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All idempotent and regular elements in the monoid of generalized hypersubstitutions for algebraic systems of type (2; 2)

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There are two different concepts for hypersubstitutions for algebraic systems [K. Denecke and D. Phusanga, Hyperformulas and solid algebraic systems, *Studia Logica* **90**(2) (2008) 263–286; J. Koppitz and D. Phusanga, The monoid of hypersubstitutions for algebraic systems, *J. Announcements Union Sci. Sliven* **33**(1) (2018) 120–127]. In this paper, we follow the more natural and practicable one given in [J. Koppitz and D. Phusanga, The monoid of hypersubstitutions for algebraic systems, *J. Announcements Union Sci. Sliven* **33**(1) (2018) 120–127]. On the other hand, in [S. Leeratanavalee and K. Denecke, Generalized hypersubstitutions and strongly solid varieties, *General Algebra and Applications*, *Proc. of 59th Workshop on General Algebra*; *15th Conf. for Young Algebraists Potsdam 2000* (Shaker Verlag, 2000), pp. 135–145], the concept of the monoid of generalized hypersubstitutions was introduced. Following both ideas, one obtains the concept of a monoid of generalized hypersubstitutions for algebraic systems in a canonical way. The purpose of this paper is the study of the monoid of generalized hypersubstitutions for algebraic systems. We characterize the idempotent as well as regular elements in this monoid.

Keywords: Algebraic system; term; formula; relational term; generalized hypersubstitution for algebraic system.

1. Introduction and Basic Concepts

An *algebraic system* of type (τ, τ') is a triple $\mathcal{A} := (A; (f_i^{\mathcal{A}})_{i\in I}, (\gamma_j^{\mathcal{A}})_{j\in J})$ consisting of a non-empty set A, a sequence $(f_i^{\mathcal{A}})_{i\in I}$ of operations on A indexed by the index

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set I, where $f_i^{\mathcal{A}} : A^{n_i} \to A$ is an n_i-ary operation for $i \in I$ and a sequence $(\gamma_j^{\mathcal{A}})_{j\in J}$ of relations on A indexed by the index set J where $\gamma_j^{\mathcal{A}} \subseteq A^{m_j}$ is an m_j-ary relation for $j \in J$. The pair (τ, τ') with $\tau = (n_i)_{i\in I},\ \tau' = (m_j)_{j\in J}$ of sequences of integers $n_i, m_j \in \mathbb{N}^+ := \mathbb{N} \setminus \{0\}$, is called the *type* of the algebraic system $\mathcal{A}$. Due to Mal'cev, algebraic systems are related to the concepts of term and formula (see [6]). Let $n \in \mathbb{N}^+$, let $X_n = \{x_1, x_2, \ldots, x_n\}$ be a n-element set of variables, and let $X := \cup_{1\leq n} X_n = \{x_1, \ldots, x_n, \ldots\}$ be countably infinite. Then the set $W_\tau(X_n)$ of all n-ary terms of type τ is defined in the usual way by the following conditions:

(i) Every $x_i \in X_n$ is an n-ary term of type τ.
(ii) If $t_1, \ldots, t_{n_i}$ are n-ary terms of type τ and if f_i is an n_i-ary operation symbol of type τ, then $f_i(t_1, \ldots, t_{n_i})$ is an n-ary term of type τ.

Let $W_\tau(X) := \cup_{n\geq 1} W_\tau(X_n)$ be the set of all terms of type τ.

To define formulas of type (τ, τ'), we need the logical connectives $\neg$ (for negation), $\vee$ (for disjunction), the equation symbol $\approx$ and the quantifier $\exists$. The classical definition of an n-ary formula is given by Mal'cev [6, pp. 115–116]. The variable x_i occurs freely in a formula F means that the quantifier $\exists$ does not occur in front of x_i in F. Otherwise x_i is called bound.

Let $n \geq 1$. An n-ary formula of type (τ, τ') is defined in the following inductive way:

(i) If t_1, t_2 are n-ary terms of type τ, then the equation $t_1 \approx t_2$ is an n-ary formula of type (τ, τ'). All variables in $t_1 \approx t_2$ are free.
(ii) If $t_1, \ldots, t_{m_j}$ are n-ary terms of type τ and if γ_j is an m_j-ary relational symbol, then $\gamma_j(t_1, \ldots, t_{m_j})$ is an n-ary formula of type (τ, τ'). All variables in such a formula are free.
(iii) If F is an n-ary formula of type (τ, τ'), then $\neg F$ is an n-ary formula of type(τ, τ'). All free variables in F are also free in $\neg F$. All bound variables in F are also bound in $\neg F$.
(iv) If F_1 and F_2 are n-ary formulas of type (τ, τ') such that variables occurring simultaneously in both formulas are free in each of them, then $F_1 \vee F_2$ is an n-ary formula of type (τ, τ'). Variables that are free in at least one of the formulas F_1 or F_2 are also free in $F_1 \vee F_2$. Variables that are bound in either F_1 or F_2 are also bound in $F_1 \vee F_2$.
(v) If F is an n-ary formula of type (τ, τ') and $x_i \in X_n$ occurs freely in F, then $\exists x_i(F)$ is an n-ary formula of type (τ, τ').

Let $\mathcal{F}_{\tau'}(W_\tau(X_n))$ be the set of all n-ary formulas of type (τ, τ') and let $\mathcal{F}_{\tau'}(W_\tau(X)) := \cup_{n\geq 1}\ \mathcal{F}_{\tau'}(W_\tau(X_n))$ be the set of all formulas of type (τ, τ'). A formula which is defined by (i) and (ii), we will call *an atomic formula* of type (τ, τ'). A formula having the form $\gamma_j(t_1, \ldots, t_{m_j})$, we will call *a relational term* of type (τ, τ'). Let $\mathcal{F}^*{}_{\tau'}(W_\tau(X))$ be the set of all relational terms of type (τ, τ').

First, we recall the concept of a generalized superposition of terms (see [5]). Let $n \in \mathbb{N}^+$. The operation

$$S^n : W_\tau(X) \times (W_\tau(X))^n \to W_\tau(X),$$

is defined by the following steps:

(i) If $t = x_i; 1 \leq i \leq n$, then $S^n(x_i, t_1, \ldots, t_n) := t_i$.
(ii) If $t = x_i; n < i \in \mathbb{N}^+$, then $S^n(x_i, t_1, \ldots, t_n) := x_i$.
(iii) If $t = f_i(s_1, \ldots, s_{n_i})$, then $S^n(f_i(s_1, \ldots, s_{n_i}), t_1, \ldots, t_n)$ $:= f_i(S^n(s_1, t_1, \ldots, t_n), \ldots, S^n(s_{n_i}, t_1, \ldots, t_n))$.

supposed that $S^n(s_k, t_1, \ldots, t_n)$ are already defined for $1 \leq k \leq n_i$.

Now, we want to extend this generalized superposition to relational terms. If we substitute variables occurring in a relational term by terms, we obtain a new relational term.

Definition 1.1. The operation

$$R^n : (W_\tau(X) \cup \mathcal{F}^*{}_{\tau'}(W_\tau(X))) \times (W_\tau(X))^n \to W_\tau(X) \cup \mathcal{F}^*{}_{\tau'}(W_\tau(X))$$

is defined by the following inductive steps:

Let $t_1, \ldots, t_n \in W_\tau(X)$.

(i) If $t \in W_\tau(X)$, then we defined $R^n(t, t_1, \ldots, t_n) := S^n(t, t_1, \ldots, t_n)$.
(ii) If $j \in J$ and $s_1, \ldots, s_{m_j} \in W_\tau(X)$, then $R^n(\gamma_j(s_1, \ldots, s_{m_j}), t_1, \ldots, t_n) := \gamma_j(S^n(s_1, t_1, \ldots, t_n), \ldots, S^n(s_{m_j}, t_1, \ldots, t_n))$.

In [8], the properties of R^n corresponding to the clone properties (FC1) and (FC3) (see, e.g. [3]) are proved as follows.

Theorem 1.2 ([8]). *Let $\beta \in W_\tau(X) \cup \mathcal{F}^*{}_{\tau'}(W_\tau(X))$. The operation R^n satisfies*:

(FC1) $R^n(R^n(\beta, t_1, \ldots, t_n), s_1, \ldots, s_n) = R^n(\beta, R^n(t_1, s_1, \ldots, s_n), \ldots,$ $R^n(t_n, s_1, \ldots, s_n))$, *whenever* $t_1, \ldots, t_n, s_1, \ldots, s_n \in W_\tau(X)$.
(FC3) $R^n(\beta, x_1, \ldots, x_n) = \beta$.

2. The Monoid of Generalized Hypersubstitutions for Algebraic Systems

In [5], Leeratanavalee and Denecke introduced the concept generalized hypersubstitutions of a given type τ for universal algebras. Further, a binary operation $\circ_G$ on the set $\mathrm{Hyp}_G(\tau)$ of all generalized hypersubstitutions of type τ was introduced such that $(\mathrm{Hyp}_G(\tau); \circ_G)$ is a monoid. This monoid was studied intensively for several types τ. All maximal idempotent subminoids of $\mathrm{Hyp}_G(2)$ was studied by Wongpinit and Leeratanavalee [11]. Puninagool and Leeratanavalee generalized these results for all regular elements in $\mathrm{Hyp}_G(2)$ [10]. The order of generalized hypersubstitutions of type $\tau = (2)$ are determined by Puninagool and Leeratanavalee [9].

On the other hand, we can consider algebraic systems in the sense of Mal'cev [6]. An algebraic system $(A; (f_i^A)_{i\in I}, (\gamma_j^A)_{j\in J})$ of type (τ, τ') with $\tau' = (m_j)_{j\in J}$ consists of a universe A, a family $(f_i^A)_{i\in I}$ of operations on A, and a family $(\gamma_j^A)_{j\in J}$ of relations on A, where γ_j^A is an m_j-ary relation on A. There were first attempts to define a concept of the hypersubstitution for algebraic systems. The concept of a hypersubstitution, introduced in [6], does not be practicable enough. Another attempt to define a hypersubstitution for algebraic systems was done by one the authors of the present paper in her Ph.D. Thesis [7]. But also this concept has not proven to be impractical.

In [4], the authors introduce the new concept of a hypersubstitution for algebraic system of type (τ, τ') in a canonical way. Later, Phusanga and Kamtornpipattanakul extended this concept for generalized hypersubstitution for algebraic systems of type (τ, τ') [8]. An operation f_i^A on a set A corresponds to the term operation $f_i(x_1, \ldots, x_{n_i})^A$ and a relation γ_j^A corresponds to a so-called "relational term operation" $\gamma_j(x_1, \ldots, x_{m_j})^A$.

Definition 2.1. Any mapping $\sigma: \{f_i \mid i \in I\} \cup \{\gamma_j \mid j \in J\} \to W_\tau(X) \cup \mathcal{F}^*{}_{\tau'}(W_\tau(X))$, which maps operation symbols to terms does not necessarily preserve the arity and relational symbols to relational terms does not necessarily preserve the arity is called a generalized hypersubstitution for algebraic systems of type (τ, τ').

Any generalized hypersubstitution for algebraic systems of type (τ, τ') consists of an operational part $\sigma\,|_{\{f_i \mid i\in I\}}: \{f_i \mid i \in I\} \to W_\tau(X)$ which is a usual hypersubstitution. We denote the set of all generalized hypersubstitutions for algebraic systems by $\mathrm{Hyp}_G(\tau, \tau')$. We define the extension of $\sigma \in \mathrm{Hyp}_G(\tau, \tau')$ as follows:

$$\widehat{\sigma}: W_\tau(X) \cup \mathcal{F}^*{}_{\tau'}(W_\tau(X)) \to W_\tau(X) \cup \mathcal{F}^*{}_{\tau'}(W_\tau(X)),$$

inductively defined as follows:

(i) $\widehat{\sigma}[x] := x$ for any variable $x \in X$,
(ii) $\widehat{\sigma}[f_i(s_1, \ldots, s_{n_i})] := S^{n_i}(\sigma(f_i), \widehat{\sigma}[s_1], \ldots, \widehat{\sigma}[s_{n_i}])$ for $i \in I$ and $s_1, \ldots, s_{n_i} \in W_\tau(X)$,
(iii) $\widehat{\sigma}[\gamma_j(s_1, \ldots, s_{m_j})] := R^{m_j}(\sigma(\gamma_j), \widehat{\sigma}[s_1], \ldots, \widehat{\sigma}[s_{m_j}])$ for $j \in J$ and $s_1, \ldots, s_{m_j} \in W_\tau(X)$.

Then, $\widehat{\sigma}$ is called the extension of a generalized hypersubstitution for algebraic system σ. In [8], an important property for the extension is proved.

Proposition 2.2 ([8]). *Let $\sigma \in \mathrm{Hyp}_G(\tau, \tau')$, let $n \in \mathbb{N}^+$ and let $t_1, \ldots, t_n \in W_\tau(X)$. Then*

$$\widehat{\sigma}[R^n(\beta, t_1, \ldots, t_n)] = R^n(\widehat{\sigma}[\beta], \widehat{\sigma}[t_1], \ldots, \widehat{\sigma}[t_n]),$$

for any $\beta \in W_\tau(X) \cup \mathcal{F}^*{}_{\tau'}(W_\tau(X))$.

In [8], the authors define a binary operation $\circ_g$ on $\mathrm{Hyp}_G\,(\tau,\tau')$ by $\sigma\circ_g\sigma' := \widehat{\sigma}\circ\sigma'$, where "$\circ$" denotes the usual composition of mappings and $\sigma,\sigma'\in \mathrm{Hyp}_G(\tau,\tau')$.

Let σ_{id} be the generalized hypersubstitution for algebraic systems mapping the operation symbols f_i to the terms $f_i(x_1,\ldots,x_{n_i})$ for all $i\in I$, and the relational symbols γ_j to the relational terms $\gamma_j(x_1,\ldots,x_{m_j})$ for all $j\in J$.

Lemma 2.3 ([8]). *For any term $t\in W_\tau(X)$ and relational term $F\in \mathcal{F}^*{}_{\tau'}(W_\tau(X))$, we have*

$$\widehat{\sigma}_{id}[t]=t \quad \text{and} \quad \widehat{\sigma}_{id}[F]=F.$$

Altogether, we obtain a monoid.

Theorem 2.4 ([8]). *The set $\mathrm{Hyp}_G(\tau,\tau')$ forms the monoid*

$$\mathcal{H}\mathrm{yp}_G(\tau,\tau') := (\mathrm{Hyp}_G(\tau,\tau');\circ_g,\sigma_{id}).$$

An important class of algebraic systems is the class of semigroups equipped with a binary relation, for example with a partial order. This justifies that we focus us to algebraic systems of type $(\tau,\tau')=((2);(2))$. In this paper, we extended the study in [4] to generalized hypersubstitutions for algebraic systems of type $((2),(2))$. We will write $(2;2)$ instead $((2),(2))$.

3. Idempotent and Regular Elements in $\mathcal{H}\mathbf{yp}_G(2;2)$

In this section, we characterize idempotent and regular elements in the monoid of generalized hypersubstitutions for algebraic systems of type $(2;2)$. We have only one binary operation symbol, say f and one binary relational symbol, say γ. A generalized hypersubstitution for algebraic systems which maps f to a term t and γ to a relational term F is denoted by $\sigma_{t,F}$, where σ_t denotes the operational part of σ. For $t\in W_{(2)}(X)$ and $F\in\mathcal{F}^*{}_{(2)}(W_{(2)}(X))$, let $\mathrm{var}(t)$ be the set of all variables occurring in term t and $\mathrm{var}(F)$ be the set of all variables occurring in relational term F. For $Y\subseteq X$, let $W_{(2)}(Y):=\{t\in W_{(2)}(X)\colon \mathrm{var}(t)\subseteq Y\}$.

Recall, a $\sigma_{t,F}\in\mathrm{Hyp}_G(2;2)$ is idempotent if $\sigma_{t,F}\circ_g\sigma_{t,F}=\sigma_{t,F}$, i.e. $\widehat{\sigma}_t[t]=t$ and $\widehat{\sigma}_{t,F}[F]=F$. In particular, σ_{id} is an idempotent element. The following lemma will show that $\sigma_{t,F}$ is an idempotent element if $t=f(t_1,t_2)$ and $F=\gamma(s_1,s_2)$, whenever $t_1,t_2,s_1,s_2\in W_{(2)}(X\setminus X_2)$.

Lemma 3.1. *Let $t,t_1,t_2\in W_{(2)}(X\setminus X_2)$ and $s_1,s_2\in W_{(2)}(X)$. Then, $S^2(t,s_1,s_2)=t$ and $R^2(\gamma(t_1,t_2),s_1,s_2)=\gamma(t_1,t_2)$. In particular, $\widehat{\sigma}_t[t]=t$ and $\widehat{\sigma}_{t,F}[F]=F$.*

Proof. If $t\in X\setminus X_2$ then $S^2(t,s_1,s_2)=t$ and $\widehat{\sigma}_t[t]=t$. Let $t=f(r_1,r_2)\in W_{(2)}(X\setminus X_2)$ and suppose that $S^2(r_i,s_1,s_2)=r_i$ for $i=1,2$. Then, $S^2(f(r_1,r_2),s_1,s_2)=f(S^2(r_1,s_1,s_2),S^2(r_2,s_1,s_2))=f(r_1,r_2)$ and $\widehat{\sigma}_t[f(r_1,r_2)]=S^2(f(r_1,r_2),\widehat{\sigma}_t[r_1],\widehat{\sigma}_t[r_2])=f(r_1,r_2)$. In particular, it holds $R^2(\gamma(t_1,t_2),s_1,s_2)=\gamma(S^2(t_1,s_1,s_2),S^2(t_2,s_1,s_2))=\gamma(t_1,t_2)$ and $\widehat{\sigma}_{t,F}[F]=R^2(\gamma(t_1,t_2),\widehat{\sigma}_t[t_1],\widehat{\sigma}_t[t_2])=\gamma(t_1,t_2)=F$. $\square$

In [9], the authors determine all the terms $t \in W_{(2)}(X)$ with $\widehat{\sigma}_t[t] = t$. In fact, the set of all $t \in W_{(2)}(X)$ with $\widehat{\sigma}_t[t] = t$ is $E_f(\mathrm{Hyp}_G(2)) = E_1 \cup E_2 \cup E_3 \cup X$, whenever

$$E_1 := \{f(x_1, s_2) \mid s_2 \in W_{(2)}(X \setminus \{x_2\}) \cup \{x_2\}\},$$

$$E_2 := \{f(s_1, x_2) \mid s_1 \in W_{(2)}(X \setminus \{x_1\} \cup \{x_1\}\}, \quad \text{and}$$

$$E_3 := \{f(s_1, s_2) \mid s_1, s_2 \in W_{(2)}(X \setminus X_2)\}.$$

Proposition 3.2. *Let $t \in E_f(\mathrm{Hyp}_G(2))$ and let $F = \gamma(x_1, r_2)$ with $r_2 \in W_{(2)}(X \setminus \{x_2\}) \cup \{x_2\}$ or $F = \gamma(r_1, x_2)$ with $r_1 \in W_{(2)}(X \setminus \{x_1\}) \cup \{x_1\}$ or $F = \gamma(r_1, r_2)$ with $r_1, r_2 \in W_{(2)}(X \setminus X_2)$. Then, $\sigma_{t,F}$ is an idempotent element.*

Proof. Suppose that $F = \gamma(x_1, r_2)$ with $r_2 \in W_{(2)}(X \setminus \{x_2\}) \cup \{x_2\}$. Here, we have $\widehat{\sigma}_{t,F}[F] = R^2(\sigma_{t,F}(\gamma), \widehat{\sigma}_t[x_1], \widehat{\sigma}_t[r_2])$

$$= \begin{cases} R^2(\gamma(x_1, r_2), x_1, \widehat{\sigma}_t[r_2]) = \gamma(x_1, r_2) = F & \text{if} \quad r_2 \neq x_2 \\ R^2(\gamma(x_1, r_2), x_1, x_2) = \gamma(x_1, r_2) = F & \text{if} \quad r_2 = x_2. \end{cases}$$

If $F = \gamma(r_1, x_2)$ with $r_1 \in W_{(2)}(X \setminus \{x_1\}) \cup \{x_1\}$ then, we obtain $\widehat{\sigma}_{t,F}[F] = F$, dually.

Suppose that $F = \gamma(r_1, r_2)$ with $r_1, r_2 \in W_{(2)}(X \setminus X_2)$. Then, we obtain $\widehat{\sigma}_{t,F}[F] = F$ by Lemma 3.1. □

Proposition 3.3. *Let $t \in E_f(\mathrm{Hyp}_G(2))$ and let $F = \gamma(s_1, s_2)$ with $\mathrm{var}(\gamma(s_1, s_2)) \cap X_2 \neq \emptyset$ such that s_1 is a composite term with $x_1 \in \mathrm{var}(s_1)$ or s_2 is a composite term with $x_2 \in \mathrm{var}(s_2)$ or both $x_1 \neq s_1$ and $x_2 \neq s_2$. Then, $\sigma_{t,F}$ is not an idempotent element.*

Proof. We have $\widehat{\sigma}_{t,F}[F] = R^2(\gamma(s_1, s_2), \widehat{\sigma}_t[s_1], \widehat{\sigma}_t[s_2)] = \gamma(S^2(s_1, \widehat{\sigma}_t[s_1], \widehat{\sigma}_t[s_2]), S^2(s_2, \widehat{\sigma}_t[s_1], \widehat{\sigma}_t[s_2]))$. If s_1 is a composite term with $x_1 \in \mathrm{var}(s_1)$, then it is easy to check that $S^2(s_1, \widehat{\sigma}_t[s_1], \widehat{\sigma}_t[s_2]) \neq s_1$. Dually, we have $S^2(s_2, \widehat{\sigma}_t[s_1], \widehat{\sigma}_t[s_2]) \neq s_2$, whenever s_2 is a composite term with $x_2 \in \mathrm{var}(s_2)$. Suppose that $x_1 \neq s_1$. From $\mathrm{var}(\gamma(s_1, s_2)) \cap X_2 \neq \emptyset$ it follows without loss of generality that $x_1 \in \mathrm{var}(s_1)$ or $x_1 \in \mathrm{var}(s_2)$. If $x_1 \in \mathrm{var}(s_1)$, then, $S^2(s_1, \widehat{\sigma}_t[s_1], \widehat{\sigma}_t[s_2]) \neq s_1$ and if $x_1 \in \mathrm{var}(s_2)$ then, $S^2(s_2, \widehat{\sigma}_t[s_1], \widehat{\sigma}_t[s_2]) \neq s_2$, since $x_1 \neq s_1$. This provides $\widehat{\sigma}_{t,F}[F] \neq \gamma(s_1, s_2)$ $(= F)$. □

Let $F = \gamma(s_1, s_2)$. Suppose that $F \notin E_\gamma(\mathrm{Hyp}_G(2;2)) := \{\gamma(x_1, r_2) \mid r_2 \in W_{(2)}(X \setminus \{x_2\}) \cup \{x_2\}\} \cup \{\gamma(r_1, x_2) \mid r_1 \in W_{(2)}(X \setminus \{x_1\}) \cup \{x_1\}\} \cup \{\gamma(r_1, r_2) \mid r_1, r_2 \in W_{(2)}(X \setminus X_2)\}$. Then, it is easy to verify that $\gamma(s_1, s_2) \notin E_\gamma(\mathrm{Hyp}_G(2,2))$ implies $\mathrm{var}(\gamma(s_1, s_2)) \cap X_2 \neq \emptyset$ such that s_1 is a composite term with $x_1 \in \mathrm{var}(s_1)$ or s_2 is a composite term with $x_2 \in \mathrm{var}(s_2)$ or both $x_1 \neq s_1$ and $x_2 \neq s_2$. Hence, Propositions 3.2 and 3.3 provide the characterization of the idempotent elements in $\text{Hyp}_G(2;2)$.

Corollary 3.4. *$\sigma_{t,F} \in \text{Hyp}_G(2;2)$ is idempotent if and only if $t \in E_f(\text{Hyp}_G(2))$ and $F \in E_\gamma(\text{Hyp}_G(2;2))$.*

We finish this section with the characterization of the regular elements. An element $\sigma_{t,F} \in \text{Hyp}_G(2;2)$ is called a regular element if there exists $\sigma_{t',F'} \in \text{Hyp}_G(2;2)$ such that

$$\sigma_{t,F} \circ_g \sigma_{t',F'} \circ_g \sigma_{t,F} = \sigma_{t,F}.$$

Theorem 3.5. *Let $t \in X \cup \{f(t_1,t_2) \mid t_1,t_2 \in W_{(2)}(X), \text{var}(f(t_1,t_2)) \cap X_2 = \{t_1,t_2\} \cap X_2\}$ and let $F \in \{\gamma(t_1,t_2) \mid t_1,t_2 \in W_{(2)}(X), \text{var}(\gamma(t_1,t_2)) \cap X_2 = \{t_1,t_2\} \cap X_2\}$. Then, $\sigma_{t,F}$ is a regular element.*

Proof. If $\text{var}(\gamma(t_1,t_2)) \cap X_2 = \emptyset$ then by Lemma 3.1, we have that $(\sigma_{r_1,F} \circ_g \sigma_{r_2,F} \circ_g \sigma_{r_1,F})(\gamma) = \sigma_{r_1,F}(\gamma)$ for any $r_1,r_2 \in W_{(2)}(X)$. In particular, we put $G := F$. Suppose that there are $t_1,t_2 \in W_{(2)}(X)$ with $\text{var}(\gamma(t_1,t_2)) \cap X_2 = \{t_1,t_2\} \cap X_2 = \{x\}$ for some $x \in X_2$ such that $F = \gamma(t_1,t_2)$. In particular, there is $i \in \{1,2\}$ such that $x = t_i$. We put $G = \gamma(x_i,x_i)$. For any $r_1,r_2 \in W_{(2)}(X)$, we have $(\sigma_{r_2,G} \circ_g \sigma_{r_1,F})(\gamma) = \widehat{\sigma}_{r_2,G}[\gamma(t_1,t_2)] = R^2(\sigma_{r_2,G}(\gamma), \widehat{\sigma}_{r_2}[t_1], \widehat{\sigma}_{r_2}[t_2]) = R^2(\gamma(x_i,x_i), \widehat{\sigma}_{r_2}[t_1], \widehat{\sigma}_{r_2}[t_2]) = \gamma(S^2(x_i, \widehat{\sigma}_{r_2}[t_1], \widehat{\sigma}_{r_2}[t_2]), S^2(x_i, \widehat{\sigma}_{r_2}[t_1], \widehat{\sigma}_{r_2}[t_2])) = \gamma(\widehat{\sigma}_{r_2}[t_i], \widehat{\sigma}_{r_2}[t_i]) = \gamma(x,x)$ since $t_i = x$. Because $\text{var}(\gamma(t_1,t_2)) \cap X_2 = \{x\}$, the only variable from X_2 occurring in t_1 as well as in t_2 is x. Hence, by the definition of S^2, we obtain $S^2(t_1,x,x) = t_1$ and $S^2(t_2,x,x) = t_2$. So, we can conclude that $(\sigma_{r_1,F} \circ_g \sigma_{r_2,G} \circ_g \sigma_{r_1,F})(\gamma) = \widehat{\sigma}_{r_1,F}[\gamma(x,x)] = R^2(\sigma_{r_1,F}(\gamma),x,x) = \gamma(S^2(t_1,x,x), S^2(t_2,x,x)) = \gamma(t_1,t_2) = \sigma_{r_1,F}(\gamma)$. It remains the case that $F = \gamma(t_1,t_2)$ with $\text{var}(\gamma(t_1,t_2)) \cap X_2 = \{t_1,t_2\} \cap X_2 = X_2$, i.e. $F = \gamma(x_1,x_2)$ or $F = \gamma(x_2,x_1)$. We put $G := F$ and obtain by straightforward calculations that $(\sigma_{r_1,F} \circ_g \sigma_{r_2,G} \circ_g \sigma_{r_1,F})(\gamma) = F = \sigma_{r_1,F}(\gamma)$ for any $r_1,r_2 \in W_{(2)}(X)$. Altogether, we have shown

$$(\sigma_{r_1,F} \circ_g \sigma_{r_2,G} \circ_g \sigma_{r_1,F})(\gamma) = \sigma_{r_1,F}(\gamma) \quad \text{for any } r_1,r_2 \in W_{(2)}(X). \tag{1}$$

In a second step, first, we suppose that $t \in X$ or $t = f(t_1,t_2)$ with $\text{var}(t) \cap X_2 = \emptyset$. Then, by the definition of σ_t and Lemma 3.1, respectively, we get $\widehat{\sigma}_t[t] = t$. Then $(\sigma_{t,F} \circ_g \sigma_{t,G} \circ_g \sigma_{t,F})(f) = t = \sigma_{t,F}(f)$. Suppose now that $t = f(t_1,t_2)$ with $\text{var}(f(t_1,t_2)) \cap X_2 = \{t_1,t_2\} \cap X_2 = X_2$, i.e. $t = f(x_1,x_2\}$ or $t = f(x_2,x_1\}$. By straightforward calculations, we obtain $(\sigma_{t,F} \circ_g \sigma_{t,G} \circ_g \sigma_{t,F})(f) = t = \sigma_{t,F}(f)$. We put $s := t$. It remains the case that $t = f(t_1,t_2)$ with $\text{var}(f(t_1,t_2)) \cap X_2 = \{t_1,t_2\} \cap X_2 = \{x\}$ for some $x \in X_2$. Here, there is $j \in \{1,2\}$ with $x = t_j$ and we put $s := f(x_j,x_j)$. Then, $(\sigma_{t,F} \circ_g \sigma_{s,G} \circ_g \sigma_{t,F})(f) = \widehat{\sigma}_{t,F}[\widehat{\sigma}_{s,G}[f(t_1,t_2)]] = \widehat{\sigma}_{t,F}[S^2(f(x_j,x_j), \widehat{\sigma}_s[t_1], \widehat{\sigma}_s[t_2])] = \widehat{\sigma}_{t,F}[f(x,x)] = S^2(f(t_1,t_2),x,x) = f(t_1,t_2)$ since $\text{var}(f(t_1,t_2)) \cap X_2 = \{x\}$.

By (1) with $r_1 = t$ and $r_2 = s$, we also have $(\sigma_{t,F} \circ_g \sigma_{s,G} \circ_g \sigma_{t,F})(\gamma) = \sigma_{t,F}(\gamma)$. Consequently, $\sigma_{t,F} \circ_g \sigma_{s,G} \circ_g \sigma_{t,F} = \sigma_{t,F}$. □

Proposition 3.6. *Let $\sigma_{t,F} \in \mathrm{Hyp}_G(2;2)$. Then $\sigma_{t,F}$ is regular if and only if $t \in X \cup \{f(t_1,t_2) \,|\, t_1,t_2 \in W_{(2)}(X), \mathrm{var}(f(t_1,t_2)) \cap X_2 = \{t_1,t_2\} \cap X_2\}$ and $F \in \{\gamma(t_1,t_2) \,|\, t_1,t_2 \in W_{(2)}(X), \mathrm{var}(\gamma(t_1,t_2)) \cap X_2 = \{t_1,t_2\} \cap X_2\}$.*

Proof. Assume that $t \notin X \cup \{f(t_1,t_2) : t_1,t_2 \in W_{(2)}(X), \mathrm{var}(f(t_1,t_2)) \cap X_2 = \{t_1,t_2\} \cap X_2\}$ or $F \notin \{\gamma(t_1,t_2) : t_1,t_2 \in W_{(2)}(X), \mathrm{var}(\gamma(t_1,t_2)) \cap X_2 = \{t_1,t_2\} \cap X_2\}$. Then, there is $\rho \in \{f,\gamma\}$ and $r_1,r_2 \in W_{(2)}(X)$ such that $\mathrm{var}(\rho(r_1,r_2)) \cap X_2 \neq \{r_1,r_2\} \cap X_2$, i.e. without loss of generality we have $x_1 \notin \{r_1,r_2\}$ but $x_1 \in \mathrm{var}(\rho(r_1,r_2))$. Since $\sigma_{t,F}$ is regular, there is $\rho(u_1,u_2) \in W_{(2)}(X) \cup \mathcal{F}^*{}_{(2)}(W_{(2)}(X))$ such that $\rho(r_1,r_2) = \widehat{\sigma}_{t,F}[\rho(u_1,u_2)] = R^2(\rho(r_1,r_2), \widehat{\sigma}_t[u_1], \widehat{\sigma}_t[u_2])$. Note that $\rho(u_1,u_2) = \widehat{\sigma}_{s,G}[\sigma_{t,F}(\rho)]$ for suitable $s \in W_{(2)}(X)$ and $G \in \mathcal{F}^*{}_{(2)}(W_{(2)}(X))$. Hence, there is $\rho(v_1,v_2) \in W_{(2)}(X) \cup \mathcal{F}^*{}_{(2)}(W_{(2)}(X))$ such that $s = f(v_1,v_2)$ or $G = \gamma(v_1,v_2)$, i.e. $\widehat{\sigma}_{s,G}[\rho(r_1,r_2)] = R^2(\rho(v_1,v_2), \widehat{\sigma}_s[r_1], \widehat{\sigma}_s[r_2])$. This, implies $x_1 \in \{\widehat{\sigma}_s[r_1], \widehat{\sigma}_s[r_2]\}$. But this is only possible if $x_1 \in \{r_1,r_2\}$, a contradiction. □

It is not hard to see that the set of regular elements does not form a semigroup. But a description of the maximal regular subsemigroups of $\mathrm{Hyp}_G(2,2)$ is a still open problem.

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