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On the semigroup of all partial fence-preserving injections on a finite set

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For $n \in \mathbb{N}$, let $X_n = \{a_1, a_2, \ldots, a_n\}$ be an n-element set and let $\mathbf{F} = (X_n; <_f)$ be a fence, also called a zigzag poset. As usual, we denote by I_n the symmetric inverse semigroup on X_n. We say that a transformation $\alpha \in I_n$ is *fence-preserving* if $x <_f y$ implies that $x\alpha <_f y\alpha$, for all x, y in the domain of α. In this paper, we study the semigroup PFI_n of all partial fence-preserving injections of X_n and its subsemigroup $IF_n = \{\alpha \in PFI_n : \alpha^{-1} \in PFI_n\}$. Clearly, IF_n is an inverse semigroup and contains all regular elements of PFI_n. We characterize the Green's relations for the semigroup IF_n. Further, we prove that the semigroup IF_n is generated by its elements with rank $\geq n-2$. Moreover, for $n \in 2\mathbb{N}$, we find the least generating set and calculate the rank of IF_n.

Keywords: Finite transformation semigroup; fence-preserving transformations; inverse semigroup; Green's relations; generators; rank.

Mathematics Subject Classification: 20M20, 20M18

1. Introduction and Preliminaries

For $n \in \mathbb{N}$, let $X_n = \{a_1, a_2, \ldots, a_n\}$ be an n-element set. As usual, we denote by I_n the symmetric inverse semigroup on X_n, i.e. the partial one-to-one transformation semigroup on X_n under composition of mappings. The importance of I_n to inverse

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semigroup theory may be likened to that of the symmetric group S_n to group theory. Every finite inverse semigroup S is embeddable in I_n, the analogue of Cayley's theorem for finite groups, and to the regular representation of finite semigroups. Thus, just as the study of symmetric, alternating and dihedral groups has made a significant contribution to group theory, so has the study of various subsemigroups of I_n, see for example [1, 3, 5, 6, 13].

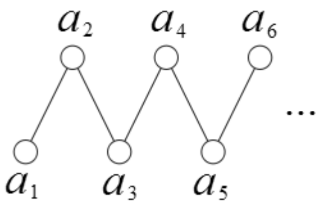
Let $\mathbf{F} = (X_n; <_f)$ be a *fence*, also called a *zigzag poset*, i.e. a partially ordered set in which the order relation forms a path with alternating orientations:

$$a_1 <_f a_2 >_f a_3 <_f \cdots a_n$$

or

$$a_1 >_f a_2 <_f a_3 >_f \ldots a_n.$$

Every element of $\mathbf{F}$ is either maximal or minimal. A fence $\mathbf{F}$ is called an up-fence (respectively a down-fence) if $a_1 <_f a_2$ (respectively $a_1 >_f a_2$). In this paper, without loss of generality, we consider an up-fence.



Several authors have investigated the number of order-preserving maps from fences to themselves, or to fences of other sizes, see for example [2, 4, 10, 11]. Recently, regular semigroups of transformations preserving a fence were characterized in [8, 12].

We begin by recalling some notations and definitions that will be used in the paper. For standard concepts in semigroup and symmetric inverse semigroup theory, see for example [7] and [9]. We denote by dom α and im α the domain and the image (range) of $\alpha \in I_n$, respectively. The natural number rank $\alpha := |\text{im } \alpha|$ is called the rank of α. The inverse element of α is denoted by α^{-1}. For a subset $Y \subseteq X_n$, we denote by $id|_Y$ the identity mapping on Y. Clearly, if $Y = X_n$ then $id|_{X_n} =: id$ is the identity mapping on X_n. For a subset $A \subseteq I_n$, we denote by $\langle A \rangle$ the subsemigroup of I_n generated by A. We say that a transformation $\alpha \in I_n$ is *fence-preserving* if $x <_f y$ implies that $x\alpha <_f y\alpha$, for all $x, y \in \text{dom } \alpha$. We denote by PFI_n the subsemigroup of I_n of all partial fence-preserving injections of X_n. Note that the semigroup PFI_n is not inverse. For example,

$$\alpha = \begin{pmatrix} 1 & 2 & 4 & 5 & 6 \\ 3 & 2 & 6 & 5 & 4 \end{pmatrix} \in PFI_6, \quad \text{but } \alpha^{-1} = \begin{pmatrix} 2 & 3 & 4 & 5 & 6 \\ 2 & 1 & 6 & 5 & 4 \end{pmatrix} \notin PFI_6.$$

Let IF_n be the set of all $\alpha \in PFI_n$ such that $\alpha^{-1} \in PFI_n$. Clearly, IF_n is the set of all $\alpha \in PFI_n$ with $x <_f y$ if and only if $x\alpha <_f y\alpha$, for all $x, y \in \text{dom } \alpha$.

Hence, IF_n is an inverse subsemigroup of PFI_n and contains all regular elements of PFI_n. In Sec. 2, we characterize the Green's relations for the inverse semigroup IF_n. Further, we prove that the semigroup IF_n is generated by its elements with rank $\geq n-2$. Moreover, for $n \in 2\mathbb{N}$ we find the least generating set and calculate the rank of IF_n.

2. Green's Relations

In this section, we characterize the Green's relations $\mathcal{R}$, $\mathcal{L}$, $\mathcal{H}$ and $\mathcal{J}$ on IF_n. Since IF_n is an inverse subsemigroup of I_n, for $\alpha, \beta \in IF_n$, it holds:

(1) $\alpha\mathcal{L}\beta$ if and only if $\mathrm{im}\ \alpha = \mathrm{im}\ \beta$.
(2) $\alpha\mathcal{R}\beta$ if and only if $\mathrm{dom}\ \alpha = \mathrm{dom}\ \beta$.
(3) $\alpha\mathcal{H}\beta$ if and only if $\mathrm{dom}\ \alpha = \mathrm{dom}\ \beta$ and $\mathrm{im}\ \alpha = \mathrm{im}\ \beta$.

It remains to describe the relation $\mathcal{J}$, since this relation is different for the semigroups I_n and IF_n. For example, let

$$\alpha = \begin{pmatrix} 1 & 4 & 5 & 6 \\ 2 & 6 & 5 & 4 \end{pmatrix}, \quad \beta = \begin{pmatrix} 1 & 2 & 5 & 6 \\ 5 & 6 & 1 & 2 \end{pmatrix} \in IF_6.$$

Then $\mathrm{rank}\ \alpha = \mathrm{rank}\ \beta$, but α and β are not $\mathcal{J}$ related.

Definition 2.1. For $Y \subseteq X$, let Y_S be the set of all subsets

$$\{a_i, a_{i+1}, \ldots, a_{i+r}\}\ (i, r \in \{1, \ldots, n\})$$

of Y such that $a_{i-1} \notin Y$ (or $i = 1$) and $a_{i+r+1} \notin Y$ (or $i + r = n$).

Definition 2.2. Let $\alpha \in IF_n$ and let $k \in \mathbb{N}$. Then we put

$$\alpha(k) := \{A \in (\mathrm{dom}\ \alpha)_S : |A| = k\},$$

$$\alpha^o(2k+1) := \{\{a_i, \ldots, a_{i+2k}\} \in \alpha(2k+1) : i \in 2\mathbb{N} - 1\}.$$

Note that $\alpha^o(2k+1) \subseteq \alpha(2k+1)$.

For a set M of natural numbers, let $\max M$ (let $\min M$) be the greatest (the least) natural number in M with respect to the natural order in $\mathbb{N}$.

Proposition 2.3. *Let $\alpha, \beta \in IF_n$. Then the following statements are equivalent*:

(i) $\alpha\mathcal{J}\beta$.
(ii) $|\alpha(k)| = |\beta(k)|$ *and* $|\alpha^o(2k+1)| = |\beta^o(2k+1)|$ *for all* $k \in \mathbb{N}$.

Proof. Suppose that $\alpha\mathcal{J}\beta$. Then there are $\gamma, \delta, \gamma_1, \delta_1, \in IF_n$ such that $\beta = \gamma\alpha\delta$ and $\alpha = \gamma_1\beta\delta_1$. We have $\mathrm{rank}\ \alpha = \mathrm{rank}\ \beta$ since $IF_n \leq I_n$. Then from $\alpha = \gamma_1\beta\delta_1$ and $\beta = \gamma\alpha\delta$, we obtain $|(\mathrm{dom}\ \alpha)_S| = |(\mathrm{dom}\ \beta)_S|$, and in particular, $|\alpha(k)| = |\beta(k)|$ for all $k \in \mathbb{N}$. Moreover, if $k \in \mathbb{N}$ and $B \in \beta(k)$, then we observe $B\gamma \in (\mathrm{dom}\ \alpha)_S$ and thus $B\gamma \in \alpha(k)$.

Let $k \in 2\mathbb{N}+1$ and $B := \{a_i, \dots, a_{i+k-1}\} \in \beta^o(k)$ (for some $i \in 2\mathbb{N}-1$). We have $B\gamma \in \alpha(k)$ and we will show that $B\gamma \in \alpha^o(k)$. Since i is odd, we have $a_i <_f a_{i+1} >_f \cdots <_f a_{i+k-2} >_f a_{i+k-1}$. This implies $a_i\gamma <_f a_{i+1}\gamma >_f \cdots <_f a_{i+k-2}\gamma >_f a_{i+k-1}\gamma$ and there is $l \in \{1, \dots, n\}$ with either $a_i\gamma = a_l$ and $a_{i+k-1}\gamma = a_{l+k-1}$ or $a_i\gamma = a_{l+k-1}$ and $a_{i+k-1}\gamma = a_l$. This gives $a_l <_f a_{l+1}$ and $l \in 2\mathbb{N}-1$, and consequently, $B\gamma \in \alpha^o(k)$. This shows $|\beta^o(k)| \leq |\alpha^o(k)|$. Dually, we can verify the converse inequation. Thus, $|\alpha^o(k)| = |\beta^o(k)|$.

Conversely, let $|\alpha(k)| = |\beta(k)|$ and $|\alpha^o(2k+1)| = |\beta^o(2k+1)|$ for all $k \in \mathbb{N}$. Then for all $k \in \mathbb{N}$, there is a bijection $f_k : \beta(k) \to \alpha(k)$ such that $f_{2k+1}(B) \in \alpha^o(2k+1)$ for all $B \in \beta^o(2k+1)$. We define now a mapping $\gamma : \text{dom } \beta \to \text{dom } \alpha$. For $k \in \mathbb{N}$, $B = \{a_i, \dots, a_{i+k-1}\} \in \beta(k)$ and $f_k(B) = \{a_l, \dots, a_{l+k-1}\}$ (with $i, l \in \{1, \dots, n\}$) let

$$a_{i+r}\gamma := \begin{cases} a_{l+r}, & \text{if } k = 1 \text{ or } i \text{ and } l \text{ have the same parity,} \\ a_{l+k-(r+1)}, & \text{otherwise} \end{cases}$$

for $0 \leq r \leq k-1$. The mapping γ is well defined since $\text{dom } \beta = \bigcup_{j=1}^{p} \beta(j)$, where $p := \max\{k \in \mathbb{N} : \beta(k) \neq \emptyset\}$.

We have to show that $\gamma \in IF_n$. For this let again $B = \{a_i, \dots, a_{i+k-1}\} \in \beta(k)$ and $\{a_l, \dots, a_{l+k-1}\} = f_k(B)$ for some $i, l, k \in \{1, \dots, n\}$.

We consider here the case $i \in 2\mathbb{N}-1$, the case $i \in 2\mathbb{N}$ can be handled in the same matter. Suppose that $k \in 2\mathbb{N}+1$. Then $B \in \beta^o(k)$ and $f_k(B) \in \alpha^o(k)$, i.e. $l \in 2\mathbb{N}-1$. Since i and l are odd, we have $a_i <_f a_{i+1} >_f \cdots <_f a_{i+k-2} >_f a_{i+k-1}$ and $a_l <_f a_{l+1} >_f \cdots <_f a_{l+k-2} >_f a_{l+k-1}$, i.e. $a_i\gamma <_f a_{i+1}\gamma >_f \cdots <_f a_{i+k-2}\gamma >_f a_{i+k-1}\gamma$.

Now, suppose that $k \in 2\mathbb{N}$. Since i is odd, we have $a_i <_f a_{i+1} >_f \cdots >_f a_{i+k-2} <_f a_{i+k-1}$. If l is odd, then $a_l <_f a_{l+1} >_f \cdots >_f a_{l+k-2} <_f a_{l+k-1}$, i.e. $a_i\gamma <_f a_{i+1}\gamma >_f \cdots >_f a_{i+k-2}\gamma <_f a_{i+k-1}\gamma$. If l is even then $a_l >_f a_{l+1} <_f \cdots <_f a_{l+k-2} >_f a_{l+k-1}$, i.e. $a_{i+k-1}\gamma >_f a_{i+k-2}\gamma <_f \cdots <_f a_{i+1}\gamma >_f a_i\gamma$.

This shows that $\gamma \in PFI_n$. Let $r \in \{1, \dots, n-1\}$ with $a_r, a_{r+1} \in A$ for some $A \in (\text{im } \gamma)_S$. We observe that $\{B\gamma : B \in (\text{dom } \beta)_S\} = (\text{dom } \alpha)_S$. Thus, there is $B \in (\text{dom } \beta)_S$ such that $B\gamma = A$ and there is $s \in \{1, \dots, n\}$ with $a_s = a_r\gamma^{-1}$. Then $a_{r+1}\gamma^{-1} \in \{a_{s+1}, a_{s-1}\}$. If r is odd then $a_r <_f a_{r+1}$. Assume that $a_r\gamma^{-1} >_f a_{r+1}\gamma^{-1}$. Then s is even, i.e. $a_s\gamma >_f a_{s+1}\gamma$ (if $a_{r+1}\,\gamma^{-1} = a_{s+1}$) and $a_{s-1}\gamma <_f a_s\gamma$ (if $a_{r+1}\,\gamma^{-1} = a_{s-1}$). This gives $a_r >_f a_{r+1}$, a contradiction. If r is even then $a_r >_f a_{r+1}$ and we obtain $a_r\gamma^{-1} >_f a_{r+1}\gamma^{-1}$ by the same arguments. This provides $\gamma^{-1} \in PFI_n$, i.e. $\gamma \in IF_n$.

Finally, we define $\delta : \text{im } \alpha \to \text{im } \beta$ by

$$\delta := \alpha^{-1}\gamma^{-1}\beta.$$

Since $\alpha, \beta, \gamma \in IF_n$, we have $\delta = \alpha^{-1}\gamma^{-1}\beta \in IF_n$.

There holds $\beta = \gamma\alpha\delta$. In fact, for $a \in \text{dom } \beta$, we obtain $a\gamma\alpha\delta = a\gamma\alpha\alpha^{-1}\gamma^{-1}\beta = a\beta$ since $\text{dom } \alpha = \text{im } \gamma$ and $\text{dom } \gamma = \text{dom } \beta$. $\square$

3. Generating Sets

For convenience, we arrange such that X_n is the set of the first positive integers n for some $n \in \mathbb{N}$, i.e. $X_n = \{1, \ldots, n\}$ with

$$1 <_f 2 >_f 3 <_f \cdots n.$$

Clearly, the minimal elements of the fence $\mathbf{F} = (X_n, <_f)$ are odd and maximal elements are even. For $a, b \in X_n$, we will write $a \equiv b \pmod 2$ or shorter $a \equiv_2 b$ if a and b have the same parity. Further, we denote by ε_i the identity mapping on $X_n \backslash \{i\}$ for $i = 1, \ldots, n$, i.e. $\varepsilon_i := id|_{X_n \backslash \{i\}}$.

Notation 3.1. Let us put

$$J := \{\alpha \in IF_n : \text{rank } \alpha \geq n-2\}.$$

The aim of this section is to show that J is a generating set for the semigroup IF_n. Note, $\varepsilon_i^{-1} = \varepsilon_i \in J$ for $1 \leq i \leq n$.

Lemma 3.2. *Let $m, p \in \mathbb{N}$ with $m + p \leq n$ and $m \equiv_2 m + p$. Then*

$$\alpha = \begin{pmatrix} 1 & \cdots & m-2 & m & \cdots & m+p & m+p+2 & \cdots & n \\ 1 & \cdots & m-2 & m+p & \cdots & m & m+p+2 & \cdots & n \end{pmatrix} \in J$$

and $\alpha^{-1} \in J$.

Proof. By simple calculations, one can see that $\alpha \in IF_n$. Since rank $\alpha = n-2$ and $\alpha^{-1} = \alpha$, we obtain $\alpha, \alpha^{-1} \in J$. □

Lemma 3.3. *Let $m, p \in \mathbb{N}$ such that $m + p + 2 \leq n$. Then*

$$\alpha = \begin{pmatrix} 1 & \cdots & m-2 & m & \cdots & m+p & m+p+4 & \cdots & n \\ 1 & \cdots & m-2 & m+2 & \cdots & m+p+2 & m+p+4 & \cdots & n \end{pmatrix} \in \langle J \rangle$$

and $\alpha^{-1} \in \langle J \rangle$.

Proof. We have to consider two cases.

(1) Suppose that p is even. Then $m \equiv_2 m + p$ and we consider the following transformations with rank $\geq n-2$:

$$\beta_1 = \begin{pmatrix} 1 & \cdots & m-2 & m & \cdots & m+p+2 & m+p+4 & \cdots & n \\ 1 & \cdots & m-2 & m+p+2 & \cdots & m & m+p+4 & \cdots & n \end{pmatrix}$$

and

$$\beta_2 = \begin{pmatrix} 1 & \cdots & m & m+2 & \cdots & m+p+2 & m+p+4 & \cdots & n \\ 1 & \cdots & m & m+p+2 & \cdots & m+2 & m+p+4 & \cdots & n \end{pmatrix}.$$

Clearly, $\beta_1, \beta_2 \in J$ by Lemma 3.2 and it is easy to verify that

$$\alpha = \beta_1 \beta_2 \varepsilon_m \text{ and } \alpha^{-1} = \varepsilon_m \beta_2 \beta_1,$$

where $\varepsilon_m \in J$. Thus, we obtain $\alpha, \alpha^{-1} \in \langle J \rangle$.

(2) Now suppose that p is odd. Then $m \not\equiv_2 m+p$ and we consider the following transformations with rank $\geq n-2$:

$$\beta_3 = \begin{pmatrix} 1 & \cdots & m-2 & m & \cdots & m+p+1 & m+p+3 & \cdots & n \\ 1 & \cdots & m-2 & m+p+1 & \cdots & m & m+p+3 & \cdots & n \end{pmatrix}$$

and

$$\beta_4 = \begin{pmatrix} 1 & \cdots & m-1 & m+1 & \cdots & m+p+2 & m+p+4 & \cdots & n \\ 1 & \cdots & m-1 & m+p+2 & \cdots & m+1 & m+p+4 & \cdots & n \end{pmatrix}.$$

Clearly, $\beta_3, \beta_4 \in J$ by Lemma 3.2 and it is easy to verify that

$$\alpha = \beta_3\beta_4 \text{ and } \alpha^{-1} = \beta_4\beta_3.$$

Thus, $\alpha, \alpha^{-1} \in \langle J \rangle$. $\square$

Corollary 3.4. *Let $m, p, k \in \mathbb{N}$ such that $m+p+2k \leq n$. Then*

$$\alpha = \begin{pmatrix} 1 & \cdots & m-2 & m & \cdots & m+p & m+p+2k+2 & \cdots & n \\ 1 & \cdots & m-2 & m+2k & \cdots & m+p+2k & m+p+2k+2 & \cdots & n \end{pmatrix} \in \langle J \rangle$$

and $\alpha^{-1} \in \langle J \rangle$.

Proof. For $0 \leq i < k$ we define the transformations

$$\beta_i = \begin{pmatrix} 1 & \cdots & m+2i-2 & m+2i & \cdots & m+p+2i & m+p+2i+4 & \cdots & n \\ 1 & \cdots & m+2i-2 & m+2i+2 & \cdots & m+p+2i+2 & m+p+2i+4 & \cdots & n \end{pmatrix}.$$

Note that $\beta_i, \beta_i^{-1} \in \langle J \rangle$ $(0 \leq i < k)$ by Lemma 3.3. It is easy to verify that $\alpha = \beta_0 \cdots \beta_{k-1} \in \langle J \rangle$ and $\alpha^{-1} = \beta_{k-1}^{-1} \cdots \beta_0^{-1} \in \langle J \rangle$. $\square$

Lemma 3.5. *Let $m, p \in \mathbb{N}$ such that p is odd and $m+p+1 \leq n$. Then*

$$\alpha = \begin{pmatrix} 1 & \cdots & m-2 & m & \cdots & m+p & m+p+3 & \cdots & n \\ 1 & \cdots & m-2 & m+p+1 & \cdots & m+1 & m+p+3 & \cdots & n \end{pmatrix} \in \langle J \rangle$$

and $\alpha^{-1} \in \langle J \rangle$.

Proof. We define a transformation

$$\beta_1 = \begin{pmatrix} 1 & \cdots & m-2 & m & \cdots & m+p+1 & m+p+3 & \cdots & n \\ 1 & \cdots & m-2 & m+p+1 & \cdots & m & m+p+3 & \cdots & n \end{pmatrix}.$$

Clearly, $\beta_1 \in J$ by Lemma 3.2. Then we can verify that $\alpha = \beta_1\varepsilon_m \in \langle J \rangle$ and $\alpha^{-1} = \varepsilon_m\beta_1 \in \langle J \rangle$. $\square$

Corollary 3.6. *Let $m, p, k \in \mathbb{N}$ such that p is odd and $m+p+2k-1 \leq n$. Then*

$$\alpha = \begin{pmatrix} 1 & \cdots & m-2 & m & \cdots & m+p & m+p+2k+1 & \cdots & n \\ 1 & \cdots & m-2 & m+p+2k-1 & \cdots & m+2k-1 & m+p+2k+1 & \cdots & n \end{pmatrix} \in \langle J \rangle$$

and $\alpha^{-1} \in \langle J \rangle$.

Proof. Let

$$\beta_1 = \begin{pmatrix} 1 & \cdots & m-2 & m & \cdots & m+p & m+p+2k & \cdots & n \\ 1 & \cdots & m-2 & m+2k-2 & \cdots & m+p+2k-2 & m+p+2k & \cdots & n \end{pmatrix}$$

and

$$\beta_2 = \begin{pmatrix} 1 & \cdots & m+2k-4 & m+2k-2 & \cdots & m+p+2k-2 & m+p+2k+1 & \cdots & n \\ 1 & \cdots & m+2k-4 & m+p+2k-1 & \cdots & m+2k-1 & m+p+2k+1 & \cdots & n \end{pmatrix}.$$

Note that $\beta_1 \in \langle J \rangle$ (by Corollary 3.4) and $\beta_2 \in \langle J \rangle$ (by Lemma 3.5). It is easy to verify that $\alpha = \beta_1\beta_2$ and $\alpha^{-1} = \beta_2^{-1}\beta_1^{-1}$, and thus $\alpha, \alpha^{-1} \in \langle J \rangle$. □

Lemma 3.7. *Let $m, p, k \in \mathbb{N}$ such that p is even and $m + p + 2k \leq n$. Then*

$$\alpha = \begin{pmatrix} 1 & \cdots & m-2 & m & \cdots & m+p & m+p+2k+2 & \cdots & n \\ 1 & \cdots & m-2 & m+p+2k & \cdots & m+2k & m+p+2k+2 & \cdots & n \end{pmatrix} \in \langle J \rangle$$

and $\alpha^{-1} \in \langle J \rangle$.

Proof. Let

$$\beta_1 = \begin{pmatrix} 1 & \cdots & m-2 & m & \cdots & m+p & m+p+2k+2 & \cdots & n \\ 1 & \cdots & m-2 & m+2k & \cdots & m+p+2k & m+p+2k+2 & \cdots & n \end{pmatrix}$$

and

$$\beta_2 = \begin{pmatrix} 1 & \cdots & m+2k-2 & m+2k & \cdots & m+p+2k & m+p+2k+2 & \cdots & n \\ 1 & \cdots & m+2k-2 & m+p+2k & \cdots & m+2k & m+p+2k+2 & \cdots & n \end{pmatrix}.$$

Note that $\beta_1 \in \langle J \rangle$ (by Corollary 3.4) and $\beta_2 \in J$ (by Lemma 3.2). It is easy to verify that $\alpha = \beta_1\beta_2$ and $\alpha^{-1} = \beta_2\beta_1^{-1}$, and thus $\alpha, \alpha^{-1} \in \langle J \rangle$. □

Lemma 3.8. *Let $Y \subseteq X_n$. Then $id|_{X_n \setminus Y} \in \langle J \rangle$.*

Proof. If $Y = \emptyset$, i.e. $X_n \setminus Y = X_n$, then $id|_{X_n} = id \in J$. Let $\emptyset \neq Y := \{i_1, \ldots, i_k\} \subseteq X_n$ with $k \in \{1, \ldots, n\}$. Then it is easy to verify that $id|_{X_n \setminus Y} = \varepsilon_{i_1} \cdots \varepsilon_{i_k} \in \langle J \rangle$. □

Proposition 3.9. *Let $\alpha \in IF_n$. Then there are transformations $\eta_1, \ldots, \eta_k, \eta_{k+1}, \ldots, \eta_l \in J$ $(k < l \in \mathbb{N})$ such that $\eta_1^{-1}, \ldots, \eta_k^{-1}, \eta_{k+1}^{-1}, \ldots, \eta_l^{-1} \in J$,* dom $\alpha \subseteq$ im $(\eta_1 \cdots \eta_k)$, im $\alpha \subseteq$ dom $(\eta_{k+1} \cdots \eta_l)$ *and $x(\eta_1 \ldots \eta_k \alpha \eta_{k+1} \ldots \eta_l) \equiv_2 x$ for all $x \in$* dom $(\eta_1 \ldots \eta_k \alpha \eta_{k+1} \ldots \eta_l)$.

Proof. If $a \equiv_2 a\alpha$ for all $a \in \text{dom } \alpha$ then $id|_{\text{dom } \alpha}\alpha\ id|_{\text{im } \alpha} = \alpha$. This shows the assertion, since $id|_{\text{dom } \alpha}, id|_{\text{im } \alpha} \in \langle J \rangle$ by Lemma 3.8.

Let $a \in \text{dom } \alpha$ such that $a \not\equiv_2 a\alpha$. Then it is clear that $a - 1, a + 1 \notin \text{dom } \alpha$.

If a is even then we put

$$\eta = \begin{pmatrix} 1 & 3 & \cdots & a & a+2 & \cdots & n \\ a & 1 & \cdots & a-2 & a+2 & \cdots & n \end{pmatrix} \in J.$$

We observe that $\eta^{-1} \in \langle J \rangle$. Moreover, it is easy to see that $\text{im } \alpha = \text{im } (\eta\alpha)$, $x\alpha^{-1} \equiv_2 x(\eta\alpha)^{-1}$ for all $x \in \text{im } \alpha \backslash \{a\alpha\}$ and $a\alpha\alpha^{-1} = a \not\equiv_2 1 = a\eta^{-1} = a\alpha(\eta\alpha)^{-1}$. This shows that

$$|\{x \in \text{im } \alpha : x \not\equiv_2 x(\eta\alpha)^{-1}\}| = |\{x \in \text{im } \alpha : x \not\equiv_2 x\alpha^{-1}\}| - 1.$$

If a is odd, then $a\alpha$ is even and we put

$$\eta = \begin{pmatrix} 1 & \cdots & a\alpha-2 & a\alpha & a\alpha+2 & \cdots & n \\ 3 & \cdots & a\alpha & 1 & a\alpha+2 & \cdots & n \end{pmatrix} \in J$$

with $\eta^{-1} \in \langle J \rangle$. By dual arguments, we obtain

$$|\{x \in \text{dom } \alpha : x \not\equiv_2 x(\alpha\eta)\}| = |\{x \in \text{dom } \alpha : x \not\equiv_2 x\alpha\}| - 1.$$

Continuing in this way, starting with the even cases, we obtain transformations $\eta_1, \ldots, \eta_k, \eta_{k+1}, \ldots, \eta_l \in J$ $(k < l \in \mathbb{N})$ such that $\eta_1^{-1}, \ldots, \eta_k^{-1}, \eta_{k+1}^{-1}, \ldots, \eta_l^{-1} \in J$ and $x(\eta_1 \ldots \eta_k \alpha \eta_{k+1} \ldots \eta_l) \equiv_2 x$ for all $x \in \text{dom } (\eta_1 \ldots \eta_k \alpha \eta_{k+1} \ldots \eta_l)$. $\square$

Notation 3.10. Let $\alpha \in PFI_n$ and let $A, B \in (\text{dom } \alpha)_S$ (or $A, B \in (\text{im } \alpha)_S$). Then we write $A < B$ if all elements in A are less than any element in B with respect to the natural order of $\mathbb{N}$. Further, we write

$$A \prec B$$

if $A < B$ and for each $C \in (\text{dom } \alpha)_S$ (for each $C \in (\text{im } \alpha)_S$, respectively) the following implication holds: $A \leq C \leq B \Rightarrow A = C$ or $B = C$.

Any transformation $\alpha \in IF_n$ with $a \equiv_2 a\alpha$ for all $a \in \text{dom } \alpha$ can be written in the following form:

Notation 3.11. Let

$$\alpha = \begin{pmatrix} A_1 & \prec \cdots \prec & A_{i-1} & \prec & A_i & \prec \cdots \prec & A_p \\ A_1 & \prec \cdots \prec & A_{i-1} & < & B_i & \cdots & B_p \end{pmatrix} \in IF_n$$

with $i \leq p \in \{1, \ldots, n\}$, and $a \equiv_2 a\alpha$ for all $a \in \text{dom } \alpha$ such that $i = 1$ or

(i) $a\alpha = a$ for all $a \in A_1 \cup \cdots \cup A_{i-1}$ and
(ii) $A_{i-1} < B_l$ for all $l \in \{i, \ldots, p\}$.

Further, let

$$r_j := \min A_j, \quad s_j := \max A_j, \quad t_j := \min B_j, \quad u_j := \max B_j$$

for $1 \leq j \leq p$.

Proposition 3.12. *Let α be as in Notation 3.11. Then there exist $\omega_1, \omega_2 \in \langle J\rangle$ with $\omega_1^{-1}, \omega_2^{-1} \in \langle J\rangle$, $\mathrm{dom}\ \alpha \subseteq \mathrm{im}\ \omega_1$, $\mathrm{im}\ \alpha \subseteq \mathrm{dom}\ \omega_2$ such that $\omega_1\alpha\omega_2$ has the form*

$$\omega_1\alpha\omega_2 = \begin{pmatrix} A_1 & \prec\cdots\prec & A_{i-1} & \prec & A'_i & \prec & A'_{i+1} & \prec\cdots\prec & A'_p \\ A_1 & \prec\cdots\prec & A_{i-1} & \prec & B'_i & < & B'_{i+1} & \cdots & B'_p \end{pmatrix} \in IF_n$$

with $a \equiv_2 a(\omega_1\alpha\omega_2)$ for all $a \in \mathrm{dom}\ (\omega_1\alpha\omega_2)$, and $B'_i < B'_l$ for all $l \in \{i+1, \ldots, p\}$ such that $i = 1$ or $a(\omega_1\alpha\omega_2) = a$ for all $a \in A_1 \cup \cdots \cup A_{i-1}$.

Proof. We will define the transformations ω_1 and ω_2 with $\mathrm{dom}\ \alpha \subseteq \mathrm{im}\ \omega_1$ and $\mathrm{im}\ \alpha \subseteq \mathrm{dom}\ \omega_2$ such that $\omega_1\alpha\omega_2$ is the required mapping of our assertion. The concrete calculations we leave to the reader.

Let $k \in \{i, \ldots, p\}$ such that $A_{i-1} \prec B_k$ if $i > 1$, and $B_k < B_l$ for all $l \in \{1, \ldots, p\}\backslash\{k\}$ if $i = 1$, respectively. Note that if $k = i$ then $\omega_1 = \omega_2 = id \in J$. Thus, let $k > i$. Then we consider the following seven cases. Note that the cases are not mutually exclusive (i.e. the transformation α can satisfy more than one case), but cover all the possibilities.

(1) If $r_i \equiv_2 s_k$, then we put $\omega_2 = id$ and

$$\omega_1 = \begin{pmatrix} 1 & \cdots & r_i - 2 & r_i & \cdots & s_k & s_k + 2 & \cdots & n \\ 1 & \cdots & r_i - 2 & s_k & \cdots & r_i & s_k + 2 & \cdots & n \end{pmatrix}.$$

(2) If $r_i \not\equiv_2 s_k$ and $r_i - 2 \notin \mathrm{dom}\ \alpha$ (or $r_i - 1 = 1$), then we put $\omega_2 = id$ and

$$\omega_1 = \begin{pmatrix} 1 & \cdots & r_i - 3 & r_i - 1 & \cdots & s_k & s_k + 2 & \cdots & n \\ 1 & \cdots & r_i - 3 & s_k & \cdots & r_i - 1 & s_k + 2 & \cdots & n \end{pmatrix}.$$

(3) If $r_i \not\equiv_2 s_k$ and $s_k + 2 \notin \mathrm{dom}\ \alpha$ (or $s_k + 1 = n$), then we put $\omega_2 = id$ and

$$\omega_1 = \begin{pmatrix} 1 & \cdots & r_i - 2 & r_i & \cdots & s_k + 1 & s_k + 3 & \cdots & n \\ 1 & \cdots & r_i - 2 & s_k + 1 & \cdots & r_i & s_k + 3 & \cdots & n \end{pmatrix}.$$

(4) If $u_i \equiv_2 t_k$, then we put $\omega_1 = id$ and

$$\omega_2 = \begin{pmatrix} 1 & \cdots & t_k - 2 & t_k & \cdots & u_i & u_i + 2 & \cdots & n \\ 1 & \cdots & t_k - 2 & u_i & \cdots & t_k & u_i + 2 & \cdots & n \end{pmatrix}.$$

(5) If $u_i \not\equiv_2 t_k$ and $t_k - 2 \notin \mathrm{im}\ \alpha$ (or $t_k - 1 = 1$), then we put $\omega_1 = id$ and

$$\omega_2 = \begin{pmatrix} 1 & \cdots & t_k - 3 & t_k - 1 & \cdots & u_i & u_i + 2 & \cdots & n \\ 1 & \cdots & t_k - 3 & u_i & \cdots & t_k - 1 & u_i + 2 & \cdots & n \end{pmatrix}.$$

(6) If $u_i \not\equiv_2 t_k$ and $u_i + 2 \notin \mathrm{im}\ \alpha$ (or $u_i + 1 = n$), then we put $\omega_1 = id$ and

$$\omega_2 = \begin{pmatrix} 1 & \cdots & t_k - 2 & t_k & \cdots & u_i + 1 & u_i + 3 & \cdots & n \\ 1 & \cdots & t_k - 2 & u_i + 1 & \cdots & t_k & u_i + 3 & \cdots & n \end{pmatrix}.$$

Clearly, dom $\alpha \subseteq$ im ω_1, im $\alpha \subseteq$ dom ω_2, and $\omega_1, \omega_2 \in J$ (by Lemma 3.2) for all cases 1–6.

(7) It remains the case $r_i \not\equiv_2 s_k$ and $u_i \not\equiv_2 t_k$ and $r_i - 2, s_k + 2 \in$ dom α and $t_k - 2, u_i + 2 \in$ im α, where $1 = r_1 \in$ dom α and $1 = t_k \in$ im α in the case $i = 1$.

(7.1) Let $k = i + 1$. First, we will show that $r_i = t_{i+1}$. In the case $i = 1$, it is clear. For the case $i > 1$, we have that $t_{i+1} = u_{i-1} + 2$ (since $A_{i-1} \prec B_{i+1}$ and $t_{i+1} - 2 \in$ im α), $r_i = s_{i-1} + 2$ (since $A_{i-1} < A_i$ and $r_i - 2 \in$ dom α) and $u_{i-1} + 2 = s_{i-1} + 2$ (since $a\alpha = a$ for all $a \in A_1 \cup \cdots \cup A_{i-1}$). Altogether, we obtain $r_i = t_{i+1}$. Since $r_i \not\equiv_2 s_{i+1}$, we have $r_i = t_{i+1} = r_{i+1}\alpha \equiv_2 r_{i+1}$. Thus, we get $r_{i+1} \not\equiv_2 s_{i+1}$ and we put $\omega_1 = \eta_1\eta_2$ and $\omega_2 = id$, where

$$\eta_1 = \begin{pmatrix} 1 & \cdots & r_i - 2 & r_i & \cdots & s_{i+1} - 1 & s_{i+1} + 1 & \cdots & n \\ 1 & \cdots & r_i - 2 & s_{i+1} - 1 & \cdots & r_i & s_{i+1} + 1 & \cdots & n \end{pmatrix}$$

$$\eta_2 = \begin{pmatrix} 1 & \cdots & r_{i+1} - 3 & r_{i+1} - 1 & \cdots & s_{i+1} - 1 & s_{i+1} + 2 & \cdots & n \\ 1 & \cdots & r_{i+1} - 3 & s_{i+1} & \cdots & r_{i+1} & s_{i+1} + 2 & \cdots & n \end{pmatrix}.$$

Clearly, $\eta_1 \in J$ by Lemma 3.2 and $\eta_2 \in \langle J \rangle$ by Lemma 3.5. Note that $r_{i+1} - 2 \notin$ dom α, since otherwise $s_i = r_{i+1} - 2 \equiv_2 r_{i+1} \equiv_2 r_i$ implies $u_i \equiv_2 r_i = t_{i+1}$ which is a contradiction. Thus, it is easy to verify that dom $\alpha \subseteq$ im ω_1.

(7.2) Let $k > i + 1$. We define a transformation τ as following:

(a) If $r_{i+1} \equiv_2 s_k$, then we put

$$\tau = \begin{pmatrix} 1 & \cdots & r_{i+1} - 2 & r_{i+1} & \cdots & s_k & s_k + 2 & \cdots & n \\ 1 & \cdots & r_{i+1} - 2 & s_k & \cdots & r_{i+1} & s_k + 2 & \cdots & n \end{pmatrix}.$$

(b) If $r_{i+1} \not\equiv_2 s_k$, i.e. $r_i \equiv_2 r_{i+1}$, then we put

$$\tau = \begin{pmatrix} 1 & \cdots & r_{i+1} - 3 & r_{i+1} - 1 & \cdots & s_k & s_k + 2 & \cdots & n \\ 1 & \cdots & r_{i+1} - 3 & s_k & \cdots & r_{i+1} - 1 & s_k + 2 & \cdots & n \end{pmatrix}.$$

By Lemma 3.2, we have $\tau, \tau^{-1} \in J$. We have to verify that $r_{i+1} - 2 \notin$ dom α. Assume the opposite that $r_{i+1} - 2 \in$ dom α. Then $s_i = r_{i+1} - 2$ and thus $s_i = r_{i+1} - 2 \equiv_2 r_{i+1} \equiv_2 r_i$. Therefore, we have $r_i \equiv_2 s_i \equiv_2 t_i \equiv_2 u_i$. Moreover, we have $r_i = t_k = 1$ in the case $i = 1$. If $i > 1$ then $u_{i-1} = s_{i-1} = r_i - 2 \equiv_2 r_i$ and $u_{i-1} \equiv_2 u_{i-1} + 2 = t_k$ implies $r_i \equiv_2 t_k$. Thus, we obtain $u_i \equiv_2 t_k$, a contradiction. Hence, dom $\alpha \subseteq$ im τ.

Now, we consider the transformation

$$\tau\alpha = \begin{pmatrix} A_1 & \cdots & A_{i-1} & A_i & A_{i+1}^* & \cdots & A_k^* & A_{k+1} & \cdots & A_p \\ A_1 & \cdots & A_{i-1} & B_i & B_{i+1}^* & \cdots & B_k^* & B_{k+1} & \cdots & B_p \end{pmatrix} \in IF_n$$

with $A_{i-1} \prec B_{i+1}^*$. For this transformation, we have the case 7.1. with corresponding transformations $\eta_1, \eta_2 \in \langle J \rangle$. Then we put $\omega_1 = \eta_1\eta_2\tau$ and $\omega_2 = id$ with $\omega_1^{-1}, \omega_2^{-1} \in \langle J \rangle$, dom $\alpha \subseteq$ im ω_1 and im $\alpha \subseteq$ dom ω_2. $\square$

Proposition 3.13. *Let α be as in Notation* 3.11 *with $A_{i-1} \prec B_i$. Then there exist $\omega_1, \omega_2 \in \langle J\rangle$ such that $\omega_1^{-1}, \omega_2^{-1} \in \langle J\rangle$, $\mathrm{dom}\ \alpha \subseteq \mathrm{im}\ \omega_1$, $\mathrm{im}\ \alpha \subseteq \mathrm{dom}\ \omega_2$, and*

$$\omega_1\alpha\omega_2 = \begin{pmatrix} A_1 & \prec\cdots\prec & A_{i-1} & \prec & A_i' & \prec & A_{i+1}' & \prec\cdots\prec & A_p' \\ A_1 & \prec\cdots\prec & A_{i-1} & \prec & A_i' & < & B_{i+1}' & \cdots & B_p' \end{pmatrix} \in IF_n$$

with $a(\omega_1\alpha\omega_2) = a$ for all $a \in (A_1 \cup \cdots \cup A_{i-1} \cup A_i')$.

Proof. If $a\alpha = a$ for all $a \in A_i$, then $\omega_1 = \omega_2 = id$. Let $a\alpha \neq a$ for some $a \in A_i$. Then we put

$$\eta_1 = \begin{pmatrix} 1 & \cdots & t_i - 2 & r_i\alpha & \cdots & s_i\alpha & s_i + 2 & \cdots & n \\ 1 & \cdots & t_i - 2 & r_i & \cdots & s_i & s_i + 2 & \cdots & n \end{pmatrix}, \quad \text{if } r_i \geq t_i$$

and

$$\eta_2 = \begin{pmatrix} 1 & \cdots & r_i - 2 & r_i\alpha & \cdots & s_i\alpha & u_i + 2 & \cdots & n \\ 1 & \cdots & r_i - 2 & r_i & \cdots & s_i & u_i + 2 & \cdots & n \end{pmatrix}, \quad \text{if } r_i \leq t_i.$$

Clearly, $\eta_1, \eta_2, \eta_1^{-1}, \eta_2^{-1} \in \langle J\rangle$ by Corollary 3.4 (if $r_i\alpha = t_i$) or Corollary 3.6 (if $r_i\alpha = u_i$ and $r_i \not\equiv_2 s_i$) or Lemma 3.7 (if $r_i\alpha = u_i$ and $r_i \equiv_2 s_i$). If $r_i \geq t_i$ then $\mathrm{dom}\ \alpha \subseteq \mathrm{im}\ \eta_1$ and we put $\omega_1 = \eta_1$ and $\omega_2 = id$. If $r_i \leq t_i$ then $\mathrm{im}\ \alpha \subseteq \mathrm{dom}\ \eta_2$ and we put $\omega_1 = id$ and $\omega_2 = \eta_2$. □

From Propositions 3.9, 3.12 and 3.13 (frequently used), we obtain

Corollary 3.14. *Let $\alpha \in IF_n$. Then there exist $\omega_1, \omega_2 \in \langle J\rangle$ such that $\omega_1^{-1}, \omega_2^{-1} \in \langle J\rangle$, $\mathrm{dom}\ \alpha \subseteq \mathrm{im}\ \omega_1$, $\mathrm{im}\ \alpha \subseteq \mathrm{dom}\ \omega_2$, and $a(\omega_1\alpha\omega_2) = a$ for all $a \in \mathrm{dom}\ (\omega_1\alpha\omega_2)$.*

Theorem 3.15. *$IF_n = \langle J\rangle$.*

Proof. Let $\alpha \in IF_n$. Then by Corollary 3.14, there exist $\omega_1, \omega_2 \in \langle J\rangle$ such that $\omega_1^{-1}, \omega_2^{-1} \in \langle J\rangle$, $\mathrm{dom}\ \alpha \subseteq \mathrm{im}\ \omega_1$, $\mathrm{im}\ \alpha \subseteq \mathrm{dom}\ \omega_2$, and $a(\omega_1\alpha\omega_2) = a$ for all $a \in \mathrm{dom}\ (\omega_1\alpha\omega_2)$. Therefore, we have

$$\omega_1\alpha\omega_2 = \varepsilon_{i_1} \cdots \varepsilon_{i_k} \in \langle J\rangle$$

(by Lemma 3.8), where $\{i_1, \ldots, i_k\} = X_n \backslash \mathrm{dom}\ (\omega_1\alpha\omega_2)$, $k \in \{1, \ldots, n\}$.

Finally, we obtain $\alpha \in \langle J\rangle$, since $\alpha = \omega_1^{-1}\omega_1\alpha\omega_2\omega_2^{-1}$. □

4. Rank of the Semigroup IF_n for Even n

Let $n \in 2\mathbb{N}+1$. Using the GAP software, we have observed that IF_n is not generated by the set $\{\alpha \in IF_n : \mathrm{rank}\ \alpha \geq n-1\}$. Moreover, there is no least generating set for IF_n. But, in the case n is even the situation is different. There is a least generating set and all its elements have rank $\geq n - 1$.

Throughout this section, let $n \in 2\mathbb{N}$ and let X_n be again the up-fence $1 <_f 2 >_f \cdots <_f n$. We describe the least generating set and calculate the rank of the semigroup IF_n.

Notation 4.1. We put

$$\sigma_1 := \begin{pmatrix} 1 & 3 & \cdots & n \\ n & 1 & \cdots & n-2 \end{pmatrix};$$

$$\sigma_2 := \sigma_1^{-1} = \begin{pmatrix} 1 & \cdots & n-2 & n \\ 3 & \cdots & n & 1 \end{pmatrix};$$

$$\gamma_i := \begin{pmatrix} 1 & \cdots & i-1 & i+1 & \cdots & n \\ i-1 & \cdots & 1 & i+1 & \cdots & n \end{pmatrix} \quad \text{for } i \in 2\mathbb{N}, 4 \leq i \leq n;$$

$$\delta_i := \begin{pmatrix} 1 & \cdots & i-1 & i+1 & \cdots & n \\ 1 & \cdots & i-1 & n & \cdots & i+1 \end{pmatrix} \quad \text{for } i \in 2\mathbb{N}-1, 1 \leq i \leq n-3;$$

$$G := \{id\} \cup \{\sigma_1, \sigma_2\} \cup \{\gamma_i : i \in 2\mathbb{N}, 4 \leq i \leq n\} \cup \{\delta_i : i \in 2\mathbb{N}-1, 1 \leq i \leq n-3\}.$$

Note that $\sigma_2^{-1} = \sigma_1, \quad \gamma_i^{-1} = \gamma_i,$ and $\delta_i^{-1} = \delta_i$.

Theorem 4.2. *$IF_n = \langle G \rangle$.*

Proof. From Theorem 3.15, we have $IF_n = \langle J \rangle$. It remains to show that $J \subseteq \langle G \rangle$. For this, we have to show that all transformations ε_i for $i \in \{1, \ldots, n\}$ as well as all transformations which are used in Propositions 3.9, 3.12 and 3.13 belong to $\langle G \rangle$.

We observe that $\varepsilon_i = \gamma_i\gamma_i$ for $i \in 2\mathbb{N}$, $4 \leq i \leq n$ and $\varepsilon_i = \delta_i\delta_i$ for $i \in 2\mathbb{N}-1$, $1 \leq i \leq n-3$ as well as $\varepsilon_2 = \sigma_1\sigma_2$ and $\varepsilon_{n-1} = \sigma_2\sigma_1$.

For the transformations in Proposition 3.9, we have

$$\eta = \begin{pmatrix} 1 & 3 & \cdots & i & i+2 & \cdots & n \\ i & 1 & \cdots & i-2 & i+2 & \cdots & n \end{pmatrix} = \delta_{i+1}\sigma_1\delta_{i-1} \in \langle G \rangle$$

if $i = a$ is even and $i < n$. If $i = n$ then $\eta = \sigma_1$. Further, we have

$$\eta = \begin{pmatrix} 1 & \cdots & i-2 & i & i+2 & \cdots & n \\ 3 & \cdots & i & 1 & i+2 & \cdots & n \end{pmatrix} = \delta_{i-1}\sigma_2\delta_{i+1} \in \langle G \rangle$$

if $i = a\alpha$ is even.

For the transformations in Proposition 3.12, we put

$$\beta_{i,j} := \begin{pmatrix} 1 & \cdots & i-1 & i+1 & \cdots & j-1 & j+1 & \cdots & n \\ 1 & \cdots & i-1 & j-1 & \cdots & i+1 & j+1 & \cdots & n \end{pmatrix} = \beta_{i,j}^{-1}$$

for $1 \le i < j \le n$ and $i \equiv_2 j$. Clearly, $\beta_{i,j} \in \langle G \rangle$ since

$$\beta_{i,j} = \begin{cases} \delta_i \delta_{n-j+i+1} \delta_i, & \text{if } i \text{ and } j \text{ are odd;} \\ \gamma_j \gamma_{j-i} \gamma_j, & \text{if } i \text{ and } j \text{ are even.} \end{cases}$$

It is easy to verify that ω_1, ω_2, η_1 and τ are all of the form $\beta_{i,j}$ for suitable i and j.

Further, we have

$$\eta_2 = \begin{pmatrix} 1 & \cdots & i-1 & i+1 & \cdots & j-2 & j+1 & \cdots & n \\ 1 & \cdots & i-1 & j-1 & \cdots & i+2 & j+1 & \cdots & n \end{pmatrix}$$

for suitable i and j, and $i+1 \not\equiv_2 j-2$. Clearly, $\eta_2 \in \langle G \rangle$ since $\eta_2 = \beta_{i,j}\varepsilon_{i+1}$.

For the transformations in Proposition 3.13, we have $\eta_1, \eta_2 \in \langle J \rangle$ by Corollary 3.4 (if $r_i\alpha = t_i$) or Corollary 3.6 (if $r_i\alpha = u_i$ and $r_i \not\equiv_2 s_i$) or Lemma 3.7 (if $r_i\alpha = u_i$ and $r_i \equiv_2 s_i$).

For the transformation α in Corollary 3.4, we have

$$\alpha = \begin{pmatrix} 1 & \dots & i-1 & i+1 & \dots & j-1-2k & j+1 & \dots & n \\ 1 & \dots & i-1 & i+1+2k & \dots & j-1 & j+1 & \dots & n \end{pmatrix}$$

with $m = i+1$ and $m+p = j-1-2k$. Hence, we obtain $\alpha \in \langle G \rangle$ since

$$\alpha = \begin{cases} \beta_{i,j}\beta_{i+2k,j}\varepsilon_{i+1}\dots\varepsilon_{i+2k-1}, & \text{if } i \equiv_2 j; \\ \beta_{i,j-1}\beta_{i+2k-1,j}\varepsilon_{i+1}\dots\varepsilon_{i+2k-2}, & \text{if } i \not\equiv_2 j. \end{cases}$$

For the transformation α in Corollary 3.6, we have

$$\alpha = \begin{pmatrix} 1 & \dots & i-1 & i+1 & \dots & j-2k & j+1 & \dots & n \\ 1 & \dots & i-1 & j-1 & \dots & i+2k & j+1 & \dots & n \end{pmatrix}$$

with $m = i+1$ and $m+p = j-2k$. Then we can verify that $\alpha \in \langle G \rangle$ since $\alpha = \beta_{i,j}\varepsilon_{i+1}\dots\varepsilon_{i+2k-1}$.

For the transformation α in Lemma 3.7, we have

$$\alpha = \begin{pmatrix} 1 & \dots & i-1 & i+1 & \dots & j-2k-1 & j+1 & \dots & n \\ 1 & \dots & i-1 & j-1 & \dots & i+2k+1 & j+1 & \dots & n \end{pmatrix}$$

with $m = i+1$ and $m+p = j-2k-1$. We have $\alpha \in \langle G \rangle$ since $\alpha = \beta_{i,j}\varepsilon_{i+1}\dots\varepsilon_{i+2k}$.

$\square$

Proposition 4.3. *The set G is the least generating set for IF_n.*

Proof. Theorem 4.2 shows that G is a generating set for IF_n. Let $\alpha, \beta \in G$ with $\alpha \ne \beta$ and $\{\alpha, \beta\} \ne \{\sigma_1, \sigma_2\}$. It is easy to verify that rank $\alpha\beta = n-2$. Moreover, we observe rank $\sigma_1^2 =$ rank $\sigma_2^2 = n-2$. Let $\alpha = \alpha_1 \cdots \alpha_m$ with $\alpha_1, \dots, \alpha_m \in IF_n$, $2 \le m \in \mathbb{N}$, such that rank $\alpha = n-1$. Without loss of generality, we can assume that $\alpha_i \ne id$ for $1 \le i \le m$. Then $\alpha_1, \dots, \alpha_m \in \{\beta \in IF_n : \text{rank } \beta = n-1\}$. Since G is a generating set for $\{\beta \in IF_n : \text{rank } \beta = n-1\}$, there is $\rho \in G$

such that $\alpha_1, \ldots, \alpha_m \in \{\rho^j : j \in \mathbb{N}\}$ or $\alpha_1, \ldots, \alpha_m \in \{\sigma_1, \sigma_2\}$ with $\alpha_i \neq \alpha_{i+1}$ for $1 \leq i \leq m-1$. This shows that any $\alpha \in G$ cannot be generated by a set without this α. Thus, each generating set of IF_n has to contain G and the assertion is shown.

$\square$

Since $|G| = n+1$ from Theorem 4.2 and Proposition 4.3, we obtain

Theorem 4.4. *Let* $n \in 2\mathbb{N}$. *Then* rank $IF_n = n+1$.

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