DOI 10.1007/s11253-019-01601-2
Ukrainian Mathematical Journal, Vol. 70, No. 11, April, 2019 (Ukrainian Original Vol. 70, No. 11, November, 2018)

FREE PRODUCTS OF n-TUPLE SEMIGROUPS

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UDC 512.579

We construct a free product of arbitrary n-tuple semigroups, introduce the notion of n-bands of n-tuple semigroups and, in terms of this notion, describe the structure of the free product. We also construct a free commutative n-tuple semigroup of any rank and characterize one-generated free commutative n-tuple semigroups. Moreover, we describe the least commutative congruence on a free n-tuple semigroup and prove that the semigroups of the constructed free commutative n-tuple semigroup are isomorphic and that its automorphism group is isomorphic to a symmetric group.

1. Introduction

Recall that a nonempty set G is called an n-tuple semigroup [1] if there are n binary operations (denoted by $\boxed{1}, \boxed{2}, \ldots, \boxed{n}$) defined on this set and satisfying the axioms $(x \boxed{r} y) \boxed{s} z = x \boxed{r} (y \boxed{s} z)$ for all $x, y, z \in G$ and $r, s \in \{1, 2, \ldots, n\}$. It is clear that each semigroup is an n-tuple semigroup for $n = 1$. However, there are many examples of n-tuple semigroups that are not semigroups.

The notion of n-tuple semigroups was used in [1] in the investigation of associative n-tuple algebras. The identities of n-tuple semigroups were used in [2, 3]. Various aspects and properties of n-tuple semigroups were studied by several authors (see, e.g., [4–14]). Thus, n-tuple semigroups are closely connected with doppelsemigroups [4–7], interassociative semigroups [8–10], restrictive bisemigroups [15, 16], commutative dimonoids [11, 12], and commutative trioids [13]. For more historical information, see [17, 18], where the dimonoids and trioids were considered. Examples of n-tuple semigroups with important applications can be found in [14]. Namely, it was shown that every commutative dimonoid (trioid) is a 2-tuple (3-tuple) semigroup. Moreover, the independence of the axioms of n-tuple semigroup was proved and a free n-tuple semigroup was constructed.

In the present paper, we continue the investigations originated in [14].

In Sec. 2, we construct the free product of arbitrary n-tuple semigroups.

In Sec. 3, for the first time, we give an example of an n-band with different operations and, hence, answer an open question posed by the first author in [5]. This enables us to introduce the notion of n-band of n-tuple semigroups that generalizes the well-known notion of a band of semigroups introduced in [19] and proves to be efficient for the description of structural properties of n-tuple semigroups. In terms of the n-band of n-tuple semigroups, we describe the structure of the free product of n-tuple semigroups.

In Sec. 4, we construct a free commutative n-tuple semigroup of arbitrary rank and characterize one-generated free commutative n-tuple semigroups. In addition, it is shown that the semigroups of a free commutative n-tuple semigroup are isomorphic and that its group of automorphisms is isomorphic to a symmetric group.

In Sec. 5, we describe the least commutative congruence on a free n-tuple semigroup and present criteria for the coincidence of operations of an n-tuple semigroup.

The results obtained in the present paper generalize some results from [4].

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Translated from Ukrains'kyi Matematychnyi Zhurnal, Vol. 70, No. 11, pp. 1484–1498, November, 2018. Original article submitted November 27, 2017.

 0041-5995/19/7011–1710

2. Construction of the Free Product

We now construct a free product of arbitrary n-tuple semigroups.

Let R be a class of universal algebras $A_\beta,\ \beta \in \Omega$. A free product in the class R of algebras $A_\beta,\ \beta \in \Omega$, is defined as an algebra A from the class R that contains all A_β as subalgebras and is such that any collection of homeomorphisms of the algebras A_β into any algebra B from R can be extended to a homeomorphism of the algebra A into B. The free product always exists if R is a manifold of universal algebras and each free algebra is a free product of one-generated free algebras. The structures of free products in manifolds of groups, semigroups, dimonoids, etc. are known in the general algebra.

In a natural way, we arrive at the problem of construction of the free product in a manifold of n-tuple semigroups.

As usual, by $\mathbb{N}$ we denote the set of all natural numbers.

The lemma presented below is used in the proof of the main result of this section.

Lemma 1 ([14], Lemma 1). *In the n-tuple semigroup $\big(G, \boxed{1}, \boxed{2}, \ldots, \boxed{n}\big)$, for any $1 < m \in \mathbb{N}$, any $x_i \in G$, $1 \le i \le m+1$, and any $*_j \in \big\{\boxed{1}, \boxed{2}, \ldots, \boxed{n}\big\}$, $1 \le j \le m$, an arbitrary arrangement of brackets in*

$$x_1 *_1 x_2 *_2 \ldots *_m x_{m+1}$$

gives the same element from G.

Let X be an arbitrary nonempty set and let v be an arbitrary word in the alphabet X. By l_v we denote the length of v. By definition, the length of an empty word is equal to 0.

Let $\mathrm{Fr}[T_i]_{i\in I}$ be the free product of arbitrary semigroups $T_i,\ i \in I$. For every $w \in \mathrm{Fr}[T_i]_{i\in I}$, by $w^{(0)}$ (resp., $w^{(1)}$) we denote the first (resp., the last) letter in the word w. We fix $n \in \mathbb{N}$. Let $Y = \{y_1, y_2, \ldots, y_n\}$ be an arbitrary set formed by n elements. By $\overline{n}$ we denote the set $\{1, 2, \ldots, n\}$. Further, let

$$\big\{\big(S_i, \boxed{i_1}, \boxed{i_2}, \ldots, \boxed{i_n}\big)\big\}_{i\in I}$$

be a family of arbitrary n-tuple pairwise disjoint semigroups, let $F^\theta[Y]$ be a free monoid on Y, and let $\theta \in F^\theta[Y]$ be an empty word. For any $j \in \overline{n}$, by $\boxed{j^*}$ we denote an operation on $\mathrm{Fr}\big[\big(S_i, \boxed{i_j}\big)\big]_{i\in I}$. We fix $j \in \overline{n}$ and define n binary operations $\boxed{1}', \boxed{2}', \ldots, \boxed{n}'$ on the set

$$V = \Big\{(w,u) \in \mathrm{Fr}\big[\big(S_i, \boxed{i_j}\big)\big]_{i\in I} \times F^\theta[Y] \ \big|\ l_w - l_u = 1\Big\}$$

by setting

$$(w_1,u_1)\boxed{r}'(w_2,u_2) = \begin{cases} (w_1w_2, u_1y_ru_2), & w_1^{(1)} \in S_k, \quad w_2^{(0)} \in S_m, \quad k,m \in I, \quad k \ne m, \\ \Big(w_1\boxed{r^*}w_2, u_1u_2\Big), & w_1^{(1)}, w_2^{(0)} \in S_k, \quad k \in I, \end{cases}$$

for all $(w_1,u_1),\ (w_2,u_2) \in V$ and $r \in \overline{n}$. These operations are well defined because, for all $r \in \overline{n}$, we have

$$l_{w_1w_2} - l_{u_1y_ru_2} = l_{w_1\boxed{r^*}w_2} - l_{u_1u_2} = 1.$$

The algebra $(V, \boxed{1}', \boxed{2}', \ldots, \boxed{n}')$ is denoted by $\mathrm{Fr}_nT(S_i)_{i\in I}$. It is clear that the structure of $\mathrm{Fr}_nT(S_i)_{i\in I}$ is independent of the choice of j in the definition of V.

Theorem 1. $\mathrm{Fr}_n T(S_i)_{i\in I}$ *is the free product of* n*-tuple semigroups* $(S_i, \boxed{i_1}, \boxed{i_2}, \ldots, \boxed{i_n})$, $i \in I$.

Proof. Let (w_1, u_1), (w_2, u_2), $(w_3, u_3) \in \mathrm{Fr}_n T(S_i)_{i\in I}$ and let $w_1^{(1)} \in S_k$, $w_2^{(0)} \in S_m$, $w_2^{(1)} \in S_f$, and $w_3^{(0)} \in S_h$. We fix $\varepsilon, e \in \overline{n}$ and consider four cases: (1) $k \neq m$, $f \neq h$; (2) $k \neq m$, $f = h$; (3) $k = m$, $f \neq h$; and (4) $k = m$, $f = h$.

Case 1:

$$\begin{aligned}
\big((w_1,u_1)\boxed{\varepsilon}'(w_2,u_2)\big)\,\boxed{e}'(w_3,u_3) &= (w_1w_2, u_1y_\varepsilon u_2)\,\boxed{e}'(w_3,u_3)\\
&= (w_1w_2w_3, u_1y_\varepsilon u_2y_eu_3)\\
&= (w_1,u_1)\boxed{\varepsilon}'\,(w_2w_3, u_2y_eu_3)\\
&= (w_1,u_1)\boxed{\varepsilon}'\,\big((w_2,u_2)\boxed{e}'(w_3,u_3)\big).
\end{aligned}$$

Case 2:

$$\begin{aligned}
\big((w_1,u_1)\boxed{\varepsilon}'(w_2,u_2)\big)\,\boxed{e}'(w_3,u_3) &= (w_1w_2, u_1y_\varepsilon u_2)\,\boxed{e}'(w_3,u_3)\\
&= \Big((w_1w_2)\,\boxed{e^*}\,w_3, u_1y_\varepsilon u_2u_3\Big)\\
&= \Big(w_1(w_2\,\boxed{e^*}\,w_3), u_1y_\varepsilon u_2u_3\Big)\\
&= (w_1,u_1)\boxed{\varepsilon}'\big(w_2\,\boxed{e^*}\,w_3, u_2u_3\big)\\
&= (w_1,u_1)\boxed{\varepsilon}'\big((w_2,u_2)\boxed{e}'(w_3,u_3)\big).
\end{aligned}$$

Case 3:

$$\begin{aligned}
\big((w_1,u_1)\boxed{\varepsilon}'(w_2,u_2)\big)\,\boxed{e}'(w_3,u_3) &= \Big(w_1\,\boxed{\varepsilon^*}\,w_2, u_1u_2\Big)\,\boxed{e}'(w_3,u_3)\\
&= \Big((w_1\,\boxed{\varepsilon^*}\,w_2)w_3, u_1u_2y_eu_3\Big)\\
&= \Big(w_1\,\boxed{\varepsilon^*}\,(w_2w_3), u_1u_2y_eu_3\Big)\\
&= (w_1,u_1)\boxed{\varepsilon}'\,(w_2w_3, u_2y_eu_3)\\
&= (w_1,u_1)\boxed{\varepsilon}'\,\big((w_2,u_2)\boxed{e}'(w_3,u_3)\big).
\end{aligned}$$

Case 4:

$$\begin{aligned}
\big((w_1,u_1)\boxed{\varepsilon}'(w_2,u_2)\big)\,\boxed{e}'(w_3,u_3) &= \Big(w_1\,\boxed{\varepsilon^*}\,w_2, u_1u_2\Big)\,\boxed{e}'(w_3,u_3)\\
&= \Big((w_1\,\boxed{\varepsilon^*}\,w_2)\,\boxed{e^*}\,w_3, u_1u_2u_3\Big)
\end{aligned}$$

$$= \Big(w_1 \boxed{\varepsilon^*} (w_2 \boxed{e^*} w_3), u_1u_2u_3\Big)$$

$$= (w_1, u_1)\boxed{\varepsilon}'\big(w_2 \boxed{e^*} w_3, u_2u_3\big)$$

$$= (w_1, u_1)\boxed{\varepsilon}'\left((w_2, u_2)\boxed{e}'(w_3, u_3)\right).$$

Hence, $\mathrm{Fr}_nT(S_i)_{i\in I}$ is an n-tuple semigroup.

If $s = 1$, then we assume that the sequence $a_1 \ldots a_{s-1} \in F^\theta[Y]$ is equal to θ.

For any n-tuple semigroup $(S_i, \boxed{i_1}, \boxed{i_2}, \ldots, \boxed{i_n})$, $i \in I$, we have

$$\big(S_i, \boxed{i_1}, \boxed{i_2}, \ldots, \boxed{i_n}\big) \cong \overline{S_i} = \big\{(w, u) \in \mathrm{Fr}_nT(S_i)_{i\in I} \mid w \in S_i\big\},$$

and all algebras $\overline{S_i}$, $i \in I$, generate $\mathrm{Fr}_nT(S_i)_{i\in I}$. In addition, it follows from the definition of the algebra $\mathrm{Fr}_nT(S_i)_{i\in I}$ that any its element $(z_{m_1} \ldots z_{m_s}, a_1 \ldots a_{s-1})$, where $z_{m_p} \in S_{m_p}$, $1 \le p \le s$, and $a_\zeta \in Y$, $1 \le \zeta \le s-1$, admits a unique representation in the form of a product of finitely many different elements from $\cup_{i\in I}\overline{S_i}$:

$$(z_{m_1} \ldots z_{m_s}, a_1 \ldots a_{s-1}) = (z_{m_1}, \theta) *_1 \ldots *_{s-1} (z_{m_s}, \theta),$$

where $*_r \in \big\{\boxed{1}', \boxed{2}', \ldots, \boxed{n}'\big\}$, $r \in \overline{s-1}$, and $*_r = \boxed{t}'$ for some $t \in \overline{n}$ if and only if $a_r = y_t$.

Further, for any $i \in I$, we consider

$$\alpha_i\colon\ \big(S_i, \boxed{i_1}, \boxed{i_2}, \ldots, \boxed{i_n}\big) \to \big(K\,, \boxed{1}^\diamond, \boxed{2}^\diamond, \ldots, \boxed{n}^\diamond\big)$$

i.e., a homomorphism from $\big(S_i, \boxed{i_1}, \boxed{i_2}, \ldots, \boxed{i_n}\big)$ into an arbitrary n-tuple semigroup $\big(K, \boxed{1}^\diamond, \boxed{2}^\diamond, \ldots, \boxed{n}^\diamond\big)$. We introduce a mapping

$$\alpha\colon\ \mathrm{Fr}_nT(S_i)_{i\in I} \to \big(K, \boxed{1}^\diamond, \boxed{2}^\diamond, \ldots, \boxed{n}^\diamond\big)$$

by the rule

$$\omega\alpha = \begin{cases} z_{m_1}\alpha_{m_1}\widetilde{a}_1 \ldots .\widetilde{a}_{s-1}z_{m_s}\alpha_{m_s} & \text{for} \quad \omega = (z_{m_1} \ldots z_{m_s}, a_1 \ldots a_{s-1}), \quad s > 1, \\ z_{m_1}\alpha_{m_1} & \text{for} \quad \omega = (z_{m_1}, \theta), \end{cases}$$

where $\widetilde{a}_r = \boxed{t}^\diamond$ for some $t \in \overline{n}$ if and only if $a_r = y_t$ $(1 \le r \le s-1,\ s > 1)$. By Lemma 1, the mapping α is well defined. By using this lemma, we can show that α is a homomorphism.

Let

$$(z_{m_1} \ldots z_{m_s}, a_1 \ldots a_{s-1}),\ \big(c_{q_1} \ldots c_{q_g}, b_1 \ldots b_{g-1}\big) \in \mathrm{Fr}_nT(S_i)_{i\in I},$$

where

$$c_{q_l} \in S_{q_l}, \quad 1 \le l \le g, \qquad \text{and} \qquad b_d \in Y, \quad 1 \le d \le g-1.$$

If $m_s \ne q_1$, then

$$\big((z_{m_1} \ldots z_{m_s}, a_1 \ldots a_{s-1})\boxed{\varepsilon}'(c_{q_1} \ldots c_{q_g}, b_1 \ldots b_{g-1})\big)\alpha$$

$$= (z_{m_1} \ldots z_{m_s}c_{q_1} \ldots c_{q_g}, a_1 \ldots a_{s-1}y_\varepsilon b_1 \ldots b_{g-1})\alpha$$

$$= z_{m_1}\alpha_{m_1}\widetilde{a}_1\dots\widetilde{a}_{s-1}z_{m_s}\alpha_{m_s}\widetilde{y}_\varepsilon c_{q_1}\alpha_{q_1}\widetilde{b}_1\dots\widetilde{b}_{g-1}c_{q_g}\alpha_{q_g}$$

$$= (z_{m_1}\alpha_{m_1}\widetilde{a}_1\dots\widetilde{a}_{s-1}z_{m_s}\alpha_{m_s})\boxed{\varepsilon}^\diamond(c_{q_1}\alpha_{q_1}\widetilde{b}_1\dots\widetilde{b}_{g-1}c_{q_g}\alpha_{q_g})$$

$$= (z_{m_1}\dots z_{m_s}, a_1\dots a_{s-1})\,\alpha\boxed{\varepsilon}^\diamond\left(c_{q_1}\dots c_{q_g}, b_1\dots b_{g-1}\right)\alpha.$$

If $v = m_s = q_1$, then, by using the homomorphisms $\alpha_i,\ i\in I$, we get

$$((z_{m_1}\dots z_{m_s}, a_1\dots a_{s-1})\boxed{\varepsilon}'(c_{q_1}\dots c_{q_g}, b_1\dots b_{g-1}))\alpha$$

$$= ((z_{m_1}\dots z_{m_s})\boxed{\varepsilon^*}(c_{q_1}\dots c_{q_g}), a_1\dots a_{s-1}b_1\dots b_{g-1})\alpha$$

$$= (z_{m_1}\dots z_{m_{s-1}}(z_{m_s}\boxed{v_\varepsilon}c_{q_1})c_{q_2}\dots c_{q_g}, a_1\dots a_{s-1}b_1\dots b_{g-1})\alpha$$

$$= z_{m_1}\alpha_{m_1}\widetilde{a}_1\dots\widetilde{a}_{s-1}(z_{m_s}\boxed{v_\varepsilon}c_{q_1})\alpha_{m_s}\widetilde{b}_1c_{q_2}\alpha_{q_2}\widetilde{b}_2\dots\widetilde{b}_{g-1}c_{q_g}\alpha_{q_g}$$

$$= z_{m_1}\alpha_{m_1}\widetilde{a}_1\dots\widetilde{a}_{s-1}z_{m_s}\alpha_{m_s}\boxed{\varepsilon}^\diamond c_{q_1}\alpha_{q_1}\widetilde{b}_1c_{q_2}\alpha_{q_2}\widetilde{b}_2\dots\widetilde{b}_{g-1}c_{q_g}\alpha_{q_g}$$

$$= (z_{m_1}\dots z_{m_s}, a_1\dots a_{s-1})\alpha\boxed{\varepsilon}^\diamond(c_{q_1}\dots c_{q_g}, b_1\dots b_{g-1})\alpha.$$

Thus, α is a homomorphism extending the homomorphisms $\alpha_i,\ i\in I$, and $\mathrm{Fr}_nT(S_i)_{i\in I}$ is the free product of n-tuple semigroups

$$(S_i, \boxed{i_1}, \boxed{i_2}, \dots, \boxed{i_n}),\quad i\in I.$$

The theorem is proved.

Theorem 1 generalizes Theorem 3.2 in [4] obtained for doppelsemigroups. It is also worth noting that some facts in the proof of Theorem 3.2 in [4] were left for independent reader's verification, unlike Theorem 1 for which we present the complete proof.

We define n binary operations $\cdot_{y_1}, \cdot_{y_2}, \dots, \cdot_{y_n}$ on the set $F^\theta[Y]$ by setting

$$u_1 \cdot_{y_r} u_2 = u_1 y_r u_2$$

for all $u_1, u_2 \in F^\theta[Y]$ and $r\in\overline{n}$. According to Corollary 1 in [14],

$$(F^\theta[Y], \cdot_{y_1}, \cdot_{y_2}, \dots, \cdot_{y_n})$$

is a free n-tuple semigroup of rank 1. Further, let $\{\Lambda_i\}_{i\in I}$ be a family of free n-tuple semigroups of rank 1. In view of the fact that every free algebra is a free product of one-generated free algebras, by Theorem 1, we arrive at the following corollary that gives a free n-tuple semigroup:

Corollary 1. *The free product $\mathrm{Fr}_nT(S_i)_{i\in I}$ of n-tuple semigroups $\Lambda_i,\ i\in I$, is a free n-tuple semigroup of rank $|I|$.*

Recall that, for the first time, a free n-tuple semigroup of any rank was constructed in [14].

3. Structure of $\mathrm{Fr}_n T(S_i)_{i\in I}$

In this section, we introduce the notion of n-band of n-tuple semigroups and use this notion to describe the structure of free products of n-tuple semigroups. In addition, we solve the open problem posed in [5] and construct, for the first time, an example of n-band with different operations.

Recall that a semigroup is called a semigroup of left (right) zeros if it satisfies the identity $xy = x \ (xy = y)$. A semigroup of idempotents is called a band. A semigroup is called a rectangular band if it satisfies the identity $xyx = x$. Equivalently, a semigroup is called a rectangular band if it satisfies the identities $x^2 = x$ and $xyz = xz$. It is known that each rectangular band is isomorphic to the Cartesian product of a semigroup of left zeros and a semigroup of right zeros. A commutative band is called a semistructure.

An n-tuple semigroup is called an n-band if each its operation is an idempotent operation. It is clear that each band can be regarded as an n-band.

The next statement gives affirmative answer to an open problem posed in [5] concerning the existence of examples of doppelsemigroups with different idempotent operations. Note that the terms “doppelsemigroup” and “double semigroup” coincide.

Let $n \in \mathbb{N}$, let

$$G = \{a, b\} \cup \{c_i : \, i \in \overline{n}\} \cup \{d_i : \, i \in \overline{n}\}$$

and, in addition, let the sets $\{a, b\}$, $\{c_i : \, i \in \overline{n}\}$, and $\{d_i : \, i \in \overline{n}\}$ be mutually disjoint. We define the operations $[j], \ j \in \overline{n}$, on the set G setting

$$a[j]a = a, \qquad b[j]b = b,$$

$$a[j]b = c_j, \qquad b[j]a = d_j,$$

$$a[j]c_k = c_k, \qquad b[j]d_k = d_k,$$

$$a[j]d_k = c_j, \qquad b[j]c_k = d_j,$$

$$c_k[j]x = c_k, \qquad d_k[j]x = d_k$$

for all $x \in G$ and $j, k \in \overline{n}$.

Proposition 1. $(G, [1], [2], \ldots, [n])$ *is an n-band. Moreover,* $(G, [i]) \cong (G, [j])$ *for any* $i, j \in \overline{n}$.

Proof. Since

$$a[j]a = a, \qquad b[j]b = b, \qquad c_k[j]c_k = c_k, \qquad d_k[j]d_k = d_k$$

for all $j, k \in \overline{n}$, the operations $[1], [2], \ldots, [n]$ are idempotent. It remains to check the axioms of n-tuple semigroups. Since

$$c_k[j]x = c_k, \qquad d_k[j]x = d_k$$

for all $x \in G, \ j, k \in \overline{n}$, it suffices to consider an equation of the form

$$a[i](x[j]y) = (a[i]x)[j]y \qquad \text{or} \qquad b[i](x[j]y) = (b[i]x)[j]y,$$

where $x, y \in G,\ i, j \in \overline{n}$. We consider only the first equation. The second equation is verified in a similar way. If $x \notin \{a, b\}$, then

$$a[i](x[j]y) = a[i]x = (a[i]x)[j]y$$

because $a[i]x \notin \{a, b\}$. It remains to consider the equation with $x \in \{a, b\}$, i.e., we get the family of equalities

$$a[i](b[j]a) = a[i]d_j = c_i = c_i[j]a = (a[i]b)[j]a,$$

$$a[i](b[j]b) = a[i]b = c_i = c_i[j]b = (a[i]b)[j]b,$$

$$a[i](a[j]b) = a[i]c_j = c_j = a[j]b = (a[i]a)[j]b,$$

$$a[i](b[j]c_k) = a[i]d_j = c_i = c_i[j]c_k = (a[i]b)[j]c_k,$$

$$a[i](b[j]d_k) = a[i]d_k = c_i = c_i[j]d_k = (a[i]b)[j]d_k,$$

$$a[i](a[j]c_k) = a[i]c_k = c_k = a[j]c_k = (a[i]a)[j]c_k,$$

$$a[i](a[j]d_k) = a[i]c_j = c_j = a_i[j]d_k = (a[i]a)[j]d_k.$$

Finally, we directly check that, for any $i, j \in \overline{n}$, the mapping

$$(G, [i]) \to (G, [j]),$$

given by the rule

$$c_i \mapsto c_j, \quad d_i \mapsto d_j, \quad c_j \mapsto c_i, \quad d_j \mapsto d_i \qquad \text{and} \qquad x \mapsto x \ \text{ otherwise},$$

is an isomorphism.

Proposition 1 is proved.

Note that, for $n = 2$, Proposition 1 yields an example of a doppelsemigroup with different idempotent operations.

The existence of n-bands with different operations enables us to introduce the notion of n-band of n-tuple semigroups.

If $\tau\colon S_1 \to S_2$ is a homomorphism of n-tuple semigroups, then we denote the corresponding congruence on S_1 by Δ_τ.

Let $(S, \boxed{1}, \boxed{2}, \ldots, \boxed{n})$ be an arbitrary n-tuple semigroup, let $(J, \boxed{1}', \boxed{2}', \ldots, \boxed{n}')$ be an n-band, and let

$$\alpha\colon (S, \boxed{1}, \boxed{2}, \ldots, \boxed{n}) \to (J, \boxed{1}', \boxed{2}', \ldots, \boxed{n}')\colon x \mapsto x\alpha$$

be a homomorphism. Then each class of congruence Δ_α is an n-tuple subsemigroup of the n-tuple semigroup $(S, \boxed{1}, \boxed{2}, \ldots, \boxed{n})$, namely, $(S, \boxed{1}, \boxed{2}, \ldots, \boxed{n})$ is the union of n-tuple semigroups $S_\xi,\ \xi \in J$, such that

$$x\alpha = \xi \Leftrightarrow x \in S_\xi = \Delta_\alpha^x = \{t \in S \mid (x; t) \in \Delta_\alpha\},$$

$$S_\xi \boxed{r} S_\varepsilon \subseteq S_{\xi \boxed{r}' \varepsilon} \quad \text{for all} \quad r \in \overline{n},$$

$$\xi \neq \varepsilon \Rightarrow S_\xi \cap S_\varepsilon = \varnothing.$$

In this case, we say that $(S, \boxed{1}, \boxed{2}, \ldots, \boxed{n})$ is decomposed into an n-band of n-tuple semigroups [or $(S, \boxed{1}, \boxed{2}, \ldots, \boxed{n})$ is an n-band $(J, \boxed{1}', \boxed{2}', \ldots, \boxed{n}')$ of n-tuple semigroups $S_\xi, \ \xi \in J$]. If

$$(J, \boxed{1}', \boxed{2}', \ldots, \boxed{n}')$$

is a band, i.e.,

$$\boxed{1}' = \boxed{2}' = \ldots = \boxed{n}',$$

then we say that $(S, \boxed{1}, \boxed{2}, \ldots, \boxed{n})$ is a band $(J, \boxed{1}')$ of n-tuple semigroups $S_\xi, \ \xi \in J$. If the operations of the n-band $(J, \boxed{1}', \boxed{2}', \ldots, \boxed{n}')$ coincide and it is a semigroup of left zeros (resp., a semigroup of right zeros), then we say that $(S, \boxed{1}, \boxed{2}, \ldots, \boxed{n})$ is the left (resp., right) band $(J, \boxed{1}')$ of the n-tuple semigroups $S_\xi, \ \xi \in J$.

It is worth noting that the notion of n-band of n-tuple semigroups generalizes the well-known notion of a band of semigroups [19]. The semistructural decompositions of semigroups were described in [20].

Let X be an arbitrary nonempty set and let

$$X_{\ell z} = (X, \dashv), \quad X_{rz} = (X, \vdash), \quad \text{and} \quad X_{rb} = X_{\ell z} \times X_{rz}$$

be a semigroup of left zeros, a semigroup of right zeros, and a rectangular band, respectively. It is known [21] that $X_{\ell z}$, X_{rz}, and X_{rb} are a free semigroup of left zeros, a free semigroup of right zeros, and a free rectangular band, respectively. Further, let $B(X)$ be the semistructure of all nonempty finite subsets of the set X with respect to the operation of theoretical-set union and let

$$B_{rb}(X) = \big\{((x,y), A) \in X_{rb} \times B(X) \mid x, y \in A\big\},$$

$$B_{\ell z}(X) = \big\{(x, A) \in X_{\ell z} \times B(X) \mid x \in A\big\},$$

$$B_{rz}(X) = \big\{(x, A) \in X_{rz} \times B(X) \mid x \in A\big\}.$$

It is clear that $B_{rb}(X)$, $B_{\ell z}(X)$, and $B_{rz}(X)$ are subsemigroups of the semigroups $X_{rb} \times B(X)$, $X_{\ell z} \times B(X)$, and $X_{rz} \times B(X)$, respectively. According to [21], $B(X)$, $B_{rb}(X)$, $B_{\ell z}(X)$, and $B_{rz}(X)$ are a free semistructure, a free normal band, a free left normal band, and a free right normal band, respectively.

For any element

$$w = z_{m_1} \ldots z_{m_l} \ldots z_{m_s} \in \mathrm{Fr}\big[\big(S_i, \boxed{i_j}\big)\big]_{i \in I}$$

(see Sec. 2), we set

$$\widetilde{c}(w) = \bigcup_{l=1}^{s} \{z_{m_l} j'\},$$

where

$$j' : \bigcup_{i \in I} S_i \to I : \ a \mapsto i \quad \text{for} \quad a \in S_i, \quad i \in I.$$

Let

$$\Phi^{((x,y),C)} = \big\{(w,u) \in \mathrm{Fr}_n T(S_i)_{i\in I} \mid \big((w^{(0)}j', w^{(1)}j'), \widetilde{c}(w)\big) = ((x,y),C)\big\}$$

for all $((x,y),C) \in B_{rb}(I)$,

$$\Phi^{(x,C]} = \big\{(w,u) \in \mathrm{Fr}_n T(S_i)_{i\in I} \mid (w^{(0)}j', \widetilde{c}(w)) = (x,C)\big\}$$

for all $(x,C) \in B_{\ell z}(I)$,

$$\Phi^{[x,C)} = \big\{(w,u) \in \mathrm{Fr}_n T(S_i)_{i\in I} \mid (w^{(1)}j', \widetilde{c}(w)) = (x,C)\big\}$$

for all $(x,C) \in B_{rz}(I)$,

$$\Phi^{(x,y)} = \big\{(w,u) \in \mathrm{Fr}_n T(S_i)_{i\in I} \mid (w^{(0)}j', w^{(1)}j') = (x,y)\big\}$$

for all $(x,y) \in I_{rb}$,

$$\Phi^{(x)} = \big\{(w,u) \in \mathrm{Fr}_n T(S_i)_{i\in I} \mid w^{(0)}j' = x\big\}$$

for all $x \in I_{\ell z}$,

$$\Phi^{[x]} = \big\{(w,u) \in \mathrm{Fr}_n T(S_i)_{i\in I} \mid w^{(1)}j' = x\big\}$$

for all $x \in I_{rz}$, and

$$\Phi^{C} = \big\{(w,u) \in \mathrm{Fr}_n T(S_i)_{i\in I} \mid \widetilde{c}(w) = C\big\}$$

for all $C \in B(I)$.

In terms of the notion of n-band of n-tuple semigroups, we get the following two structural theorems:

Theorem 2. *The free product* $\mathrm{Fr}_n T(S_i)_{i\in I}$ *of* n*-tuple semigroups* $(S_i, \boxed{i_1}, \boxed{i_2}, \ldots, \boxed{i_n})$, $i \in I$, *is:*

(i) a normal band $B_{rb}(I)$ *of* n*-tuple semigroups* $\Phi^{((x,y),C)}$, $((x,y),C) \in B_{rb}(I)$*;*

(ii) a left normal band $B_{\ell z}(I)$ *of* n*-tuple semigroups* $\Phi^{(x,C]}$, $(x,C) \in B_{\ell z}(I)$*;*

(iii) a right normal band $B_{rz}(I)$ *of* n*-tuple semigroups* $\Phi^{[x,C)}$, $(x,C) \in B_{rz}(I)$.

Proof. (i) We define a mapping

$$\varrho_{rb} \colon \mathrm{Fr}_n T(S_i)_{i\in I} \to B_{rb}(I)$$

by the rule

$$(w,u) \mapsto \big((w^{(0)}j', w^{(1)}j'), \widetilde{c}(w)\big), \quad (w,u) \in \mathrm{Fr}_n T(S_i)_{i\in I}.$$

It is easy to see that

$$\widetilde{c}(w \star \omega) = \widetilde{c}(w) \cup \widetilde{c}(\omega),$$

$$(w \star \omega)^{(0)}j' = w^{(0)}j', \qquad (w \star \omega)^{(1)}j' = \omega^{(1)}j'$$

for all $w, \omega \in \mathrm{Fr}\big[(S_i, \boxed{i_j})\big]_{i\in I}$ and $\star \in \{\boxed{1^*}, \boxed{2^*}, \ldots, \boxed{n^*}\}$.

By using the previous equalities, for any elements $(w_1, u_1),\ (w_2, u_2) \in \mathrm{Fr}_n T(S_i)_{i\in I}$, and $r \in \overline{n}$, we get

$$
\begin{aligned}
&((w_1, u_1)\,\boxed{r}'(w_2, u_2))\varrho_{rb}\\
&\quad = \begin{cases} (w_1w_2, u_1y_ru_2)\,\varrho_{rb}, & w_1^{(1)} \in S_k, \quad w_2^{(0)} \in S_m, \quad k, m \in I,\ k \neq m,\\[2ex] \big(w_1\boxed{r^*}w_2, u_1u_2\big)\,\varrho_{rb}, & w_1^{(1)}, \quad w_2^{(0)} \in S_k, \quad k \in I, \end{cases}\\
&\quad = \begin{cases} \big(\big(((w_1w_2)^{(0)})j', ((w_1w_2)^{(1)})j'\big), \widetilde{c}(w_1w_2)\big), & w_1^{(1)} \in S_k, \quad w_2^{(0)} \in S_m, \quad k, m \in I,\ k \neq m,\\[2ex] \big(\big(((w_1\boxed{r^*}w_2)^{(0)})j', ((w_1\boxed{r^*}w_2)^{(1)})j'\big), \widetilde{c}(w_1\boxed{r^*}w_2)\big), & w_1^{(1)}, w_2^{(0)} \in S_k, \quad k \in I, \end{cases}\\
&\quad = ((w_1^{(0)}j', w_2^{(1)}j'), \widetilde{c}(w_1) \cup \widetilde{c}(w_2))\\
&\quad = ((w_1^{(0)}j', w_1^{(1)}j'), \widetilde{c}(w_1))((w_2^{(0)}j', w_2^{(1)}j'), \widetilde{c}(w_2))\\
&\quad = (w_1, u_1)\varrho_{rb}(w_2, u_2)\varrho_{rb}.
\end{aligned}
$$

Hence, ϱ_{rb} is a surjective homomorphism. It is easy to see that $\Phi^{((x,y),C)},\ ((x, y), C) \in B_{rb}(I)$, is a class of congruence $\Delta_{\varrho_{rb}}$, which is an n-tuple subsemigroup of the algebra $\mathrm{Fr}_n T(S_i)_{i\in I}$. This implies that $\mathrm{Fr}_n T(S_i)_{i\in I}$ is a normal band $B_{rb}(I)$ of n-tuple semigroups $\Phi^{((x,y),C)},\ ((x, y), C) \in B_{rb}(I)$.

(ii) The analysis similar to the proof of assertion (i) shows that the mapping

$$\varrho_{\ell z}\colon\ \mathrm{Fr}_n T(S_i)_{i\in I} \to B_{\ell z}(I)$$

given by the rule

$$(w, u) \mapsto (w^{(0)}j', \widetilde{c}(w)), \quad (w, u) \in \mathrm{Fr}_n T(S_i)_{i\in I}$$

is a surjective homomorphism. This implies that $\Phi^{(x,C]},\ (x, C) \in B_{\ell z}(I)$, is a class of congruence $\Delta_{\varrho_{\ell z}}$, which is an n-tuple subsemigroup of the algebra $\mathrm{Fr}_n T(S_i)_{i\in I}$. Hence, $\mathrm{Fr}_n T(S_i)_{i\in I}$ is a left normal band $B_{\ell z}(I)$ of n-tuple semigroups $\Phi^{(x,C]},\ (x, C) \in B_{\ell z}(I)$.

(iii) We define the mapping

$$\varrho_{rz}\colon\ \mathrm{Fr}_n T(S_i)_{i\in I} \to B_{rz}(I)$$

by the rule

$$(w, u) \mapsto (w^{(1)}j', \widetilde{c}(w)), \quad (w, u) \in \mathrm{Fr}_n T(S_i)_{i\in I}.$$

As in the proof of assertion (i), we conclude that ϱ_{rz} is a surjective homomorphism and $\Phi^{[x,C)},\ (x, C) \in B_{rz}(I)$, is class of congruence $\Delta_{\varrho_{rz}}$, which is an n-tuple subsemigroup of the algebra $\mathrm{Fr}_n T(S_i)_{i\in I}$. Thus, $\mathrm{Fr}_n T(S_i)_{i\in I}$ is a right normal band $B_{rz}(I)$ of n-tuple semigroups $\Phi^{[x,C)},\ (x, C) \in B_{rz}(I)$.

The theorem is proved.

The proof of the following theorem is similar to the proof of Theorem 2:

Theorem 3. *The free product* $\mathrm{Fr}_n T(S_i)_{i\in I}$ *of* n*-tuple semigroups* $(S_i, \boxed{i_1}, \boxed{i_2}, \ldots, \boxed{i_n})$, $i \in I$, *is:*

(i) *a rectangular band* I_{rb} *of* n*-tuple semigroups* $\Phi^{(x,y)}$, $(x, y) \in I_{rb}$;

(ii) *a left band* $I_{\ell z}$ *of* n*-tuple semigroups* $\Phi^{(x)}$, $x \in I_{\ell z}$;

(iii) *a right band* I_{rz} *of* n*-tuple semigroups* $\Phi^{[x]}$, $x \in I_{rz}$;

(iv) *a semistructure* $B(I)$ *of* n*-tuple semigroups* Φ^C, $C \in B(I)$.

We describe one congruence on $\mathrm{Fr}_n T(S_i)_{i\in I}$ and use it to get a free product of semigroups from the free product of n-tuple semigroups.

Let γ be an arbitrary fixed congruence on the free product $\mathrm{Fr}\big[(S_i, \boxed{i_j})\big]_{i\in I}$. We define the operation $\widetilde{\gamma}$ on $\mathrm{Fr}_n T(S_i)_{i\in I}$ by the rule

$$(w_1, u_1)\widetilde{\gamma}(w_2, u_2) \Leftrightarrow w_1 \gamma w_2$$

for all $(w_1, u_1), (w_2, u_2) \in \mathrm{Fr}_n T(S_i)_{i\in I}$.

The following statement can be easily proved:

Proposition 2. *The operation* $\widetilde{\gamma}$ *is a congruence on the free product* $\mathrm{Fr}_n T(S_i)_{i\in I}$ *of* n*-tuple semigroups* $\big(S_i, \boxed{i_1}, \boxed{i_2}, \ldots, \boxed{i_n}\big)$, $i \in I$, *and the operations of the quotient algebra* $\mathrm{Fr}_n T(S_i)_{i\in I}/\widetilde{\gamma}$ *coincide.*

Proposition 2 yields the following corollary:

Corollary 2. *If* γ *is a diagonal of* $\mathrm{Fr}\big[(S_i, \boxed{i_j})\big]_{i\in I}$, *then* $\mathrm{Fr}_n T(S_i)_{i\in I}/\widetilde{\gamma}$ *is the free product of semigroups.*

4. Free Commutative n-Tuple Semigroups

In this section, we construct a free commutative n-tuple semigroup of an arbitrary rank and consider one-generated free commutative n-tuple semigroups. In addition, we show that semigroups of a free commutative n-tuple semigroup are isomorphic and its automorphism group is isomorphic to a symmetric group.

In the proof of the main result of this section, we use the following three statements:

Let G be an arbitrary n-tuple semigroup with operations $\boxed{1}, \boxed{2}, \ldots, \boxed{n}$ and $a_1, a_2, \ldots, a_n \in G$. We define a new operations $\boxed{1}_{a_1}, \boxed{2}_{a_2}, \ldots, \boxed{n}_{a_n}$ on G by the rule

$$x \boxed{i}_{a_i} y = x \boxed{i} a_i \boxed{i} y$$

for all $x, y \in G$ and $i \in \overline{n}$.

Proposition 3 ([14], Proposition 3). $\big(G, \boxed{1}_{a_1}, \boxed{2}_{a_2}, \ldots, \boxed{n}_{a_n}\big)$ *is an* n*-tuple semigroup.*

An n-tuple semigroup $\big(G, \boxed{1}_{a_1}, \boxed{2}_{a_2}, \ldots, \boxed{n}_{a_n}\big)$ is called either a version of G, or (alternatively) a sandwich n-tuple semigroup of the algebra G relative to the sandwich elements $a_1, a_2, \ldots, a_n$, or an n-tuple semigroup with deformed multiplications. The operations $\boxed{1}_{a_1}, \boxed{2}_{a_2}, \ldots, \boxed{n}_{a_n}$ are called sandwich operations [14].

We call an n-tuple semigroup commutative if all its operations are commutative. The class of all commutative n-tuple semigroups forms a submanifold in a manifold of n-tuple semigroups. An n-tuple semigroup, which is free in the manifold of commutative n-tuple semigroups, is called a free commutative n-tuple semigroup.

Lemma 2. *In the commutative n-tuple semigroup $\big(G, \boxed{1}, \boxed{2}, \ldots, \boxed{n}\big)$, the equality*

$$(x \diamond y) \circ z = x \circ (y \diamond z)$$

holds for all $x, y, z \in G$ and $\diamond, \circ \in \big\{\boxed{1}, \boxed{2}, \ldots, \boxed{n}\big\}$.

Proof. For any $x, y, z \in G$, we get

$$(x \diamond y) \circ z = z \circ (x \diamond y) = (z \circ x) \diamond y$$

$$= (x \circ z) \diamond y = x \circ (z \diamond y) = x \circ (y \diamond z)$$

in view of the commutativity of the operations $\diamond$ and $\circ$ and the axiom of n-tuple semigroup.

The lemma is proved.

Lemma 3. *In the commutative n-tuple semigroup $\big(G, \boxed{1}, \boxed{2}, \ldots, \boxed{n}\big)$, for any $m \in \mathbb{N}$ and any $x_i \in G$, $1 \leq i \leq m+1$, i $*_j \in \big\{\boxed{1}, \boxed{2}, \ldots, \boxed{n}\big\}$, $1 \leq j \leq m$, the following equality is true:*

$$x_1 *_1 x_2 *_2 \ldots *_m x_{m+1} = x_{1\pi} *_{1\pi'} x_{2\pi} *_{2\pi'} \ldots *_{m\pi'} x_{(m+1)\pi},$$

where π and π' are permutations on $\overline{m+1}$ and $\overline{m}$, respectively.

Proof. The proof of the lemma follows from Lemmas 1 and 2 and the commutativity of the operations $*_j$, $1 \leq j \leq m$.

We fix $n \in \mathbb{N}$ and, as above, assume that X is an arbitrary nonempty set and $Y = \{y_1, y_2, \ldots, y_n\}$ is an arbitrary set of n elements. Further, let $F^*[X]$ be a free commutative semigroup on X, let $F_*^\theta[Y]$ be a free commutative monoid on Y, and let $\theta \in F_*^\theta[Y]$ be an empty word. We define n binary operations $\boxed{1}, \boxed{2}, \ldots, \boxed{n}$ on the set

$$XY_{(n)} = \big\{(w, u) \in F^*[X] \times F_*^\theta[Y] \;\big|\; l_w - l_u = 1\big\}$$

and set

$$(w_1, u_1) \boxed{i} (w_2, u_2) = (w_1 w_2, u_1 \cdot_{y_i} u_2) \tag{1}$$

for all $(w_1, u_1), (w_2, u_2) \in XY_{(n)}$ and $i \in \overline{n}$, where $\cdot_{y_i}$ is a sandwich operation on $F_*^\theta[Y]$. The operations thus defined are correct because $l_{w_1 w_2} - l_{u_1 y_i u_2} = 1$ for all $i \in \overline{n}$. By $FC_nS(X)$ we denote the algebra $(XY_{(n)}, \boxed{1}, \boxed{2}, \ldots, \boxed{n})$.

Theorem 4. *$FC_nS(X)$ is a free commutative n-tuple semigroup.*

Proof. The proof of the theorem is similar to the proof of Theorem 4.3 in [4]. In this case, we use Proposition 3 and Lemmas 1 and 3.

Note that, for $n = 2$, the Theorem 4 yields Theorem 4.3 in [4].

Corollary 3. *$\big(F_*^\theta[Y], \cdot_{y_1}, \cdot_{y_2}, \ldots, \cdot_{y_n}\big)$ is a free commutative n-tuple semigroup of rank 1.*

Proof. If $X=\{r\}$, then we can easily show that the mapping

$$\delta\colon \left(F_*^{\theta}[Y], \cdot_{y_1}, \cdot_{y_2}, \ldots, \cdot_{y_n}\right) \to FC_nS(X)$$

given by the rule $u\delta=(r^{l_u+1},u)$ for all $u\in F_*^{\theta}[Y]$ is an isomorphism.

The corollary is proved.

The following statement establishes relationships between the semigroups of the free commutative n-tuple semigroup $FC_nS(X)$:

Proposition 4. *For any $i,j\in\{1,2,\ldots,n\}$, the semigroups $(XY_{(n)}, \boxed{i})$ and $(XY_{(n)}, \boxed{j})$ are isomorphic.*

The proof of this proposition is similar to the proof of Lemma 2 in [14].

By $\Im[X]$ we denote a symmetric group on the set X and by $\operatorname{Aut} G'$ we denote a group of automorphisms of the n-tuple semigroup G'.

A free commutative n-tuple semigroup $FC_nS(X)$ is defined to within an isomorphism of cardinality of the set X because the generating set $FC_nS(X)$ has the same cardinality as X. This yields the following description of the group of automorphisms of the free commutative n-tuple semigroup:

Proposition 5. $\operatorname{Aut} FC_nS(X)\cong \Im[X]$.

We construct a congruence on $FC_nS(X)$ and use it to get a free commutative semigroup from the free commutative n-tuple semigroup.

Let ζ be an arbitrary fixed congruence on the free commutative semigroup $F^*[X]$. We define an operation $\widetilde{\zeta}$ on $FC_nS(X)$ by the rule

$$(w_1,u_1)\widetilde{\zeta}(w_2,u_2) \Leftrightarrow w_1\zeta w_2$$

for all $(w_1,u_1),(w_2,u_2)\in FC_nS(X)$.

The following statement can be easily proved:

Proposition 6. *The operation $\widetilde{\zeta}$ is a congruence on the free commutative n-tuple semigroup $FC_nS(X)$ and the operations of the quotient algebra $FC_nS(X)/\widetilde{\zeta}$ coincide.*

Proposition 6 yields the following corollary:

Corollary 4. *If ζ is a diagonal of $F^*[X]$, then $FC_nS(X)/\widetilde{\zeta}$ is a free commutative semigroup.*

5. Least Commutative Congruence on the Free n-Tuple Semigroup

We describe the least commutative congruence on the free n-tuple semigroup and present criteria for the coincidence of operations of the n-tuple semigroup.

We now recall the structure of free n-tuple semigroup [14]. To do this, we use the notation introduced in the previous section.

In the structure of $FC_nS(X)$, we replace the free commutative semigroup $F^*[X]$ on X by the free semigroup $F[X]$ on X and the free commutative monoid $F_*^{\theta}[Y]$ on Y by the free monoid $F^{\theta}[Y]$ on Y with empty word θ. In this case, by $F_nTS(X)$ we denote the algebra $\left(XY_{(n)}, \boxed{1}, \boxed{2}, \ldots, \boxed{n}\right)$ with operations specified by condition (1). By Theorem 2 in [14], $F_nTS(X)$ is a free n-tuple semigroup.

If ρ is a congruence on the n-tuple semigroup G' such that G'/ρ is a commutative n-tuple semigroup, then we say that ρ is a commutative congruence. In this section, by $\star$ (resp., $\cdot$) we denote the operation on $F^*[X]$ $\big($resp., on $F_*^\theta[Y]\big)$.

We take

$$(x_1x_2\dots x_s, b_1b_2\dots b_{s-1}), (z_1z_2\dots z_k, c_1c_2\dots c_{k-1}) \in F_nTS(X),$$

where $x_d, z_r \in X,\ 1 \le d \le s,\ 1 \le r \le k,\ b_p, c_j \in Y,\ 1 \le p \le s-1,\ 1 \le j \le k-1$. If $s = 1$, then we assume that the sequence $h_1h_2\dots h_{s-1}$, where $h_i \in Y,\ i \in \overline{s-1}$, is equal to θ. We define the operation λ on $F_nTS(X)$ by the rule

$$(x_1x_2\dots x_s, b_1b_2\dots b_{s-1})\lambda(z_1z_2\dots z_k, c_1c_2\dots c_{k-1})$$

if and only if

$$(x_1 \star x_2 \star \dots \star x_s, b_1 \cdot b_2 \cdot \dots \cdot b_{s-1}) = (z_1 \star z_2 \star \dots \star z_k, c_1 \cdot c_2 \cdot \dots \cdot c_{k-1}).$$

Theorem 5. *The operation λ on the free n-tuple semigroup $F_nTS(X)$ is the least commutative congruence.*

Proof. Let $\omega = (x_1x_2\dots x_s, b_1b_2\dots b_{s-1}) \in F_nTS(X)$, where $x_d \in X,\ 1 \le d \le s,\ b_p \in Y$, and $1 \le p \le s-1$.

We define a mapping

$$\pi\colon F_nTS(X) \to FC_nS(X)$$

by the rule

$$\omega\pi = \begin{cases} (x_1 \star x_2 \star \dots \star x_s, b_1 \cdot b_2 \cdot \dots \cdot b_{s-1}) & \text{for} \quad s > 1, \\ \omega & \text{for} \quad \omega = (x_1, \theta). \end{cases}$$

It is easy to see that π is a surjective homomorphism. By Theorem 4, $FC_nS(X)$ is a free commutative n-tuple semigroup. Thus, Δ_π (see Sec. 3) is the least commutative congruence on $F_nTS(X)$. The definition of π implies that $\Delta_\pi = \lambda$.

The theorem is proved.

Note that, for $n = 2$, the last theorem yields the first part of Theorem 4.9 in [4].

At the end of the section, we formulate conditions under which the operations of an arbitrary (commutative) n-tuple semigroup coincide.

Proposition 7. *The operations of a commutative n-tuple semigroup coincide if they are idempotent.*

Proof. Let $\big(G, \boxed{1}, \boxed{2}, \dots, \boxed{n}\big)$ be an arbitrary commutative n-tuple semigroup. By Lemma 2, $(x \diamond y) \circ z = (x \circ y) \diamond z$ for all $x, y, z \in G$ and $\diamond,\ \circ \in \big\{\boxed{1}, \boxed{2}, \dots, \boxed{n}\big\}$. For $x = y$, in view of the idempotence of the operations $\diamond$ and $\circ$, we obtain $x \circ z = x \diamond z$.

The proposition is proved.

Proposition 8. *The operations of the n-tuple semigroup $(G, \boxed{1}, \boxed{2}, \dots, \boxed{n})$ coincide if one of the following conditions is satisfied:*

(i) $(G, \boxed{i})$ *is a semigroup of left zeros for some* $i \in \overline{n}$;

(ii) $(G, \boxed{i})$ *is a semigroup of right zeros for some* $i \in \overline{n}$;

(iii) $(G, \boxed{i})$ *is a rectangular band for some* $i \in \overline{n}$;

(iv) $(G, \boxed{i})$ *is a semigroup with zero multiplication for all* $i \in \overline{n}$.

Proof. Let $x, y, z \in G$ and let $\diamond, \circ \in \{\boxed{1}, \boxed{2}, \ldots, \boxed{n}\}$.

(i) Setting $x \diamond y = x$ for all $x, y \in G$ and using the axiom of n-tuple semigroup, we obtain

$$(x \diamond y) \circ z = x \diamond (y \circ z) = x = x \circ z.$$

This yields $x \circ z = x$ for all $x, z \in G$.

Case (ii) is proved similarly.

(iii) Let $(G, \circ)$ be a rectangular band. By the definition of rectangular band (see Sec. 3), we find

$$x \circ y \circ x = x, \quad y \circ x \circ y = y, \quad x \circ z \circ y = x \circ y.$$

By using these equalities and the axioms of n-tuple semigroup, we get

$$x \diamond y = (x \circ y \circ x) \diamond (y \circ x \circ y)$$

$$= x \circ ((y \circ x) \diamond (y \circ x \circ y))$$

$$= x \circ ((y \circ x) \diamond (y \circ x)) \circ y = x \circ y.$$

(iv) Let 0 and $0'$ be zero elements of the semigroups $(G, \diamond)$ and $(G, \circ)$, respectively. Thus, we get

$$0 = (0 \circ 0) \diamond 0 = 0 \circ (0 \diamond 0) = 0'.$$

The proposition is proved.

Let V be a manifold of semigroups and let $u, v \in F[X]$. By IdV we denote the set of all identities $u \approx v$ such that $u \approx v \in IdV$ provided that each semigroup $S \in V$ satisfies the identity $u \approx v$. By $c(u)$ we denote the set of all elements $x \in X$ contained in the word u. Let C be a manifold of semigroups with zero multiplication.

Proposition 9. *Suppose that* V *is a manifold of semigroups such that* $C \subseteq V$, G *is an arbitrary set with* $|G| \geq 4$, *and* $n > 1$. *The following statements are equivalent:*

(i) for any n*-tuple semigroup* $(G, [1], [2], \ldots, [n])$ *with* $(G, [j]) \in V$ *and all* $j \in \overline{n}$, *the following assertion is true: if there exists* $i' \in \overline{n}$ *such that* $(G, [i']) \in C$, *then* $[i] = [j]$ *for all* $i, j \in \overline{n}$;

(ii) there exists $u \in F[X]$ *with* $c(u) = \{x, y, z\}$ *such that* $xy \approx u \in IdV$.

Proof. (i) $\Rightarrow$ (ii). Let c and a_0 be different elements from G. We define an operation $[2]$ on G by the rule

$$a[2]c = c[2]a = c \quad \text{for all} \quad a \in G,$$

$$a[2]b = b[2]a = a_0 \quad \text{for all} \quad a, b \in G \setminus \{c\}.$$

Further, let $(G, [1])$ be a semigroup with zero multiplication and null element c. We can easily show that

$$(G, [1], [2], \ldots, [n]),$$

where $[k] = [2]$ for $3 \leq k \leq n$ is an n-tuple semigroup, and $(G, [k])$, $k \in \overline{n}$, satisfies any identity $u \approx v$ with $u, v \in F[X]$ and $c(u) = c(v)$. Thus, there exists $u \approx v \in IdV$ with $c(u) \neq c(v)$, i.e., there exists an element $u_1 \in F[X]$ with $c(u_1) = \{x, y\}$ such that $u_1 \approx x^k \in IdV$ for some $k \geq 2$. Otherwise, $(G, [j]) \in V$ for all $j \in \overline{n}$ and, according to assertion (i), we get $[i] = [j]$ for all $i, j \in \overline{n}$. We arrive at a contradiction.

Now let a_0, b_0, d_0, and c be different elements from G and let the operation $[1]$ be defined as above. We define the operation $[2]$ on G as follows:

$$a[2]b = c \quad \text{for all} \quad (a, b) \in G \times G \quad \text{with} \quad (a, b), (b, a) \neq (a_0, b_0),$$

$$a_0[1]b_0 = b_0[1]a_0 = d_0.$$

Further, let $[k] = [2]$ for $3 \leq k \leq n$. It is easy to see that $(G, [1], [2], \ldots, [n])$ is an n-tuple semigroup such that, for all $k \in \overline{n}$, the semigroup $(G, [k])$ satisfies the commutative law and any of the identities $u \approx v$, $u, v \in F[X]$ with $l_u, l_v \geq 3$ or $l_u \geq 3$ and $v = \upsilon\upsilon$ for some $\upsilon \in X$ (or conversely). Thus, there exists $u_2 \in F[X]$ with $c(u_2) = \{x, y\}$ such that $u_2 \approx xy \in IdV$. Otherwise, for all $u \approx v \in IdV$ of length $l_u, l_v \geq 3$ or $l_u \geq 3$ and $v = \upsilon\upsilon$ for some $\upsilon \in X$ (or conversely), we conclude that, according to assertion (i), $[i] = [j]$ for all $i, j \in \overline{n}$. We arrive at a contradiction. As a result of direct calculations, we conclude that the equalities $u_1 \approx x^k \in IdV$ and $u_2 \approx xy \in IdV$ imply that $u_3 \approx xy \in IdV$ for some $u_3 \in F[X]$ with $c(u_3) = \{x, y, z\}$.

(ii) $\Rightarrow$ (i). Let $a, b \in G$ and let $(G, [1], [2], \ldots, [n])$ be an n-tuple semigroup such that $(G, [j]) \in V$ for all $j \in \overline{n}$. Assume that there exists $i' \in \overline{n}$ such that $(G, [i']) \in C$. Further, let $j \in \overline{n} \setminus \{i'\}$. Replacing x with a, y with b, and z with $a[i']b$ in $xy \approx u \in IdV$, we obtain $a[j]b = a_1[i']b_1$ for some $a_1, b_1 \in G$. This implies that $[i'] = [j]$.

Proposition 9 is proved.

Proposition 10. *Suppose that* $(G, \boxed{1}, \boxed{2}, \ldots, \boxed{n})$ *is an* n*-tuple semigroup and* $i, j \in \overline{n}$. *The semigroup* $(G, \boxed{i})$ *is a semigroup with the null element* 0 *if and only if* $(G, \boxed{j})$ *is a semigroup with the null element* 0.

Proof. Let $0 \boxed{i} x = 0$ for all $x \in G$. Then, for all $y \in G$, we get

$$\big(0 \boxed{i} x\big) \boxed{j} y = 0 \boxed{i} \big(x \boxed{j} y\big) = 0 = 0 \boxed{j} y.$$

This yields $0 \boxed{j} y = 0$ for all $y \in G$. Similarly, we can prove that $y \boxed{j} 0 = 0$ for all $y \in G$.

The converse assertion is proved in a similar way.

Proposition 10 is proved.

The present paper was written in the course of scientific training of the first author at the Potsdam University (Germany) within the framework of the program of the German Service of Academic Exchange (DAAD).

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