEnd}(P_n) \leq |\mathscr{B}| = \begin{cases} 3+1=4 & \text{if } n=3 \\ n-1+\lceil \frac{n}{2}\rceil - 1 = n + \lceil \frac{n}{2}\rceil - 2 & \text{if } n \geq 4. \end{cases}$$

On the other hand, by Lemmas 12 and 16 , we have

$$\mathrm{rank}\,\mathrm{IEnd}(P_n) \geq \begin{cases} 3+1=4 & \text{if } n=3 \\ n-1+\lceil \frac{n}{2}\rceil - 1 = n + \lceil \frac{n}{2}\rceil - 2 & \text{if } n \geq 4, \end{cases}$$

as required. □

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