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On relative ranks of the semigroup of orientation-preserving transformations on infinite chains

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In this paper, we determine the relative rank of the semigroup $\mathcal{OP}(X)$ of all orientation-preserving transformations on infinite chains modulo the semigroup $\mathcal{O}(X)$ of all order-preserving transformations.

Keywords: Transformation semigroups on infinite chains; order-preserving transformations; orientation-preserving transformations; relative rank.

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1. Introduction and Preliminaries

The *rank* of a semigroup S is the minimum cardinality of a generating set of S. For a finitely generated semigroup S, determining the rank of S is a natural question and obviously yields some information about the semigroup. The ranks of various well-known semigroups have been calculated [9, 11, 12, 17]. By the famous theorem of Sierpiński [22], we know that any countable semigroup can be embedded in a semigroup with rank less than three. On the other hand, if the semigroup is uncountable then the rank is equal to the cardinality of the semigroup. In this case, the notion of rank provides us with no information. Here, we use a different rank property, namely the notation of a relative rank. It was introduced by Ruškuc [21] in order to describe the generating sets of semigroups with infinite rank. The *relative rank* of a semigroup S modulo a subset A of S, denoted by $\operatorname{rank}(S : A)$, is the minimal cardinality of a set $B \subseteq S$ such that $A \cup B$ generates S. It follows immediately from the definition that $\operatorname{rank}(S : \emptyset) = \operatorname{rank} S$, $\operatorname{rank}(S : S) = 0$, $\operatorname{rank}(S : A) = \operatorname{rank}(S : \langle A \rangle)$, and $\operatorname{rank}(S : A) = 0$ if and only if A is a generating set for S.

Let X be a nonempty infinite set and denote by $\mathcal{T}(X)$ the monoid of all full transformations of X (under composition). As a consequence of the main result in [22], that every countable subset of $\mathcal{T}(X)$ is contained in a two-generated subsemigroup of $\mathcal{T}(X)$, Banach obtained that the relative rank of $\mathcal{T}(X)$ modulo $A \subseteq \mathcal{T}(X)$ is either uncountable or at most two [2]. Relative ranks of subsemigroups of $\mathcal{T}(X)$ were considered in [16, 18]. Howie *et al.* showed that the relative rank of the full transformation semigroup $\mathcal{T}(X)$ modulo the set of all idempotent elements and modulo the symmetric group, respectively, is two. Moreover, they showed that the relative rank of $\mathcal{T}(X)$ modulo the top $\mathcal{J}$-class of $\mathcal{T}(X)$ is zero. This study was continued in [14] and was extended to subsemigroups of the monoid of all binary relations and the symmetric inverse monoid (that is, all injective partial mappings) over infinite sets. Similar properties were considered in the context of groups in [7, 10, 20], where subgroups of symmetric groups are studied. Recently, Tinpun and Koppitz considered generating sets of infinite full transformation semigroups with restricted range [25].

Now, let $X = (X, \leq)$ be a chain (totally ordered set). A function $f : A \to X$ from a subchain A of X into X is said to be *order-preserving* if $x \leq y$ implies $xf \leq yf$, for all $x, y \in A$. Note that given two subchains A and B of X and an order-isomorphism (i.e. an order-preserving bijection) $f : A \to B$, then the inverse function $f^{-1} : B \to A$ is also an order-isomorphism. In this case, the subchains A and B are called *order-isomorphic*. We denote by $\mathcal{O}(X)$ the submonoid of $\mathcal{T}(X)$ of all order-preserving transformations of X. The relative rank of $\mathcal{T}(X)$ modulo $\mathcal{O}(X)$ was considered by Higgins *et al.* [13]. They showed that $\operatorname{rank}(\mathcal{T}(X) : \mathcal{O}(X)) = 1$, when X is an arbitrary countable chain or an arbitrary well-ordered set, while $\operatorname{rank}(\mathcal{T}(\mathbb{R}) : \mathcal{O}(\mathbb{R}))$ is uncountable, by considering the usual order of the real numbers $\mathbb{R}$. In [15], the relative ranks $\operatorname{rank}(\mathcal{T}(X) : \mathcal{O}(X))$ were considered, where X is a countably infinite partially ordered set. In [4], Dimitrova *et al.* studied particular conditions for a countable infinite chain such that the monoid $\mathcal{O}(X)$ is generated by the set J of its transformations with infinite range (image), i.e. $\operatorname{rank}(\mathcal{O}(X) : J) = 0$.

A generalization of the concept of an order-preserving transformation is the concept of an orientation-preserving transformation, which was introduced in 1998 by McAlister [19] and, independently, one year later by Catarino and Higgins [3], but only for finite chains. In [8], Fernandes, Jesus, and Singha introduced the concept of an orientation-preserving transformation on an infinite chain. It generalizes the concept for a finite chain.

Definition 1 ([8]). Let $\alpha \in \mathcal{T}(X)$. We say that α is an *orientation-preserving* transformation if there exists a non-empty subset X_1 of X such that:

(1) α is order-preserving both on X_1 and on $X_2 = X \backslash X_1$;
(2) for all $a \in X_1$ and $b \in X_2$, we have $a < b$ and $a\alpha \geq b\alpha$.

In this paper, we will keep this meaning of X_1 and X_2 for an orientation-preserving transformation if it is clear from the context. We call to such a subset X_1 an *ideal* of α. Note that X_1 is an order ideal of X. Recall that a subset I of X is called an order ideal of X if $x \leq a$ implies $x \in I$, for all $x \in X$ and all $a \in I$.

Denote by $\mathcal{OP}(X)$ the subset of $\mathcal{T}(X)$ of all orientation-preserving transformations. In [8], Fernandes *et al.* proved that $\mathcal{OP}(X)$ is a semigroup. Moreover, they proved that if $\alpha \in \mathcal{OP}(X)$ is a non-constant transformation then α admits a unique ideal. Clearly, $\mathcal{O}(X) \subseteq \mathcal{OP}(X)$ and we have $\alpha \in \mathcal{O}(X)$ if and only if $\alpha \in \mathcal{OP}(X)$ and α admits X as an ideal.

In contrast to the infinite case, the finite case is well studied. Several properties of the monoid of orientation-preserving transformations on a finite chain have been investigated in [3, 19]. A presentation for this monoid, in terms of two (its rank) generators, was found by Arthur and Ruškuc [1]. The ranks of its ideals were determined by Zhao and Fernandes [26]. In [24], Tinpun and Koppitz considered the relative rank of the finite full transformation semigroup with restricted range. In [5], Dimitrova *et al.* studied rank properties of the semigroup of orientation-preserving transformations with restricted range on a finite chain. In [6], Dimitrova and Koppitz determined the relative rank of the finite full transformation semigroup with restricted range $\mathcal{T}(X, Y)$ modulo its subsemigroup $\mathcal{OP}(X, Y)$ of all orientation-preserving transformation with restricted range. Moreover, they considered the relative rank of the semigroup $\mathcal{OP}(X, Y)$ modulo the monoid $\mathcal{O}(X, Y)$ of all order-preserving transformations in $\mathcal{OP}(X, Y)$. The number of papers on full transformation semigroups and their subsemigroups for an infinite set X is many times smaller than in the finite case. In this paper, we determine the relative rank of the semigroup $\mathcal{OP}(X)$ of all orientation-preserving transformations on infinite chains modulo the subsemigroup $\mathcal{O}(X)$ of all order-preserving transformations. This corresponds to the second part of Problem 5.1 given by Fernandes *et al.* in [8]. We will give a complete answer for a certain class of infinite chains, which contains classical infinite chains.

We begin by recalling some notations and definitions that will be used in the paper. For every transformation $\alpha \in \mathcal{T}(X)$, we denote by $\operatorname{dom} \alpha$ and $\operatorname{im} \alpha$ the domain and the image (range) of α, respectively. The inverse of α is denoted by α^{-1}. For a subset $A \subseteq \mathcal{T}(X)$, we denote by $\langle A \rangle$ the subsemigroup of $\mathcal{T}(X)$ generated by A. For a subset $C \subseteq X$, we denote by $\alpha|_C$ the restriction of α to C and by id_C the identity mapping on C. A subset C of X is called a *convex* subset of X if $z \in X$ and $x < z < y$ imply $z \in C$, for all $x, y \in C$. Let C, D be convex subsets of X. We will write $C < D$ (respectively, $C \leq D$) if $c < d$ (respectively, $c \leq d$) for all $c \in C$

and all $d \in D$. If $C = \{c\}$ or $D = \{d\}$, we write $c < D$, $C < d$ (respectively, $c \leq D$, $C \leq d$) instead of $\{c\} < D$ or $C < \{d\}$ (respectively, $\{c\} \leq D$ or $C \leq \{d\}$).

For $A < B \subseteq X$ and $a, b \in X$ with $a < b$, $a < B$, and $A < b$, we put

$$(A, B) = \{x \in X : A < x < B\},$$

for a convex subset of X which has no minimum and no maximum;

$$[a, B) = \{x \in X : a \leq x < B\},$$

for a convex subset of X which has a minimum a but no maximum;

$$(A, b] = \{x \in X : A < x \leq b\},$$

for a convex subset of X which has no minimum but a maximum b;

$$[a, b] = \{x \in X : a \leq x \leq b\},$$

for a convex subset of X which has a minimum a and a maximum b.

Note that if $A = \emptyset$ or $B = \emptyset$ then, we have

$$(A, \emptyset) = \{x \in X : A < x\}, \quad (\emptyset, B) = \{x \in X : x < B\},$$
$$[a, \emptyset) = \{x \in X : a \leq x\} \quad \text{and} \quad (\emptyset, b] = \{x \in X : x \leq b\}.$$

A total order $<$ on a set X is said to be dense if, for all $x, y \in X$ with $x < y$, there is a $z \in X$ such that $x < z < y$. For the remaining part of the paper, X will be an infinite densely totally ordered set and any two convex subsets of X of the type (A, B) (i.e. has no minimum and no maximum) are order-isomorphic. We will consider the following cases:

(1) X has no minimum and no maximum.
(2) X has a minimum and a maximum.
(3) X has a minimum but no maximum.
(4) X has no minimum but a maximum.

The first case, when X has no minimum and no maximum, was considered by Tinpun in his thesis [23]. He proved that $\text{rank}(\mathcal{OP}(X) : \mathcal{O}(X)) = 2$. We will consider the other cases.

If X has a minimum and a maximum, we put $a = \min X$ and $b = \max X$. Then we have $X = [a, b]$.

If X has a minimum but no maximum, we put $a = \min X$. Then we have $X = [a, \emptyset)$.

If X has no minimum but a maximum, we put $b = \max X$. Then we have $X = (\emptyset, b]$.

Lemma 2. *Let X has a minimum a and let $\alpha \in \mathcal{OP}(X) \backslash \mathcal{O}(X)$ with ideal X_1. Then $a\alpha$ is the minimum of $X_1\alpha$.*

Proof. Since a is the minimum of X, we have $a \leq c$ for all $c \in X = X_1 \cup X_2$. Moreover, $\alpha \in \mathcal{OP}(X)$ implies $\alpha|_{X_1}$ is order-preserving and thus, $a\alpha \leq c\alpha$, for all $c \in X_1$. Hence, $a\alpha$ is the minimum of $X_1\alpha$. □

Dually, one can verify.

Lemma 3. *Let X has a maximum b and let $\alpha \in \mathcal{OP}(X)\backslash\mathcal{O}(X)$ with ideal X_1. Then $b\alpha$ is the maximum of $X_2\alpha$.*

Lemma 4. *Let $\alpha \in \mathcal{OP}(X)\backslash\mathcal{O}(X)$ with ideal X_1 and let X_1 has a maximum or X_2 has a minimum. If $c \in X$ is the maximum of X_1 (respectively, the minimum of X_2) then $c\alpha$ is the maximum of $\operatorname{im}\alpha$ (respectively, $c\alpha$ is the minimum of $\operatorname{im}\alpha$).*

Proof. Let $c \in X$ be the maximum of X_1 and let $y \in \operatorname{im}\alpha$. Then there is $x \in X$ such that $x\alpha = y$. If $x \in X_1$ then $x\alpha \leq c\alpha$, since c is the maximum of X_1 and $\alpha|_{X_1}$ is order-preserving. If $x \in X_2$ then $x\alpha \leq z\alpha$, for all $z \in X_1$, since $X_2\alpha \leq X_1\alpha$ (α is orientation-preserving). Therefore, we obtain $y \leq c\alpha$, i.e. $c\alpha$ is the maximum of $\operatorname{im}\alpha$.

Now, let $c \in X$ be the minimum of X_2. Dually, we obtain that $c\alpha$ is the minimum of $\operatorname{im}\alpha$. □

Proposition 5 ([8]). *Let $\alpha \in \mathcal{OP}(X)$ with ideal X_1. If $X_1\alpha \cap X_2\alpha \neq \emptyset$ then $X_1\alpha \cap X_2\alpha = \{c\}$, for some $c \in X$. Moreover, in this case, $X_1\alpha$ has a minimum, $X_2\alpha$ has a maximum, and both of these elements coincide with c.*

From Proposition 5 and Lemma 2, we obtain the following corollary.

Corollary 6. *Let $X = [a, \emptyset)$ and let $\alpha \in \mathcal{OP}(X)\backslash\mathcal{O}(X)$ with ideal X_1. If $X_1\alpha \cap X_2\alpha \neq \emptyset$ then $X_1\alpha \cap X_2\alpha = \{a\alpha\}$.*

From Proposition 5 and Lemma 3, we obtain the following corollary.

Corollary 7. *Let $X = (\emptyset, b]$ and let $\alpha \in \mathcal{OP}(X)\backslash\mathcal{O}(X)$ with ideal X_1. If $X_1\alpha \cap X_2\alpha \neq \emptyset$ then $X_1\alpha \cap X_2\alpha = \{b\alpha\}$.*

From Proposition 5, Lemma 2 and 3, we obtain the following corollary.

Corollary 8. *Let $X = [a, b]$ and let $\alpha \in \mathcal{OP}(X)\backslash\mathcal{O}(X)$ with ideal X_1. If $X_1\alpha \cap X_2\alpha \neq \emptyset$ then $a\alpha = b\alpha$ and $X_1\alpha \cap X_2\alpha = \{a\alpha\}$.*

2. X has a Minimum and a Maximum

In this section, we consider the relative rank of the semigroup $\mathcal{OP}(X)$ modulo the semigroup $\mathcal{O}(X)$, whence the set X has a minimum and a maximum. Recall that $a = \min X$, $b = \max X$, and $X = [a, b]$.

Let $c \in (a, b)$ and $d \in (c, b)$, i.e. $a < c < d < b$. Since any two convex subsets of X of the type (A, B) are order-isomorphic, we have that there are order-isomorphisms $\mu_1 : (a, c) \to (d, b)$ and $\mu_2 : (c, b) \to (a, c)$.

We define the transformation $\gamma : X \to X$ by

$$x\gamma = \begin{cases} d, & x = a, \\ x\mu_1, & x \in (a, c), \\ a, & x = c, \\ x\mu_2, & x \in (c, b), \\ c, & x = b. \end{cases}$$

Since μ_1 is an order-isomorphism and $a \leq y < c$ for all $y \in [a, c)$ as well as $d \leq y < b$ for all $y \in [d, b)$, we have that $\gamma|_{[a,c)}$ is order-preserving. Since μ_2 is an order-isomorphism and $c \leq y \leq b$ for all $y \in [c, b]$ as well as $a \leq y \leq c$ for all $y \in [a, c]$, we have that $\gamma|_{[c,b]}$ is order-preserving. Moreover, for all $x_1 \in [a, c)$ and all $x_2 \in [c, b]$, we have $x_1 < x_2$ and $x_2\gamma \leq c < d \leq x_1\gamma$ (since $x_2\gamma \in [a, c]$ and $x_1\gamma \in [d, b)$). Therefore, $\gamma \in \mathcal{OP}(X)$ with the ideal $X_1^* = [a, c)$. Clearly, $\gamma \notin \mathcal{O}(X)$ since $X_2^* = X \backslash X_1^* = [c, b] \neq \emptyset$.

Theorem 9. $\text{rank}(\mathcal{OP}(X) : \mathcal{O}(X)) = 1$.

Proof. Let $\alpha \in \mathcal{OP}(X) \backslash \mathcal{O}(X)$ with ideal X_1. Moreover, from $X = [a, b]$, Lemma 2 and 3, it follows $a\alpha = \min X_1\alpha$ and $b\alpha = \max X_2\alpha$.

We will consider two cases:

(a) For any decomposition $X = \tilde{X}_1 \cup \tilde{X}_2$ with $\tilde{X}_1 < \tilde{X}_2$ holds $\tilde{X}_1$ has a maximum or $\tilde{X}_2$ has a minimum.
(b) There is a decomposition $X = \tilde{X}_1 \cup \tilde{X}_2$ with $\tilde{X}_1 < \tilde{X}_2$ such that $\tilde{X}_1$ has no maximum and $\tilde{X}_2$ has no minimum.

Note that the case $\tilde{X}_1$ has a maximum and $\tilde{X}_2$ has a minimum is not possible for a decomposition $\tilde{X}_1 < \tilde{X}_2$, since X is densely ordered set.

Case (a). There is $m \in X$ such that $X_1 = [a, m]$, $X_2 = (X_1, b] = (m, b]$ or $X_1 = [a, X_2) = [a, m)$, $X_2 = [m, b]$.

First, we will consider the case $X_1 = [a, m)$ and $X_2 = [m, b]$, i.e.

$$\alpha = \begin{pmatrix} X_1 < X_2 \\ X_1\alpha \geq X_2\alpha \end{pmatrix} = \begin{pmatrix} [a, m) < [m, b] \\ [a, m)\alpha \geq [m, b]\alpha \end{pmatrix}.$$

Note that if $m = b$ then $X_2 = [b, b] = \{b\}$.

Since any two convex subsets of X of the type (A, B) are order-isomorphic, we have that there are order-isomorphisms $\lambda_1 : (a, m) \to (a, c)$ and $\lambda_2 : (m, b) \to (c, b)$.

We define the transformation $\beta : X \to X$ by

$$x\beta = \begin{cases} a, & x = a \neq m, \\ x\lambda_1, & x \in (a, m), \\ c, & x = m, \\ x\lambda_2, & x \in (m, b), \\ b, & x = b \neq m. \end{cases}$$

Clearly, β is an order-preserving bijection on X, i.e. $\beta \in \mathcal{O}(X)$.

Further, we define the transformation $\delta : X \to X$ by

$$x\delta = \begin{cases} m\alpha, & x = a, \\ x\mu_2^{-1}\lambda_2^{-1}\alpha, & x \in (a, c), \\ b\alpha, & x = c, \\ a\alpha, & x \in (c, d], \\ x\mu_1^{-1}\lambda_1^{-1}\alpha, & x \in (d, b), \\ b, & x = b. \end{cases}$$

Note that $\alpha \in \mathcal{OP}(X)\backslash\mathcal{O}(X)$ implies $X_2\alpha = [m, b]\alpha \leq X_1\alpha = [a, m)\alpha$. From Lemma 4, we have $m\alpha = \min \operatorname{im} \alpha$. Thus, $m\alpha = \min X_2\alpha \leq (m, b)\alpha \leq b\alpha = \max X_2\alpha \leq a\alpha = \min X_1\alpha \leq (a, m)\alpha \leq b = \max X$. Moreover, λ_i, μ_i, $i = 1, 2$, are order-isomorphisms and $\alpha|_{(m,b)}$, $\alpha|_{(a,m)}$ are order-preserving imply $\delta|_{(a,c)}$ and $\delta|_{(d,b)}$ are order-preserving. Therefore, we obtain $\delta \in \mathcal{O}(X)$.

Now, we will show that $\alpha = \beta\gamma\delta$. Let $x \in X$.

If $x = a$ then $a\beta\gamma\delta = a\gamma\delta = d\delta = a\alpha$.

If $x \in (a, m)$ then $x\beta\gamma\delta = (x\lambda_1)\gamma\delta = (x\lambda_1\mu_1)\delta = x\lambda_1\mu_1\mu_1^{-1}\lambda_1^{-1}\alpha = x\alpha$, since $\mu_1\mu_1^{-1} = \mathrm{id}_{(a,c)}$, $\lambda_1 \,\mathrm{id}_{(a,c)}\, \lambda_1^{-1} = \mathrm{id}_{(a,m)}$, and $x \,\mathrm{id}_{(a,m)}\, \alpha = x\alpha$.

If $x = m$ then $m\beta\gamma\delta = c\gamma\delta = a\delta = m\alpha$.

If $x \in (m, b)$ then $x\beta\gamma\delta = (x\lambda_2)\gamma\delta = (x\lambda_2\mu_2)\delta = x\lambda_2\mu_2\mu_2^{-1}\lambda_2^{-1}\alpha = x\alpha$, since $\mu_2\mu_2^{-1} = \mathrm{id}_{(c,b)}$, $\lambda_2 \,\mathrm{id}_{(c,b)}\, \lambda_2^{-1} = \mathrm{id}_{(m,b)}$, and $x \,\mathrm{id}_{(m,b)}\, \alpha = x\alpha$.

If $x = b$ then $b\beta\gamma\delta = b\gamma\delta = c\delta = b\alpha$.

Therefore, we have $\alpha \in \langle \mathcal{O}(X), \gamma \rangle$.

Now, consider the case $X_1 = [a, m]$ and $X_2 = (m, b]$, i.e.

$$\alpha = \begin{pmatrix} X_1 < X_2 \\ X_1\alpha \geq X_2\alpha \end{pmatrix} = \begin{pmatrix} [a, m] < (m, b] \\ [a, m]\alpha \geq (m, b]\alpha \end{pmatrix}.$$

Note that if $a = m$ then $X_1 = [a, a] = \{a\}$.

Let $c' \in (a, c)$, i.e. $(a, c') \subseteq X_1^* = [a, c)$. Recall that $c < d < b$, i.e. $d \in X_2^*$ and $(c, d) \subseteq X_2^* = [c, b]$.

Since any two convex subsets of X of the type (A, B) are order-isomorphic, we have that there is an order-isomorphism $\tau_1 : (a, c) \to (a, c')$.

We define the transformation $\eta_1 : X \to X$ by

$$x\eta_1 = \begin{cases} a, & x = a, \\ x\tau_1, & x \in (a, c), \\ c', & x = c, \\ x, & x \in (c, b]. \end{cases}$$

Since $a\eta_1 = a = \min X < (a, c)\eta_1 = (a, c)\tau_1 = (a, c') < c' = c\eta_1 < (c, b) < b$, $\eta_1|_{(c,b]} = \mathrm{id}_{(c,b]}$, and τ_1 is an order-isomorphism, we have that η_1 is an order-preserving transformation on X, i.e. $\eta_1 \in \mathcal{O}(X)$.

Further, since any two convex subsets of X of the type (A, B) are order-isomorphic, we have that there is an order-isomorphism $\tau_2 : (d, c'\mu_1) \to (d, b)$.

We define the transformation $\eta_2 : X \to X$ by

$$x\eta_2 = \begin{cases} x, & x \in [a, c], \\ d, & x \in (c, d], \\ x\tau_2, & x \in (d, c'\mu_1), \\ b, & x \in [c'\mu_1, b]. \end{cases}$$

Since $\eta_2|_{[a,c]} = \mathrm{id}_{[a,c]}$, $c < (c, d]\eta_2 = d < (d, c'\mu_1)\eta_2 = (d, c'\mu_1)\tau_2 = (d, b) < b = [c'\mu_1, b]\eta_2$, and τ_2 is an order-isomorphism, we have that η_2 is an order-preserving transformation on X, i.e. $\eta_2 \in \mathcal{O}(X)$.

Since μ_1 is an order-isomorphism, $x\gamma = x\mu_1$, for all $x \in (a, c)$, and $c' \in (a, c)$, we have $c'\gamma = c'\mu_1 \in (d, b)$ and thus $d < c'\gamma < b$. Therefore, we can write the transformation γ in the following way:

$$\gamma = \begin{pmatrix} a < (a, c') < c' < (c', c) < c < (c, b) < b \\ d < (a, c')\mu_1 < c'\mu_1 < (c', c)\mu_1 > a < (c, b)\mu_2 < c \end{pmatrix}.$$

Further, we define the transformation $\delta_1 : X \to X$ by

$$x\delta_1 = \begin{cases} a, & x = a, \\ x\mu_2^{-1}\lambda_2^{-1}\alpha, & x \in (a, c), \\ b\alpha, & x = c, \\ a\alpha, & x \in (c, d], \\ x\tau_2^{-1}\mu_1^{-1}\tau_1^{-1}\lambda_1^{-1}\alpha, & x \in (d, b), \\ m\alpha, & x = b. \end{cases}$$

Since $\lambda_1, \lambda_2, \mu_1, \mu_2, \tau_1, \tau_2$ are order-isomorphisms, $a = \min X$, $\alpha|_{(m,b]}$ is order-preserving ($X_2 = (m, b]$), $b\alpha = \max X_2\alpha$ (by Lemma 3), $a\alpha = \min X_1\alpha$ (by Lemma 2), $X_2\alpha \leq X_1\alpha$ ($\alpha \in \mathcal{OP}(X)$), $\alpha|_{[a,m]}$ is order-preserving ($X_1 = [a, m]$ is the ideal of α), and $m\alpha = \max \operatorname{im} \alpha$ (by Lemma 4), i.e. $\min X = a \leq X_2\alpha \leq b\alpha = \max X_2\alpha \leq a\alpha = \min X_1\alpha \leq X_1\alpha \leq m\alpha = \max \operatorname{im} \alpha$, it follows that $\delta_1 \in \mathcal{O}(X)$.

Finally, we will show that $\alpha = \beta\eta_1\gamma\eta_2\delta_1$. Let $x \in X$.

If $x = a \neq m$ then $a\beta\eta_1\gamma\eta_2\delta_1 = a\eta_1\gamma\eta_2\delta_1 = a\gamma\eta_2\delta_1 = d\eta_2\delta_1 = d\delta_1 = a\alpha$.

If $x \in (a, m)$ then $x\beta\eta_1\gamma\eta_2\delta_1 = (x\lambda_1)\eta_1\gamma\eta_2\delta_1 = (x\lambda_1\tau_1)\gamma\eta_2\delta_1 = (x\lambda_1\tau_1\mu_1)\eta_2\delta_1 = (x\lambda_1\tau_1\mu_1\tau_2)\delta_1 = x\lambda_1\tau_1\mu_1\tau_2\tau_2^{-1}\mu_1^{-1}\tau_1^{-1}\lambda_1^{-1}\alpha$. Since $\tau_2\tau_2^{-1} = \mathrm{id}_{(d,c'\mu_1)}$, $\mu_1|_{(a,c')}\,\mathrm{id}_{(d,c'\mu_1)}\,\mu_1^{-1} = \mathrm{id}_{(a,c')}$, $\tau_1\,\mathrm{id}_{(a,c')}\,\tau_1^{-1} = \mathrm{id}_{(a,c)}$, and $\lambda_1\,\mathrm{id}_{(a,c)}\,\lambda_1^{-1} = \mathrm{id}_{(a,m)}$, we have $x\lambda_1\tau_1\mu_1\tau_2\tau_2^{-1}\mu_1^{-1}\tau_1^{-1}\lambda_1^{-1}\alpha = x\,\mathrm{id}_{(a,m)}\,\alpha = x\alpha$.

If $x = m$ then $m\beta\eta_1\gamma\eta_2\delta_1 = c\eta_1\gamma\eta_2\delta_1 = c'\gamma\eta_2\delta_1 = (c'\mu_1)\eta_2\delta_1 = b\delta_1 = m\alpha$.

If $x \in (m, b)$ then $x\beta\eta_1\gamma\eta_2\delta_1 = (x\lambda_2)\eta_1\gamma\eta_2\delta_1 = (x\lambda_2)\gamma\eta_2\delta_1 = (x\lambda_2\mu_2)\eta_2\delta_1 = (x\lambda_2\mu_2)\delta_1 = x\lambda_2\mu_2\mu_2^{-1}\lambda_2^{-1}\alpha = x\alpha$, since $\mu_2\mu_2^{-1} = \mathrm{id}_{(c,b)}$, $\lambda_2\,\mathrm{id}_{(c,b)}\,\lambda_2^{-1} = \mathrm{id}_{(m,b)}$, and $x\,\mathrm{id}_{(m,b)}\,\alpha = x\alpha$.

If $x = b$ then $b\beta\eta_1\gamma\eta_2\delta_1 = b\eta_1\gamma\eta_2\delta_1 = b\gamma\eta_2\delta_1 = c\eta_2\delta_1 = c\delta_1 = b\alpha$.

Therefore, we have $\alpha \in \langle \mathcal{O}(X), \gamma \rangle$.

Case (b). There is a decomposition $X = X_1 \cup X_2$ with $X_1 < X_2$ such that X_1 has no maximum and X_2 has no minimum. Let $\alpha \in \mathcal{OP}(X)\backslash\mathcal{O}(X)$ with ideal X_1. Then $X_1 = [a, X_2)$ and $X_2 = (X_1, b]$, i.e.

$$\alpha = \begin{pmatrix} X_1 < X_2 \\ X_1\alpha \geq X_2\alpha \end{pmatrix} = \begin{pmatrix} [a, X_2) < (X_1, b] \\ [a, X_2)\alpha \geq (X_1, b]\alpha \end{pmatrix}.$$

Since any two convex subsets of X of the type (A, B) are order-isomorphic, we have that there are order-isomorphisms $\lambda_3 : (a, X_2) \to (a, c)$ and $\lambda_4 : (X_1, b) \to (c, b)$.

We define the transformation $\theta_1 : X \to X$ by

$$x\theta_1 = \begin{cases} a, & x = a, \\ x\lambda_3, & x \in (a, X_2), \\ x\lambda_4, & x \in (X_1, b), \\ b, & x = b. \end{cases}$$

Clearly, θ_1 is an order-preserving transformation on X, i.e. $\theta_1 \in \mathcal{O}(X)$.

Recall that $c < d < b$, i.e. $d \in X_2^*$ and $(c, d) \subseteq X_2^* = [c, b]$. We define the transformation $\theta_2 : X \to X$ by

$$x\theta_2 = \begin{cases} a, & x = a, \\ x\mu_2^{-1}\lambda_4^{-1}\alpha, & x \in (a, c), \\ b\alpha, & x = c, \\ a\alpha, & x \in (c, d], \\ x\mu_1^{-1}\lambda_3^{-1}\alpha, & x \in (d, b), \\ b, & x = b. \end{cases}$$

Since μ_1, μ_2, λ_3, λ_4 are order-isomorphisms, $a = \min X \leq (X_1, b)\alpha \leq b\alpha \leq a\alpha \leq (a, X_2)\alpha \leq b$, $\alpha|_{(X_1,b]}$ is order-preserving ($X_2 = (X_1, b]$), and $\alpha|_{[a,X_2)}$ is order-preserving ($X_1 = [a, X_2)$ is the ideal of α), it follows that $\theta_2 \in \mathcal{O}(X)$.

Now, we will show that $\alpha = \theta_1\gamma\theta_2$. Let $x \in X$.

If $x = a$ then $a\theta_1\gamma\theta_2 = a\gamma\theta_2 = d\theta_2 = a\alpha$.

If $x \in (a, X_2)$ then $x\theta_1\gamma\theta_2 = (x\lambda_3)\gamma\theta_2 = (x\lambda_3\mu_1)\theta_2 = x\lambda_3\mu_1\mu_1^{-1}\lambda_3^{-1}\alpha = x\alpha$, since $\mu_1\mu_1^{-1} = \mathrm{id}_{(a,c)}$, $\lambda_3 \,\mathrm{id}_{(a,c)}\, \lambda_3^{-1} = \mathrm{id}_{(a,X_2)}$, and $x \,\mathrm{id}_{(a,X_2)}\, \alpha = x\alpha$.

If $x \in (X_1, b)$ then $x\theta_1\gamma\theta_2 = (x\lambda_4)\gamma\theta_2 = (x\lambda_4\mu_2)\theta_2 = x\lambda_4\mu_2\mu_2^{-1}\lambda_4^{-1}\alpha = x\alpha$, since $\mu_2\mu_2^{-1} = \mathrm{id}_{(c,b)}$, $\lambda_4 \,\mathrm{id}_{(c,b)}\, \lambda_4^{-1} = \mathrm{id}_{(X_1,b)}$, and $x \,\mathrm{id}_{(X_1,b)}\, \alpha = x\alpha$.

If $x = b$ then $b\theta_1\gamma\theta_2 = b\gamma\theta_2 = c\theta_2 = b\alpha$.

Therefore, we have $\alpha \in \langle \mathcal{O}(X), \gamma \rangle$.

Altogether, we obtain $\mathcal{OP}(X) = \langle \mathcal{O}(X), \gamma \rangle$ and since $\mathcal{O}(X)$ is a proper submonoid of $\mathcal{OP}(X)$ we have $\mathrm{rank}(\mathcal{OP}(X) : \mathcal{O}(X)) = 1$. □

3. *X* has a Minimum but no Maximum

In this section, we consider the relative rank of the semigroup $\mathcal{OP}(X)$ modulo the semigroup $\mathcal{O}(X)$, whence the set X has a minimum a but no maximum, i.e. $X = [a, \emptyset)$.

Let $b \in X$, with $a < b$ and let $Y = [a, b] \subset X$. Then $\mathcal{OP}(Y)$ is the semigroup of all orientation-preserving transformations of Y and by Theorem 9, we have $\mathcal{OP}(Y) = \langle \mathcal{O}(Y), \gamma \rangle$. In fact, we will assume that γ is defined as in Sec. 2, where the set X is replaced by the set Y and $d \in Y$ is given. Recall also that $\mu_1 : (a, c) \to (d, b)$ and $\mu_2 : (c, b) \to (a, c)$ are order-isomorphisms.

Let us put

$$\mathcal{OP}^*(X) = \{\beta \in \mathcal{OP}(X) : \mathrm{im}\,\beta \subseteq [a, b] \text{ and } |[b, \emptyset)\beta| = 1\}.$$

Lemma 10. *$\mathcal{OP}^*(X)$ is isomorphic to $\mathcal{OP}(Y)$.*

Proof. We consider the mapping $h : \mathcal{OP}(Y) \to \mathcal{OP}^*(X)$, defined by

$$xh(\alpha) = \begin{cases} x\alpha, & x \in [a, b), \\ b\alpha, & x \in [b, \emptyset), \end{cases}$$

for $\alpha \in \mathcal{OP}(Y)$. Since $\beta \in \mathcal{OP}^*(X)$ is uniquely determined by the image of Y and $xh(\alpha) = x\alpha$ for all $x \in Y = [a, b]$, we can conclude that h is a bijection.

Now, let $\alpha, \beta \in \mathcal{OP}(Y)$. We will show that $h(\alpha\beta) = h(\alpha)h(\beta)$. Since $\alpha \in \mathcal{OP}(Y)$, we have $x\alpha \in Y = [a, b]$ for all $x \in Y$. Let $x \in [a, b)$. Then $xh(\alpha\beta) = x(\alpha\beta) = (x\alpha)\beta = (x\alpha)h(\beta) = xh(\alpha)h(\beta)$. Let $x \in [b, \emptyset)$. Then $xh(\alpha\beta) = b(\alpha\beta) = (b\alpha)\beta = (b\alpha)h(\beta) = xh(\alpha)h(\beta)$.

Therefore, h is an isomorphism and thus, $\mathcal{OP}^*(X)$ is isomorphic to $\mathcal{OP}(Y)$. □

Let $\mathcal{O}^*(X) = \mathcal{OP}^*(X) \cap \mathcal{O}(X)$. Then from Theorem 9 and Lemma 10, we have:

Corollary 11. $\mathcal{OP}^*(X) = \langle \mathcal{O}^*(X), h(\gamma) \rangle$.

Since any two convex subsets of X of the type (A, B) are order-isomorphic, we have that there is an order-isomorphism $\nu : (a, c) \to (c, \emptyset)$.

We define the transformation $\gamma^* : X \to X$ by

$$x\gamma^* = \begin{cases} c, & x = a, \\ x\nu, & x \in (a,c), \\ a, & x = c, \\ x\nu^{-1}, & x \in (c,\emptyset). \end{cases}$$

Since ν is an order-isomorphism, $c < x\nu \in (c,\emptyset)$, and $a < x\nu^{-1} \in (a,c)$, we have that $\gamma^*|_{[a,c)}$ and $\gamma^*|_{[c,\emptyset)}$ are order-preserving. Moreover, for all $x_1 \in [a,c)$ and $x_2 \in [c,\emptyset)$, we have $x_1 < x_2$ and $x_2\gamma^* < c \leq x_1\gamma^*$ (since $x_2\gamma^* \in [a,c)$ and $x_1\gamma^* \in [c,\emptyset)$). Therefore, $\gamma^* \in \mathcal{OP}(X)\backslash\mathcal{O}(X)$ with the same ideal $[a,c)$ as the transformation $\gamma \in \mathcal{OP}(Y)\backslash\mathcal{O}(Y)$. Moreover, γ^* is an injective orientation-preserving transformation on X and $\operatorname{im}\gamma^* = [a,c) \cup [c,\emptyset) = X$.

Proposition 12. *$\mathcal{OP}^*(X) \subseteq \langle \mathcal{O}(X), \gamma^* \rangle$.*

Proof. Since $\mathcal{OP}^*(X) = \langle \mathcal{O}^*(X), h(\gamma)\rangle$ (by Corollary 11) and $\mathcal{O}^*(X) \subseteq \mathcal{O}(X)$, it remains to show that $h(\gamma) \in \langle \mathcal{O}(X), \gamma^*\rangle$.

We define the transformation $\delta : X \to X$ by

$$x\delta = \begin{cases} a, & x = a, \\ x\nu\mu_2, & x \in (a,c) \text{ and } x\nu \in (c,b), \\ c, & x \in (a,c) \text{ and } x\nu \in [b,\emptyset), \\ d, & x = c, \\ x\nu^{-1}\mu_1, & x \in (c,\emptyset). \end{cases}$$

Since ν, μ_1, μ_2 are order-isomorphisms, $x\nu\mu_2 \in (a,c)$, for $x \in (a,c)$, $x\nu \in (c,b)$, and $x\nu^{-1}\mu_1 \in (d,b)$, for $x \in (c,\emptyset)$, i.e.

$$a < x\nu\mu_2 < c < d < x\nu^{-1}\mu_1,$$

it follows that $\delta \in \mathcal{O}(X)$.

Now, we will show that $h(\gamma) = \gamma^*\delta$. Let $x \in X$.

If $x = a$ then $a\gamma^*\delta = c\delta = d = a\gamma = ah(\gamma)$.

If $x \in (a,c)$ then $x\gamma^*\delta = x\nu\delta = x\nu\nu^{-1}\mu_1 = x\mu_1 = x\gamma = xh(\gamma)$, since $\nu\nu^{-1} = \mathrm{id}_{(a,c)}$ and $x\,\mathrm{id}_{(a,c)}\,\mu_1 = x\mu_1$.

If $x = c$ then $c\gamma^*\delta = a\delta = a = c\gamma = ch(\gamma)$.

If $x \in (c,b)$ then $x\gamma^*\delta = x\nu^{-1}\delta = x\nu^{-1}\nu\mu_2 = x\mu_2 = x\gamma = xh(\gamma)$, since $(\nu^{-1}\nu)|_{(c,b)} = \mathrm{id}_{(c,b)}$ and $x\,\mathrm{id}_{(c,b)}\,\mu_2 = x\mu_2$.

If $x \in [b,\emptyset)$ then $x\gamma^*\delta = x\nu^{-1}\delta = c = b\gamma = xh(\gamma)$.

Therefore, we obtain $h(\gamma) = \gamma^*\delta \in \langle \mathcal{O}(X), \gamma^*\rangle$ and thus, $\mathcal{OP}^*(X) = \langle \mathcal{O}^*(X), h(\gamma)\rangle \subseteq \langle \mathcal{O}(X), \gamma^*\rangle$. $\square$

Theorem 13. *$\operatorname{rank}(\mathcal{OP}(X) : \mathcal{O}(X)) = 1$.*

Proof. Let $\beta \in \mathcal{OP}(X)\backslash\mathcal{O}(X)$.

Since any two convex subsets of X of the type (A, B) are order-isomorphic, we have that there is an order-isomorphism $\tau_1 : (a, b) \to (a, \emptyset)$.

We define the transformation $\eta_1 : X \to X$ by

$$x\eta_1 = \begin{cases} a, & x = a, \\ x\tau_1, & x \in (a, b), \\ a, & x \in [b, \emptyset). \end{cases}$$

Clearly, η_1 is an orientation-preserving transformation of X with ideal $[a, b)$, i.e. $\eta_1 \in \mathcal{OP}(X)$.

Further, let $I \subseteq X$ be the smallest convex subset of X containing $\operatorname{im}\beta$, i.e. $\operatorname{im}\beta \subseteq I = \bigcap\{H : H$ is a convex subset of X with $\operatorname{im}\beta \subseteq H\}$. Let $Y' \subseteq Y = [a, b]$ be such that

- if I has a minimum and a maximum then $Y' = [a, b]$;
- if I has a minimum but no maximum then $Y' = [a, b)$;
- if I has no minimum but a maximum then $Y' = (a, b]$;
- if I has no minimum and no maximum then $Y' = (a, b)$.

Then there is an order-isomorphism $\tau_2 : I \to Y'$ and we define the transformation $\eta_2 : X \to X$ by

$$x\eta_2 = \begin{cases} a, & x \in [a, I), \\ x\tau_2, & x \in I, \\ b, & x \in (I, \emptyset). \end{cases}$$

Since τ_2 is an order-isomorphism and $a \leq x\tau_2 \in Y' \subseteq [a, b] \leq b$, we have that η_2 is an order-preserving transformation of X, i.e. $\eta_2 \in \mathcal{O}(X)$. Note that if $\operatorname{im}\beta = X$ then $I = X$, $Y' = [a, b)$, and $a\eta_2 = a\tau_2 = a$, $(a, \emptyset)\eta_2 = (a, \emptyset)\tau_2 = (a, b) = (a, \emptyset)\tau_1^{-1}$. Moreover, in the case $I = X$ we have $\tau_2 \in \mathcal{O}(X)$.

Now, we consider the transformation $\beta^* = \eta_1\beta\eta_2$. Since $\beta, \eta_1, \eta_2 \in \mathcal{OP}(X)$, it follows that $\beta^* \in \mathcal{OP}(X)$. We will show that $\beta^* \in \mathcal{OP}^*(X)$. Let $x \in X$. Then $(x\eta_1)\beta \in \operatorname{im}\beta \subseteq I$ and thus, $(x\eta_1\beta)\eta_2 = (x\eta_1\beta)\tau_2 \in Y' \subseteq [a, b]$. Thus, $x\beta^* = x\eta_1\beta\eta_2 \in [a, b]$. Hence, $\operatorname{im}\beta^* = \operatorname{im}(\eta_1\beta\eta_2) \subseteq [a, b]$. Moreover, from $[b, \emptyset)\beta^* = \{a\beta\tau_2\}$, it follows $|[b, \emptyset)\beta^*| = 1$. Therefore, $\beta^* \in \mathcal{OP}^*(X)$. From Proposition 12, it follows that $\beta^* \in \langle\mathcal{O}(X), \gamma^*\rangle$.

Further, we define the transformation $\theta_1 : X \to X$ by

$$x\theta_1 = \begin{cases} a, & x = a, \\ x\tau_1^{-1}, & x \in (a, \emptyset). \end{cases}$$

Since τ_1 is an order-isomorphism and $a < x\tau_1^{-1} \in (a, b)$, we have that θ_1 is an order-preserving transformation of X, i.e. $\theta_1 \in \mathcal{O}(X)$.

Now, we consider two cases for the set I.

(1) If $(I, \emptyset) \neq \emptyset$ then we define the transformation $\theta_2 : X \to X$ by

$$x\theta_2 = \begin{cases} a, & x = a \text{ and } a \notin Y', \\ x\tau_2^{-1}, & x \in Y', \\ p, & x \in (Y', \emptyset), \end{cases}$$

where $p \in (I, \emptyset)$. Since τ_2 is an order-isomorphism and $a \leq Y'\tau_2^{-1} = I < p \in (I, \emptyset)$, we obtain $\theta_2 \in \mathcal{O}(X)$.

(2) If $(I, \emptyset) = \emptyset$ then there is an order-isomorphism η_3 from Y' into a subset of $[a, c)$ with $(Y'\eta_3, c) = \emptyset$. We define a transformation $\theta_{2,1} : X \to X$ by

$$x\theta_{2,1} = \begin{cases} a, & x < Y', \\ x\eta_3, & x \in Y', \\ c, & x > Y'. \end{cases}$$

Since η_3 is an order-isomorphism and $a \leq Y'\eta_3 < c$, we have $\theta_{2,1} \in \mathcal{O}(X)$.

Further, we define $\theta_{2,2} : X \to X$ by

$$x\theta_{2,2} = \begin{cases} a, & x < Y'\eta_3\gamma^* \\ x(\eta_3\gamma^*)^{-1}\tau_2^{-1}, & x \in Y'\eta_3\gamma^*. \end{cases}$$

Since $(Y'\eta_3, c) = \emptyset$ and $\gamma^*|_{[a,c)}$ is an order-isomorphism, we have $(Y'\eta_3\gamma^*, \emptyset) = \emptyset$. Hence, $\theta_{2,2}$ is well defined. Since η_3 as well as $\gamma^*|_{[a,c)}$ are order-isomorphisms and $a \leq I = Y'\tau_2^{-1} = Y'(\eta_3\gamma^*)(\eta_3\gamma^*)^{-1}\tau_2^{-1}$, we have that $\theta_{2,2} \in \mathcal{O}(X)$.

We put $\theta_2 = \theta_{2,1}\gamma^*\theta_{2,2} \in \langle \mathcal{O}(X), \gamma^* \rangle$.

Note that $\theta_2|_{Y'} = (\theta_{2,1}\gamma^*\theta_{2,2})|_{Y'} = \eta_3\gamma^*(\eta_3\gamma^*)^{-1}\tau_2^{-1} = \tau_2^{-1}$.

Finally, we will show that $\beta = \theta_1\beta^*\theta_2$. Let $x \in X$.

If $x = a$ then $a\theta_1\beta^*\theta_2 = a\theta_1\eta_1\beta\eta_2\theta_2 = a\eta_1\beta\eta_2\theta_2 = a\beta\eta_2\theta_2 = a\beta\tau_2\theta_2 = a\beta\tau_2\tau_2^{-1} = a\beta$, since $\tau_2\tau_2^{-1} = \mathrm{id}_I$, $\mathrm{im}\,\beta \subseteq I$.

If $x \in (a, \emptyset)$ then $x\theta_1\beta^*\theta_2 = x\tau_1^{-1}\beta^*\theta_2 = x\tau_1^{-1}\eta_1\beta\eta_2\theta_2 = x\tau_1^{-1}\tau_1\beta\eta_2\theta_2 = x\tau_1^{-1}\tau_1\beta\tau_2\theta_2 = x\tau_1^{-1}\tau_1\beta\tau_2\tau_2^{-1}$, since $\mathrm{im}\,\beta \subseteq I$. Because $\tau_1^{-1}\tau_1 = \mathrm{id}_{(a,\emptyset)}$, $\tau_2\tau_2^{-1} = \mathrm{id}_I$, and $x\,\mathrm{id}_{(a,\emptyset)}\,\beta\,\mathrm{id}_I = x\beta$, we obtain $x\tau_1^{-1}\tau_1\beta\tau_2\tau_2^{-1} = x\beta$, i.e. $x\theta_1\beta^*\theta_2 = x\beta$.

Therefore, we have $\beta = \theta_1\beta^*\theta_2 \in \langle \mathcal{O}(X), \gamma^* \rangle$.

Altogether, we obtain $\mathcal{OP}(X) = \langle \mathcal{O}(X), \gamma^* \rangle$ and since $\mathcal{O}(X)$ is a proper submonoid of $\mathcal{OP}(X)$, we have $\mathrm{rank}(\mathcal{OP}(X) : \mathcal{O}(X)) = 1$. □

One can dually consider the case X has no minimum but a maximum. In this case, one obtains also $\mathrm{rank}(\mathcal{OP}(X) : \mathcal{O}(X)) = 1$.

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