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RESEARCH ARTICLE

The rank of the inverse semigroup of partial automorphisms on a finite fence

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Abstract
A fence is a particular partial order on a (finite) set, close to the linear order. In this paper, we calculate the rank of the semigroup $\mathscr{FI}_n$ of all order-preserving partial injections on an n-element fence. In particular, we provide a minimal generating set for $\mathscr{FI}_n$. In the present paper, n is odd since this problem for even n was already solved by I. Dimitrova and J. Koppitz.

Keywords Partial injections · Finite transformation semigroup · Fence · Rank · Generators

1 Introduction

Let $n \in \mathbb{N}$ and denote by $\mathscr{PT}_n$ the semigroup (under composition) of all partial transformations on the set $\overline{n} := \{1, \dots, n\}$ of the first n natural numbers. The set $\mathscr{I}_n$ of all partial injections on $\overline{n}$ forms an inverse subsemigroup of $\mathscr{PT}_n$. For more information about the symmetric inverse semigroup $\mathscr{I}_n$, we refer the reader to O. Ganyushkin and V. Mazorchuk's book [8].

Let $\preceq$ be any partial order on $\overline{n}$. Let $\alpha \in \mathscr{PT}_n$. Then α is called order-preserving on $\overline{n}$ with respect to $\preceq$ if $a \preceq b \Rightarrow a\alpha \preceq b\alpha$, for all $a, b \in \operatorname{dom}\alpha$. If $\alpha \in \mathscr{I}_n$ is order-preserving then it is a partial injective endomorphism on the digraph $(\overline{n}, \preceq)$. Clearly, the set $\mathscr{I}\mathrm{End}(\overline{n}, \preceq)$ of all partial injective endomorphisms on $(\overline{n}, \preceq)$ forms

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a submonoid of $\mathcal{I}_n$, which, in general, need not be inverse. A regular element α in $\mathcal{I}\mathrm{End}(\overline{n}, \preceq)$ is characterized by the following property:

$$a \preceq b \Leftrightarrow a\alpha \preceq b\alpha, \qquad \text{for all } a, b \in \operatorname{dom}\alpha.$$

Such regular elements in $\mathcal{I}\mathrm{End}(\overline{n}, \preceq)$ are called partial automorphisms on $(\overline{n}, \preceq)$. The set $\mathcal{P}\mathrm{Aut}(\overline{n}, \preceq)$ of all partial automorphisms on $(\overline{n}, \preceq)$ forms an inverse subsemigroup of $\mathcal{I}\mathrm{End}(\overline{n}, \preceq)$.

A very important particular and natural case occurs when the considered linear order is induced by the usual order $\leq$ on the natural numbers. The monoid $\mathcal{PIO}_n$ of all partial order-preserving injections on $(\overline{n}, \leq)$ has been extensively studied. Basic information about the monoid $\mathcal{PIO}_n$ can be found in [6]. In [14], the author considers generating sets of the semigroup of all partial injective decreasing maps of $(\overline{n}, \leq)$, a submonoid of $\mathcal{PIO}_n$. The maximal subsemigroups of the ideals of some semigroups of partial injections on $(\overline{n}, \leq)$ were determined by I. Dimitrova and J. Koppitz [3]. In [1], F. Al-Kharousi et al. consider distance-preserving injections on $(\overline{n}, \leq)$. They study the algebraic structure of such semigroups, in particular, Green's relations.

A non-linear order, close to a linear order in some sense, is the so-called zig-zag order. The pair $(\overline{n}, \preceq)$ is called a zig-zag poset or fence if

$$1 \prec 2 \succ \cdots \prec n-1 \succ n \text{ or } 1 \succ 2 \prec \cdots \succ n-1 \prec n \text{ if } n \text{ is odd}$$
$$\text{and } 1 \prec 2 \succ \cdots \succ n-1 \prec n \text{ or } 1 \succ 2 \prec \cdots \prec n-1 \succ n \text{ if } n \text{ is even.}$$

The definition of the partial order $\preceq$ is self-explanatory. Observe that every element in a fence is either minimal or maximal.

If the domain of an $\alpha \in \mathcal{PT}_n$ is $\overline{n}$, i.e. $\operatorname{dom}\alpha = \overline{n}$, then α is called a (full) transformation on $\overline{n}$. The set $\mathcal{T}_n$ of all full transformations on $\overline{n}$ forms a submonoid of $\mathcal{PT}_n$. The monoid $\mathcal{TF}_n$ of all order-preserving transformations within $\mathcal{T}_n$ (with respect to $\preceq$), i.e. of all endomorphisms on $(\overline{n}, \preceq)$, was first investigated by J.D. Currie and T.I. Visentin in [2] and by A. Rutkowski [12]. In [2], by using generating functions, J.D. Currie and T.I. Visentin calculate the cardinality of $\mathcal{TF}_n$ for the case that n is even. On the other hand, an exact formula for the number of endomorphisms on $(\overline{n}, \preceq)$ for even as well as odd n was given in [12]. Recently, in [7], V.H. Fernandes and the present authors determine the rank of $\mathcal{TF}_n$. Recall that the rank of a semigroup S, denoted by $\operatorname{rank} S$, is the minimal size of a generating set of S,

$$\operatorname{rank} S := \min\{|A| : A \subseteq S, A \text{ generates } S\}.$$

In particular, a concrete generating set of $\mathcal{TF}_n$ of minimal size is given in [7]. Moreover, V.H. Fernandes et al. characterize the transformations on $\overline{n}$ preserving the fence. It is worth mentioning that several other properties of monoids of order-preserving transformations of a fence were also studied. In [11, 13], R. Srithus et al. discussed the regular elements of these monoids. Coregular elements (i.e. elements α with the property $\alpha = \alpha^3$) of these monoids were determined in [10]. Some relative ranks of the monoid of all partial transformations preserving an infinite zig-zag order were determined in [5].

In this paper, we will denote the semigroup of all partial automorphisms on $(\overline{n}, \preceq)$ by $\mathcal{FI}_n$, i.e. $\mathcal{P}\mathrm{Aut}(\overline{n}, \preceq) = \mathcal{FI}_n$. This inverse semigroup was first studied by I. Dimitrova and J. Koppitz in [4]. They described Green's relations on $\mathcal{FI}_n$. In fact, they described only the $\mathcal{J}$-relation since the smaller Green's relations are clear because we have an inverse subsemigroup of $\mathcal{I}_n$. Moreover, I. Dimitrova and J. Koppitz show that $\mathcal{FI}_n$ is generated by the set

$$J_n := \{\alpha \in \mathcal{FI}_n : \operatorname{rank} \alpha \geq n-2\}.$$

Recall that the rank of a (partial) transformation α (in symbols: $\operatorname{rank}\alpha$) is the size of the range of α (in symbols: $\operatorname{im}\alpha$), i.e. $\operatorname{rank}\alpha = |\operatorname{im}\alpha|$. For the case that n is even, it is proved that $\operatorname{rank}\mathcal{FI}_n = n+1$ and a concrete generating set of $\mathcal{FI}_n$ with $n+1$ elements is given in [4]. We will summarize the results of [4] in the next section. On the other hand, the rank of $\mathcal{FI}_n$ is still an open problem, whenever n is odd. We will solve it in the present paper. We will determine the rank of $\mathcal{FI}_n$ and give a concrete generating set of $\mathcal{FI}_n$ with minimal size in the case that n is odd.

Without loss of generality, let $1 \prec 2 \succ 3 \prec \cdots \succ n$. Such fences are also called up-fences. The fence $1 \succ 2 \prec 3 \succ \cdots \prec n$ would be called down-fence. We avoid both notations up-fence and down-fence. In fact, in order to check a fence is an up-fence or down-fence, we need that 1 and 2 are comparable with respect to $\preceq$. Recall that $x, y \in \overline{n}$ are comparable with respect to $\preceq$ if $x \prec y$ or $x = y$ or $x \succ y$. Otherwise, x and y are called incomparable. But the restriction that 1 and 2 belong to the fence and are comparable is an unnecessary restriction for the concept fence since instead of $\overline{n}$ one could choose another n-element set or one could define $\preceq$ on $\overline{n}$ such that 1 and 2 are incomparable.

But if the fence $(\overline{n}, \preceq)$ is defined as above (which is the most natural way) then we observe that any $x, y \in \overline{n}$ are comparable if and only if $x \in \{y-1, y, y+1\}$. For general background on Semigroup Theory and standard notation, we refer the reader to Howie's book [9].

2 The even case

In [4, Theorem 3.15], the authors state that $\mathcal{FI}_n = \langle J_n \rangle$. In particular, to verify it, the authors show that two series of partial transformations can be generated by J_n. In [4, Corollary 3.4], the authors prove that

$$\begin{pmatrix} 1 & \cdots & m-2 & m & \cdots & m+p & m+p+2k+2 & \cdots & n \\ 1 & \cdots & m-2 & m+2k & \cdots & m+p+2k & m+p+2k+2 & \cdots & n \end{pmatrix} \in \langle J_n \rangle$$

for all $m, p, k \in \mathbb{N}$ with $m+p+2k \leq n$. The second series can be obtained by joining Corollary 3.6 and Lemma 3.7 (in [4]). For this let

$$v_p := \begin{cases} 0 & \text{if } p \text{ is even} \\ 1 & \text{otherwise,} \end{cases}$$

for any $p \in \mathbb{N}$. Then

$$\begin{pmatrix} 1 & \cdots & m-2 & m & \cdots & m+p & m+p+2k+2-v_p & \cdots & n \\ 1 & \cdots & m-2 & m+p+2k-v_p & \cdots & m+2k-v_p & m+p+2k+2-v_p & \cdots & n \end{pmatrix} \in \langle J_n \rangle$$

for $m, p, k \in \mathbb{N}$ with $m+p+2k-v_p \leq n$. Clearly, for all $U \subseteq \overline{n}$, the partial identity mapping $\mathrm{id}_U := \mathrm{id}_{\overline{n}}|_U$, i.e. the identity mapping $\mathrm{id}_{\overline{n}}$ on $\overline{n}$ restricted to U, is a partial automorphism on $(\overline{n}, \preceq)$. It is easy to verify that $\mathrm{id}_U \in \langle J_n \rangle$.

For any partial transformation α of the both previous series, x and $x\alpha$ have the same parity, for all x in the domain of α. Let $\alpha \in \mathcal{FI}_n$ be such that there is at least one $x \in \operatorname{dom}\alpha$ such that x and $x\alpha$ have different parity. Then there are $\eta_1, \ldots, \eta_k, \eta_{k+1}, \ldots, \eta_l \in J_n$ (for some $k, l \in \mathbb{N}$) such that $\eta_1^{-1}, \ldots, \eta_k^{-1}, \eta_{k+1}^{-1}, \ldots, \eta_l^{-1} \in J_n$, where x and $x(\eta_1 \ldots \eta_k \alpha \eta_{k+1} \ldots \eta_l)$ have the same parity for all $x \in \operatorname{dom}(\eta_1 \ldots \eta_k \alpha \eta_{k+1} \ldots \eta_l)$ and $\operatorname{rank}\alpha = \operatorname{rank}(\eta_1 \ldots \eta_k \alpha\, \eta_{k+1} \ldots \eta_l)$ [4, Proposition 3.9].

Using the previous facts, I. Dimitrova and J. Koppitz verify that for all $\alpha \in \mathcal{FI}_n$, there are $\omega_1, \omega_2 \in \langle J_n \rangle$ such that $\omega_1^{-1}, \omega_2^{-1} \in \langle J_n \rangle$, $\operatorname{dom}\alpha \subseteq \operatorname{im}\omega_1$, $\operatorname{im}\alpha \subseteq \operatorname{dom}\omega_2$, and $x(\omega_1\alpha\omega_2) = x$ for all $x \in \operatorname{dom}(\omega_1\alpha\omega_2)$ [4, Corollary 3.14]. This shows that $\omega_1\alpha\omega_2 = \mathrm{id}_{\operatorname{dom}(\omega_1\alpha\omega_2)} \in \langle J_n \rangle$. Consequently, $\mathcal{FI}_n = \langle J_n \rangle$.

If n is odd then using the GAP software, one can observe that $\mathcal{FI}_n$ is not generated by the set $\{\alpha \in \mathcal{FI}_n : \operatorname{rank}\alpha \geq n-1\}$. If n is even then $\mathrm{id}_{\overline{n}}$ is the only automorphism on $(\overline{n}, \preceq)$. This fact simplifies the situation and one can show that $\mathcal{FI}_n$ is generated by partial transformations with rank $\geq n-1$. It is easy to see that the only partial transformation in $\mathcal{FI}_n$ with rank $> n-1$ is the automorphism $\mathrm{id}_{\overline{n}}$. Let

$$\sigma_1 = \begin{pmatrix} 1 & 2 & 3 & \cdots & n \\ n & & 1 & \cdots & n-2 \end{pmatrix},$$

$$\sigma_2 = \begin{pmatrix} 1 & \cdots & n-2 & n-1 & n \\ 3 & \cdots & n & & 1 \end{pmatrix},$$

$$\rho_i = \begin{pmatrix} 1 & \cdots & i-1 & i & i+1 & \cdots & n \\ i-1 & \cdots & 1 & & i+1 & \cdots & n \end{pmatrix} \text{ and } \rho_i^2 \text{ for even } i \in \{4, \ldots, n\}, \text{ and}$$

$$\psi_i = \begin{pmatrix} 1 & \cdots & i-1 & i & i+1 & \cdots & n \\ 1 & \cdots & i-1 & & n & \cdots & i+1 \end{pmatrix} \text{ and } \psi_i^2 \text{ for odd } i \in \{1, \ldots, n-3\}.$$

In [4, Theorem 4.2], the authors show that $G_n = \{\mathrm{id}_{\overline{n}}, \sigma_1, \sigma_2\} \cup \{\rho_i : i \in \{4, \ldots, n\}$ is even$\} \cup \{\psi_i : i \in \{1, \ldots, n-3\}$ is odd$\}$ generates J_n. Thus, $\mathcal{FI}_n = \langle G_n \rangle$. On the other hand, also using GAP software, one can observe that $\mathcal{FI}_n$ has no (set theoretical) least generating set, whenever n is odd. Since if n is even, there is a generating set of $\mathcal{FI}_n$ consisting entirely of partial transformations with rank $\geq n-1$; it is not hard to see that there is a least generating set of $\mathcal{FI}_n$. In fact, in [4, Proposition 4.3], I. Dimitrova and J. Koppitz explain that all the members of G_n must be contained in any generating set of $\mathcal{FI}_n$. For more details about the proofs, we refer the reader to [4].

3 Main result

Let us fix now an odd natural number n. Note that $\mathscr{F}\mathscr{I}_1$ consists of the identity mapping on $\{1\}$ and the empty transformation $\emptyset$. Since neither of these partial transformations generates the other, the rank of $\mathscr{F}\mathscr{I}_1$ is 2. We suppose now that $n \geq 3$. There are exactly two automorphisms in $\mathscr{F}\mathscr{I}_n$, besides $\mathrm{id}_{\overline{n}} \in \mathscr{F}\mathscr{I}_n$, there is the reflection

$$\gamma_n := \begin{pmatrix} 1 & 2 & \cdots & n-1 & n \\ n & n-1 & \cdots & 2 & 1 \end{pmatrix}.$$

I. Dimitrova and J. Koppitz proved $\mathscr{F}\mathscr{I}_m = \langle J_m \rangle$ for all natural numbers m, which comprises several pages in [4]. For the case that n is odd, one can shorten the proof. Therefore, and for the sake of completeness, we will give a new proof for the particular case that n is odd. For this, we define a series of partial transformations of J_n. Let

$\alpha_i = \begin{pmatrix} 1 & \cdots & i-1 & i & i+1 & \cdots & n \\ 1 & \cdots & i-1 & & n & \cdots & i+1 \end{pmatrix}$ for even $i \in \{2, \dots, n-1\}$,

$\alpha_i = \mathrm{id}_{\overline{n} \backslash \{i\}}$ for odd $i \in \overline{n}$,

$\beta_2^{odd} = \begin{pmatrix} 1 & 2 & 3 & 4 & \cdots & n \\ 2 & & & 4 & \cdots & n \end{pmatrix}$, $\beta_{n-1}^{odd} = \begin{pmatrix} 1 & 2 & 3 & \cdots & n-1 & n \\ n-1 & & 1 & \cdots & n-3 & \end{pmatrix}$,

$\beta_2^{even} = \begin{pmatrix} 1 & 2 & 3 & 4 & \cdots & n \\ & 1 & & 4 & \cdots & n \end{pmatrix}$, and $\beta_{n-1}^{even} = \begin{pmatrix} 1 & \cdots & n-3 & n-2 & n-1 & n \\ 3 & \cdots & n-1 & & 1 & \end{pmatrix}$.

In the case $n \geq 5$, we define

$\alpha_{i,j} = \begin{pmatrix} 1 & \cdots & i-1 & i & i+1 & \cdots & j-1 & j & j+1 & \cdots & n \\ 1 & \cdots & i-1 & & j-1 & \cdots & i+1 & & j+1 & \cdots & n \end{pmatrix}$ for $2 \leq i < j \leq n-1$, where i and j have the same parity,

$\alpha_{1,j} = \begin{pmatrix} 1 & 2 & \cdots & j-1 & j & j+1 & \cdots & n \\ & j-1 & \cdots & 2 & & j+1 & \cdots & n \end{pmatrix}$ and

$\alpha_{j,n} = \begin{pmatrix} 1 & \cdots & j-1 & j & j+1 & \cdots & n-1 & n \\ 1 & \cdots & j-1 & & n-1 & \cdots & j+1 & \end{pmatrix}$ for odd $j \in \{3, \dots, n-2\}$, and

$\alpha_{1,n} = \begin{pmatrix} 1 & 2 & \cdots & n-1 & n \\ & n-1 & \cdots & 2 & \end{pmatrix}$.

In the case $n \geq 7$, we define

$\beta_i^{odd} = \begin{pmatrix} 1 & 2 & 3 & \cdots & i & i+1 & i+2 & \cdots & n \\ i & & 1 & \cdots & i-2 & & i+2 & \cdots & n \end{pmatrix}$ and

$\beta_i^{even} = \begin{pmatrix} 1 & \cdots & i-2 & i-1 & i & i+1 & i+2 & \cdots & n \\ 3 & \cdots & i & & 1 & & i+2 & \cdots & n \end{pmatrix}$ for even $i \in \{4, \dots, n-3\}$.

Note that $\beta_i^{even}\beta_i^{odd} = \mathrm{id}_{\overline{n} \backslash \{i-1,i+1\}}$ for each even $i \in \{2, \dots, n-1\}$, $\alpha_i\alpha_i = \mathrm{id}_{\overline{n} \backslash \{i\}}$ for each $i \in \overline{n}$, and $\alpha_{i,j}\alpha_{i,j} = \mathrm{id}_{\overline{n} \backslash \{i,j\}}$ for all $i < j \in \overline{n}$ having the same parity. Further, let Par_n be the set of all $\delta \in \mathscr{F}\mathscr{I}_n$ such that x and $x\delta$ have different parity for some $x \in \mathrm{dom}\,\delta$. First, we observe that each $\delta \in Par_n$ is generated by elements of $\mathscr{F}\mathscr{I}_n \backslash Par_n$ and $\{\mathrm{id}_{\overline{n}}\} \cup \{\beta_i^{odd}, \beta_i^{even} : i \in \{2, 4, \dots, n-1\}\} \subseteq J_n$.

Lemma 1 *Let $\delta \in Par_n$. Then there are $\widetilde{\beta} \in \mathscr{F}\mathscr{I}_n \backslash Par_n$ and $l_1, \dots, l_p, r_1, \dots, r_p \in \{\mathrm{id}_{\overline{n}}\} \cup \{\beta_i^{odd}, \beta_i^{even} : i \in \{2, 4, \dots, n-1\}\}$ with $\delta = l_1 \cdots l_p \widetilde{\beta} r_1 \cdots r_p$.*

Proof Let $x \in \operatorname{dom}\delta$ such that x and $x\delta$ have different parity. If x is odd then $x\delta$ is even, where $x\delta - 1, x\delta + 1 \notin \operatorname{im}\delta$. Let $\beta := \operatorname{id}_{\overline{n}}\delta\beta^{even}_{x\delta}$. Since by the construction of $\beta^{even}_{x\delta}$, it holds $x\beta = x\delta\beta^{even}_{x\delta} = 1$ and $\beta^{even}_{x\delta}$ does not change the parity of every $y \in \operatorname{dom}\beta^{even}_{x\delta}\backslash\{x\delta\}$, one gets immediately that $|\{w \in \operatorname{dom}\delta : w \text{ and } w\delta \text{ have different parity}\}| > |\{w \in \operatorname{dom}\beta : w \text{ and } w\beta \text{ have different parity}\}|$. Furthermore, $\beta\beta^{odd}_{x\delta} = \delta\beta^{even}_{x\delta}\beta^{odd}_{x\delta} = \delta\operatorname{id}_{\overline{n}\backslash\{x\delta-1, x\delta+1\}} = \delta$ since $\operatorname{im}\delta \subseteq \overline{n}\backslash\{x\delta - 1, x\delta + 1\}$.

If x is even then $x\delta$ is odd and $x - 1, x + 1 \notin \operatorname{dom}\delta$. In this case, let $\beta := \beta^{odd}_{x}\delta\operatorname{id}_{\overline{n}}$. Likewise, by the construction of β^{odd}_{x}, it holds $1\beta = 1\beta^{odd}_{x}\delta = x\delta$ and β^{odd}_{x} does not change the parity of every $y \in \operatorname{dom}\beta^{odd}_{x}\backslash\{1\}$, one gets immediately $|\{w \in \operatorname{dom}\delta : w \text{ and } w\delta \text{ have different parity}\}| > |\{w \in \operatorname{dom}\beta : w \text{ and } w\beta$ have different parity$\}|$. Furthermore, $\beta^{even}_{x}\beta = \beta^{even}_{x}\beta^{odd}_{x}\delta = \operatorname{id}_{\overline{n}\backslash\{x-1,x+1\}}\delta = \delta$ since $\operatorname{dom}\delta \subseteq \overline{n}\backslash\{x - 1, x + 1\}$.

Now, we can consider β instead of δ. Since the domain of δ is finite, we obtain successively after p steps $l_1, \ldots, l_p, r_1 \ldots, r_p \in \{\operatorname{id}_{\overline{n}}\} \cup \{\beta^{odd}_{i}, \beta^{even}_{i} : i \in \{2, 4, \ldots, n - 1\}\}$ and $\beta \in \mathscr{F}\mathscr{I}_n\backslash Par_n$ such that $\delta = l_1 \cdots l_p\beta r_1 \cdots r_p$. □

The following fact will be used frequently without further reference. If U is a convex subset of the domain of a $\delta \in \mathscr{F}\mathscr{I}_n$ then $U\delta = \{x\delta : x \in U\}$ is also a convex set. Next, we show that any $\delta \in \mathscr{F}\mathscr{I}_n\backslash J_n$ with a convex domain can be generated by elements of J_n and a transformation $\beta \in \mathscr{F}\mathscr{I}_n$ with $\operatorname{rank}\beta > \operatorname{rank}\delta$.

Lemma 2 *Let $\delta \in \mathscr{F}\mathscr{I}_n\backslash J_n$. If $\operatorname{dom}\delta$ is a convex set then there are $\beta \in \mathscr{F}\mathscr{I}_n$ with $\operatorname{rank}\beta = \operatorname{rank}\delta + 1$ and $\alpha \in J_n$ such that $\delta = \alpha\beta$.*

Proof Suppose that $\operatorname{dom}\delta$ is a convex set. Since both intervals $\operatorname{dom}\delta$ and $\operatorname{im}\delta$ have length less than or equal $n - 3$, there are $x, w \in \bar{n}$ such that $w - 1, w, w + 1 \in \{0, 1, \ldots, n, n + 1\}\backslash\operatorname{dom}\delta$ and $x - 1, x, x + 1 \in \{0, 1, \ldots, n, n + 1\}\backslash\operatorname{im}\delta$. We define a transformation β by

$$r\beta := r\delta \text{ for all } r \in \operatorname{dom}\delta \text{ and } w\beta := x.$$

Clearly, $\beta \in \mathscr{F}\mathscr{I}_n$ with $\operatorname{rank}\beta = \operatorname{rank}\delta + 1$ and $\delta = \operatorname{id}_{\overline{n}\backslash\{w\}}\beta$. □

Lemma 3 *Suppose $\delta \in \mathscr{F}\mathscr{I}_n\backslash Par_n$, and $\operatorname{dom}\delta$ is not a convex set. Then there exists $\widehat{\delta} \in \mathscr{F}\mathscr{I}_n\backslash Par_n$ with $\operatorname{rank}\delta = \operatorname{rank}\widehat{\delta}$, such that $\langle J_n, \widehat{\delta}\rangle$ and the following conditions hold:*

(a) There exists a convex interval $\widehat{I} = \{\widehat{z}, \widehat{z} + 1, \ldots, \widehat{x} - 1\} \subset \operatorname{dom}\widehat{\delta}$ such that $w \notin \operatorname{dom}\widehat{\delta}$ for all $w < \widehat{z}$, and $\widehat{x} \notin \operatorname{dom}\widehat{\delta}$. Moreover, for all $a \in \operatorname{dom}\widehat{\delta}$ with $a > \widehat{x}, b \in \widehat{I}$, it holds $a\widehat{\delta} > b\widehat{\delta}$.
(b) If $|\widehat{I}| > 2$ then $(\widehat{x} - 2)\widehat{\delta} < (\widehat{x} - 1)\widehat{\delta}$.
(c) $\widehat{z}\widehat{\delta} \in \{1, 2\}$.

Proof First, we will show that for a given $\delta \in \mathscr{F}\mathscr{I}_n \backslash Par_n$ there exist $\sigma \in \mathscr{F}\mathscr{I}_n \backslash Par_n$ with $\operatorname{rank}\delta = \operatorname{rank}\sigma$, such that $\langle J_n, \widehat{\delta}\rangle$ and a convex interval $\widehat{I} = \{\widehat{z}, \widehat{z}+1, \ldots, \widehat{x}-1\} \subset \operatorname{dom}\delta$ that satisfies a).

Let $\{r_1, \ldots, r_s\}$ be a convex subset of $\operatorname{dom}\delta$ with $r_1 < \cdots < r_s$ and $r_1 - 1, r_s + 1 \in \{0, 1, \ldots, n+1\} \backslash \operatorname{dom}\delta$ such that $a\delta > r_i\delta$ for all $a \in \operatorname{dom}\delta \backslash \{r_1, \ldots, r_s\}$ and $i \in \{1, \ldots, s\}$. If there is no $w \in \operatorname{dom}\delta$ with $w < r_1$ then let $\sigma := \delta$. Otherwise, there exists $w_0 \in \operatorname{dom}\delta$ with $w_0 < r_1$. Let $\{z, z+1, \ldots, x-1\} \subset \operatorname{dom}\delta$ be a convex interval with $z < z+1 < \cdots < x-1$ such that $w \notin \operatorname{dom}\delta$ for all $w < z$, and $x \notin \operatorname{dom}\delta$.

- Let r_s be odd and put $\alpha_0 := \gamma_n$. Note that $\gamma_n\alpha_{n-r_s}\gamma_n\gamma_n\alpha_{n-r_s}\gamma_n\delta = \delta$. Denote by $\sigma = \gamma_n\alpha_{n-r_s}\gamma_n\delta$, i.e. $\gamma_n\alpha_{n-r_s}\gamma_n\sigma = \delta$. Observe that if $r_s \neq n$, we have

$$\sigma = \begin{pmatrix} 1 & 2 & \cdots & r_s & r_s+1 & r_s+2 & \cdots & n-1 & n \\ r_s & r_s-1 & \cdots & 1 & & r_s+2 & \cdots & n-1 & n \end{pmatrix}\delta,$$

  whence $\operatorname{rank}\delta = \operatorname{rank}\sigma$. This shows (also for $r_s = n$) that $a\sigma < b\sigma$ for $a \leq r_s - r_1 + 1$ and $b \in \operatorname{dom}\sigma \backslash \{1, \ldots, r_s - r_1 + 1\}$. We put $\widetilde{z} := 1$ and $\widetilde{x} := r_s - r_1 + 2$.
- Let r_1 be odd and let r_s be even. Note that $\alpha_{r_1-1}\gamma_n\gamma_n\alpha_{r_1-1}\delta = \delta$. Denote by $\sigma = \gamma_n\alpha_{r_1-1}\delta$, i.e. $\alpha_{r_1-1}\gamma_n\sigma = \delta$. Observe that

$$\sigma = \begin{pmatrix} 1 & \cdots & n-r_1+1 & n-r_1+2 & n-r_1+3 & \cdots & n \\ r_1 & \cdots & n & & r_1-2 & & 1 \end{pmatrix}\delta,$$

  whence $\operatorname{rank}\sigma = \operatorname{rank}\delta$. This shows that $a\sigma < b\sigma$ for $a \leq r_s - r_1 + 1$ and $b \in \operatorname{dom}\sigma \backslash \{1, \ldots, r_s - r_1 + 1\}$. We put $\widetilde{z} := 1$ and $\widetilde{x} := r_s - r_1 + 2$.
- Let r_1 and r_s be even. Let $p := \max\{z\delta, \ldots, (x-1)\delta\}$ be odd and recall that $\alpha_0 := \gamma_n$. Note that $\delta = \delta\gamma_n\alpha_{n-p}\gamma_n\gamma_n\alpha_{n-p}\gamma_n$. Denote by $\sigma = \delta\gamma_n\alpha_{n-p}\gamma_n$, i.e. $\delta = \sigma\gamma_n\alpha_{n-p}\gamma_n$. Observe that if $p \neq n$ we have

$$\sigma = \delta\begin{pmatrix} 1 & \cdots & p & p+1 & p+2 & \cdots & n+1 & n \\ p & & 1 & & p+2 & \cdots & n+1 & n \end{pmatrix},$$

  whence $\operatorname{rank}\sigma = \operatorname{rank}\delta$. This shows (also for $p = n$) that $a\sigma < b\sigma$ for $a \in \{z, \ldots, x-1\}$ and $b \in \operatorname{dom}\delta \backslash \{z, \ldots, x-1\}$, where $\operatorname{dom}\sigma = \operatorname{dom}\delta$. We put $\widetilde{z} := z$ and $\widetilde{x} := x$.
- Let r_1, r_s, and p be even. Then $1 \notin \operatorname{im}\delta$. Note that $\delta = \delta\alpha_{1,p+1}\alpha_{1,p+1}$. Denote by $\sigma = \delta\alpha_{1,p+1}$, i.e. $\sigma\alpha_{1,p+1} = \delta$. Recall that

$$\delta\alpha_{1,p+1} = \delta\begin{pmatrix} 1 & 2 & \cdots & p & p+1 & p+2 & \cdots & n \\ & p & \cdots & 2 & & p+2 & \cdots & n \end{pmatrix},$$

  whence $\operatorname{rank}\sigma = \operatorname{rank}\delta$. This shows that $a\sigma < b\sigma$ for $a \in \{z, \ldots, x-1\}$ and $b \in \operatorname{dom}\delta \backslash \{z, \ldots, x-1\}$, where $\operatorname{dom}\sigma = \operatorname{dom}\delta$. We put $\widetilde{z} := z$ and $\widetilde{x} := x$.

Now, we will construct a transformation $\nu \in \mathscr{F}\mathscr{I}_n \backslash Par_n$ such that $\sigma \in \langle J_n, \widehat{\delta}\rangle$, $\operatorname{rank}\nu = \operatorname{rank}\sigma$, and there is a convex interval $\widehat{I} = \{\widehat{z}, \widehat{z}+1, \ldots, \widehat{x}-1\} \subset \operatorname{dom}\nu$

that satisfies a) and b) additionally. If $\widetilde{x}-1>\widetilde{z}$ and $(\widetilde{x}-2)\sigma<(\widetilde{x}-1)\sigma$ then let $\nu:=\sigma$. Suppose that $\widetilde{x}-2\in\operatorname{dom}\sigma$ with $(\widetilde{x}-2)\sigma>(\widetilde{x}-1)\sigma$.

- If $\widetilde{z}$ is odd, we define $\nu:=\sigma\gamma_n\alpha_{n-\widetilde{z}\sigma}\gamma_n=\sigma\begin{pmatrix}1&\dots&\widetilde{z}\sigma&\widetilde{z}\sigma+1&\widetilde{z}\sigma+2&\dots&n\\ \widetilde{z}\sigma&\dots&1&&\widetilde{z}\sigma+2&\dots&n\end{pmatrix}$, whence $\operatorname{rank}\nu=\operatorname{rank}\sigma$. Moreover, since it holds $\sigma=(\sigma\gamma_n\alpha_{n-\widetilde{z}\sigma}\gamma_n)\gamma_n\alpha_{n-\widetilde{z}\sigma}\gamma_n$, we get $\sigma=\nu\gamma_n\alpha_{n-\widetilde{z}\sigma}\gamma_n$. Further, we have $(\widetilde{x}-2)\nu<(\widetilde{x}-1)\nu$, where $\operatorname{dom}\nu=\operatorname{dom}\sigma$, and we put $\widehat{z}:=\widetilde{z}$ and $\widehat{x}:=\widetilde{x}$.
- If $\widetilde{z}$ is even and $\widetilde{x}-1$ is odd, we define

$$\nu:=\gamma_n\alpha_{n-\widetilde{x}+1}\gamma_n\sigma=\begin{pmatrix}1&\dots&\widetilde{x}-1&\widetilde{x}&\widetilde{x}+1&\dots&n\\ \widetilde{x}-1&\dots&1&&\widetilde{x}+1&\dots&n\end{pmatrix}\sigma,$$

  whence $\operatorname{rank}\nu=\operatorname{rank}\sigma$. Moreover, since it holds $\sigma=\gamma_n\alpha_{n-\widetilde{x}+1}\gamma_n(\gamma_n\alpha_{n-\widetilde{x}+1}\gamma_n\sigma)$, we get $\sigma=\gamma_n\alpha_{n-\widetilde{x}+1}\gamma_n\nu$. Further, we have $(\widetilde{x}-(\widetilde{z}+1)-2)\nu=(\widetilde{x}-\widetilde{z}-1)\nu<(\widetilde{x}-\widetilde{z})\nu=(\widetilde{x}-(\widetilde{z}+1)-1)\nu$, where $\operatorname{im}\nu=\operatorname{im}\sigma$, and we put $\widehat{z}:=1$ and $\widehat{x}:=\widetilde{x}-\widetilde{z}+1$.
- If $\widetilde{z}$ and $\widetilde{x}-1$ are even, we define

$$\nu:=\alpha_{\widetilde{z}-1,\widetilde{x}}\sigma.$$

  Recall that $\alpha_{\widetilde{z}-1,\widetilde{x}}\sigma=\begin{pmatrix}1&\cdots&\widetilde{z}-2&\widetilde{z}-1&\widetilde{z}&\cdots&\widetilde{x}-1&\widetilde{x}&\widetilde{x}+1&\cdots&n\\ 1&\cdots&\widetilde{z}-2&&\widetilde{x}-1&\cdots&\widetilde{z}&&\widetilde{x}+1&\cdots&n\end{pmatrix}\sigma$ if $\widetilde{z}\geq 4$ and $\alpha_{1,\widetilde{x}}\sigma=\begin{pmatrix}1&2&\cdots&\widetilde{x}-1&\widetilde{x}&\widetilde{x}+1&\cdots&n\\ &\widetilde{x}-1&\cdots&2&&\widetilde{x}+1&\cdots&n\end{pmatrix}\sigma$, whence $\operatorname{rank}\nu=\operatorname{rank}\sigma$. This shows that $(\widetilde{x}-2)\nu<(\widetilde{x}-1)\nu$ and we put $\widehat{z}:=\widetilde{z}$ and $\widehat{x}:=\widetilde{x}$.

If $\widehat{z}\nu\in\{1,2\}$ then $\widehat{\delta}:=\nu$ is the required transformation. Otherwise, let

$$\widehat{\delta}:=\nu\begin{pmatrix}3&4&\cdots&n\\1&2&\cdots&n-2\end{pmatrix}^{\left\lfloor\frac{\widehat{z}\nu-1}{2}\right\rfloor}.$$

Then $\widehat{z}\widehat{\delta}\in\{1,2\}$ as required. □

We note that the empty set is convex. So, the empty transformation $\emptyset$ is a product of two transformations with rank ≥ 1. Now, we can prove the following proposition, which is a particular case of Theorem 3.15 in [4] for n is odd.

Proposition 1 *$\mathcal{FI}_n$ is generated by J_n.*

Proof Recall that $\beta\in\langle J_n\rangle$ for all $\beta\in\mathcal{FI}_n$ with $\operatorname{rank}\beta>n-3$. By induction, we assume that $\beta\in\langle J_n\rangle$ for all $\beta\in\mathcal{FI}_n$ with $\operatorname{rank}\beta>n-r$ for some $r\in\{3,\dots,n\}$ and we will show that $\delta\in\langle J_n\rangle$ for all $\delta\in\mathcal{FI}_n$ with $\operatorname{rank}\delta=n-r$. Let now $\delta\in\mathcal{FI}_n$ with $\operatorname{rank}\delta=n-r$. By Lemma 3.1, we can restrict ourselves to the case $\delta\in\mathcal{FI}_n\backslash Par_n$. Let $\operatorname{dom}\delta$ be a convex set. By Lemma 3.2, there are $\beta\in\mathcal{FI}_n$ with $\operatorname{rank}\beta=\operatorname{rank}\delta+1$ and $\alpha\in J_n$ such that $\delta=\alpha\beta$. Since $\beta\in\langle J_n\rangle$ by the inductive assumption, we obtain $\delta=\alpha\beta\in\langle J_n\rangle$. Thus, we can assume further $\operatorname{dom}\delta$ is not a convex set. Then by Lemma 3.3, we can restrict ourselves to the case that the domain of δ contains the interval $\{z,\dots,x-1\}$ such that $w \notin \operatorname{dom}\delta$ for all $w<z$, $x \notin \operatorname{dom}\delta$, and $z\delta \in \{1,2\}$. We put now

$$a := (x-1)\delta.$$

Since $x \notin \operatorname{dom}\delta$ and $v\delta > (x-1)\delta$ for all $v \in \operatorname{dom}\delta \backslash \{z,\dots,x-1\}$, it holds that $a+1 \notin \operatorname{im}\delta$.

Case 1: $x+1 \in \operatorname{dom}\delta$.

If $(x+1)\delta = a+2$ then we define a partial transformation β_1 by $v\beta_1 = v\delta$ for $v \in \operatorname{dom}\delta$ and $x\beta_1 = a+1$. Clearly, $\beta_1 \in \mathscr{F}\mathscr{I}_n$ with $\operatorname{rank}\beta_1 > n-r$. We have $\delta = \alpha_x\alpha_x\beta_1 \in \langle J_n\rangle$. $(*)$

Now we show that there is $\delta^* \in \mathscr{F}\mathscr{I}_n$ with $v\delta^* = v\delta$ for $v \in \{z,\dots,x-1\}$ and $(x+1)\delta^* = a+2$ such that $\delta \in \langle J_n, \delta^*\rangle$. Then $\operatorname{rank}\delta^* \geq \operatorname{rank}\delta = n-r$. If $\operatorname{rank}\delta^* = n-r$ then $\delta^* \in \langle J_n\rangle$ by $(*)$. Hence, $\delta \in \langle J_n, \delta^*\rangle = \langle J_n\rangle$.

Case 1.1: Suppose that $a+2 \in \operatorname{dom}\delta$. Then there is $y \in \operatorname{dom}\delta$ with $y\delta = a+2$ since $v\delta \leq a$ for all $v \in \operatorname{dom}\delta$ with $v<x$, we conclude that $y > x+1$ (the case $y = x+1$ has been already considered in $(*)$). Note that $(x-1)\delta = a$ and $y\delta = a+2$ have the same parity. Hence, x and $y+1$ have the same parity, a principle fact for the further considerations.

(1) If $y+1 \notin \operatorname{dom}\delta$ then we put $\beta_2 := \alpha_{x,y+1}$, whenever $y<n$, and $\beta_2 := \alpha_x$, whenever $y=n$. Clearly, $\beta_2 \in J_n$. Then $\delta = \beta_2\beta_2\delta$, where $v\beta_2\delta = v\delta$ for $v \in \{z,\dots,x-1\}$ and $(x+1)\beta_2\delta = a+2$. So, $\delta^* := \beta_2\delta$ is as required.

(2) If $y+1 \in \operatorname{dom}\delta$ then $(y+1)\delta = a+3$ and $y-1 \notin \operatorname{dom}\delta$ since $a+1 \notin \operatorname{im}\delta$.

(2.1) If $y-1$ is even then $\delta = \alpha_{y-1}\alpha_{y-1}\delta$, where $v\alpha_{y-1}\delta = v\delta$ for $v \in \{z,\dots,x-1\}$ and $n\alpha_{y-1}\delta = a+2$. By (1), there is $\beta' \in J_n$ with $\beta'\beta'\alpha_{y-1}\delta = \alpha_{y-1}\delta$ and $\delta^* := \beta'\alpha_{y-1}\delta$ is as required.

(2.2) If $y-1$ is odd and there is an odd $m > y+1$ with $m \notin \operatorname{dom}\delta$ then we have $\delta = \alpha_{y-1,m}\alpha_{y-1,m}\delta$, where $v\alpha_{y-1,m}\delta = v\delta$ for $v \in \{z,\dots,x-1\}$, $(m-1)\alpha_{y-1,m}\delta = a+2$, and $m \notin \operatorname{dom}(\alpha_{y-1,m}\delta)$. Then by (1), there is $\beta' \in J_n$ with $\beta'\beta'\alpha_{y-1,m}\delta = \alpha_{y-1,m}\delta$ and $\delta^* := \beta'\alpha_{y-1,m}\delta$ is as required.

(2.3) If $y-1$ is odd and each $m \in \overline{n}\backslash \operatorname{dom}\delta$ with $m > y+1$ is even. Since $x+1 \in \operatorname{dom}\delta$, there is an even $i > a+2$ with $i \notin \operatorname{im}\delta$ and $(i+1) \notin \operatorname{im}\delta$ such that $i+2 = (x+1)\delta$ since $x+1$ is even, $(x+1)\delta > y\delta = a+2$, and $v \in \operatorname{dom}\delta$ for all odd $v > y$. Hence, there is $u \in \operatorname{dom}\delta$ with $u, u+1 \notin \operatorname{dom}\delta$, where u is even.

If $u > x$ then we have $\delta = \alpha_u\alpha_u\delta$, where $v\alpha_u\delta = v\delta$ for $v \in \operatorname{dom}\delta \cap \{z,\dots,u-1\}$ and $(n+u-y+1)\alpha_u\delta = y\delta = a+2$, where $(n+u-(y-1)+1)\alpha_u = y-1 \notin \operatorname{dom}\delta$, i.e. $((n+u-y+1)+1) \notin \operatorname{dom}(\alpha_u\delta)$. By (1), there is $\beta' \in J_n$ with $\beta'\beta'\alpha_u\delta = \alpha_u\delta$ and $\delta^* := \beta'\alpha_u\delta$ is as required.

If $u < x$ then $1,2 \notin \operatorname{dom}\delta$. We have $\alpha_2\gamma_n\gamma_n\alpha_2\delta = \delta$, where $v\gamma_n\alpha_2\delta = (v+2)\delta$ for $v \in \{1,\dots,n-2\}$ with $v+2 \in \operatorname{dom}\delta$. Then $(y-2)\gamma_n\alpha_2\delta = a+2$, $y-3 \notin \operatorname{dom}\gamma_n\alpha_2\delta$, and $n \notin \operatorname{dom}(\gamma_n\alpha_2\delta)$. As in (2.2), we can conclude that there is $\beta' \in J_n$ such that $\delta^* := \beta'\alpha_{y-1,n}\gamma_n\alpha_2\delta$ is as required.

Case 1.2: Suppose that $a+2 \notin \operatorname{dom}\delta$.

(1) If $(x+1)\delta+1 \notin \operatorname{im}\delta$ then we $\delta = \delta\alpha_{a+1,(x+1)\delta+1}\alpha_{a+1,(x+1)\delta+1}$, where $v\delta\alpha_{a+1,(x+1)\delta+1} = v\delta$ for $v \in \{z,\dots,x-1\}$ and $(x+1)\delta\alpha_{a+1,(x+1)\delta+1} = a+2$. This shows that $\delta^* := \delta\alpha_{a+1,(x+1)\delta+1}$ is as required.

(2) Suppose now that $(x+1)\delta+1 \in \operatorname{im}\delta$, i.e. $(x+1)\delta-1 \notin \operatorname{im}\delta$. If $(x+1)\delta$ is even then $a+2$ is even and we put $\beta_3 := \alpha_{a+2}$. If $(x+1)\delta$ is odd then we put $\beta_3 := \alpha_{(x+1)\delta-1}$. Clearly, $\beta_3 \in J_n$, $\delta = \delta\beta_3\beta_3$, where $v\delta\beta_3 = v\delta$ for $v \in \{z,\dots,x-1\}$, and $a+2 \notin \operatorname{im}\delta\beta_3$. Since $(x+1)\delta-1 \notin \operatorname{im}\delta$, we can calculate that $(x+1)\delta\beta_3+1 \notin \operatorname{im}\delta\beta_3$. By Case 1.2, (1), we have that $\delta^* := \delta\beta_3\alpha_{a+1,(x+1)\delta\beta_3+1}$ is as required.

Case 2: $x+1 \notin \operatorname{dom}\delta$.

(1) If $a+2 \notin \operatorname{im}\delta$, we define a partial transformation β_4 by $v\beta_4 = v\delta$ for $v \in \operatorname{dom}\delta$ and $x\beta_4 = a+1$. It is easy to verify that $\beta_4 \in \mathscr{F}\mathscr{I}_n$ with rank $\beta_4 > n-r$. Then we have $\delta = \alpha_x\alpha_x\beta_4 \in \langle J_n\rangle$.

(2) Suppose $a+2 \in \operatorname{im}\delta$, i.e. there is $y \in \operatorname{dom}\delta$ with $y\delta = a+2$.

(2.1) If $y+1 \notin \operatorname{dom}\delta$ then we obtain $\delta = \alpha_{x,y+1}\alpha_{x,y+1}\delta$, where $v\alpha_{x,y+1}\delta = v\delta$ for $v \in \{z,\dots,x-1\}$, and $(x+1)\alpha_{x,y+1}\delta = a+2$. Then by Case 1, we obtain that $\alpha_{x,y+1}\delta \in \langle J_n\rangle$, i.e. $\delta = \alpha_{x,y+1}\alpha_{x,y+1}\delta \in \langle J_n\rangle$.

(2.2) If $y+1 \in \operatorname{dom}\delta$ then $y-1 \notin \operatorname{dom}\delta$ and we take $i \in \{0,1\}$ such that $x+i$ is even. Since $x+i \notin \operatorname{dom}\delta$, we have $\delta = \alpha_{x+i}\alpha_{x+i}\delta$, where $v\alpha_{x+i}\delta = v\delta$ for $v \in \{z,\dots,x-1\}$. In particular, we have $(n-y+x+i+1)\alpha_{x+i}\delta = a+2$ and $n-y+x+i+2 \notin \operatorname{dom}(\alpha_{x+i}\delta)$. By Case 1 (if $x+1 \in \operatorname{dom}(\alpha_{x+i}\delta)$) and by Case 2, (2.1) (if $x+1 \notin \operatorname{dom}(\alpha_{x+i}\delta)$), respectively, we can conclude that $\delta \in \langle J_n\rangle$. $\square$

Now, we construct a generating set of $\mathscr{F}\mathscr{I}_n$ of minimal size. By Proposition 1 and since no element in J_n can be generated by elements which do not belong to J_n, we have to find a generating set of J_n of minimal size. For this, we define

$$G_3 := \{\gamma_3, \alpha_1, \alpha_2, \beta_2^{odd}, \beta_2^{even}\}$$

and

$$\begin{aligned} G_n := \{\gamma_n\} &\cup \{\alpha_i : i \in \{1,3,\dots,\tfrac{n+1}{2}\} \text{ is odd}\} \cup \\ &\{\alpha_i : i \in \{2,4,\dots,n-3\} \text{ is even}\} \cup \\ &\{\beta_i^{odd}, \beta_i^{even} : i \in \{2,\dots,\tfrac{n+1}{2}\} \text{ is even}\} \cup \\ &\{\alpha_{i,j} : i,j \in \overline{n} \text{ are odd with } 4 \le j-i < n-1,\ i \le n-j+1\}, \end{aligned}$$

whenever $n \ge 5$.

Lemma 4 *Let $\delta \in J_n \cap Par_n$. Then there is exactly one $x \in \operatorname{dom}\delta$ such that x and $x\delta$ have different parity. In particular, it holds $x\delta \in \{1,n\}$ or $x \in \{1,n\}$.*

Proof Since $\delta \in Par_n$, there is an $x \in \operatorname{dom}\delta$ such that x and $x\delta$ have different parity. Assume that x is even and $x\delta \notin \{1,n\}$. Then $x-1, x+1 \notin \operatorname{dom}\delta$ and $x\delta-1, x\delta+1 \notin \operatorname{im}\delta$. Since $x\delta-1, x\delta+1 \in \{2,\dots,n-1\}$, we have $\frac{n-1}{2}-2$ even elements in $\operatorname{im}\delta$. Hence, since rank $\delta \ge n-2$, there is only one $y \in \operatorname{dom}\delta$ such that y and $y\delta$ have different parity, namely x. So, we have $\frac{n+1}{2}-2$ odd elements and $\frac{n-1}{2}$ even elements in the domain of δ. Since $x\delta-1, x\delta+1 \in \{2,\dots,n-1\}$, we have $\frac{n-1}{2}-2$ even elements in the image. So, δ maps the $\frac{n-1}{2}$ even elements to $\frac{n-1}{2}-2$ even elements and one odd element, i.e. to $\frac{n-1}{2}-1$ elements, a contradiction.

If x is odd then $x\delta$ is even and by the dual argumentations we obtain that $x \in \{1, n\}$. $\square$

Lemma 5 *Let $\delta \in J_n$. Then $\delta \in \langle G_n \rangle$.*

Proof If $\operatorname{rank}\delta = n$ then $\delta = \gamma_n \in G_n$ or $\delta = \mathrm{id}_{\overline{n}} = \gamma_n\gamma_n \in \langle G_n \rangle$.

Let now $\operatorname{rank}\delta = n-1$. Then there is $i \in \overline{n} \setminus \operatorname{dom}\delta$.

Case 1: Let i be even. Then we have 8 types for δ:

1.1) $$\delta = \begin{pmatrix} 1 & \cdots & i-1 & i & i+1 & \cdots & n \\ 1 & \cdots & i-1 & & n & \cdots & i+1 \end{pmatrix} = \begin{cases} \alpha_i \in G_n & \text{if } i < n-1 \\ \alpha_{n-1} = \gamma_n\alpha_2^2\gamma_n \in \langle G_n \rangle & \text{if } i = n-1; \end{cases}$$

1.2) $\delta = \begin{pmatrix} 1 & \cdots & i-1 & i & i+1 & \cdots & n \\ 1 & \cdots & i-1 & & i+1 & \cdots & n \end{pmatrix} = \alpha_i^2 \in \langle G_n \rangle$;

1.3) $\delta = \begin{pmatrix} 1 & \cdots & i-1 & i & i+1 & \cdots & n \\ i-1 & \cdots & 1 & & i+1 & \cdots & n \end{pmatrix} = \gamma_n\alpha_{n-i+1}\gamma_n \in \langle G_n \rangle$;

1.4) $\delta = \begin{pmatrix} 1 & \cdots & i-1 & i & i+1 & \cdots & n \\ i-1 & \cdots & 1 & & n & \cdots & i+1 \end{pmatrix} = \alpha_i\gamma_n\alpha_{n-i+1}\gamma_n \in \langle G_n \rangle$;

1.5) $\delta = \begin{pmatrix} 1 & \cdots & i-1 & i & i+1 & \cdots & n \\ n & \cdots & n-i+2 & & 1 & \cdots & n-i \end{pmatrix} = \alpha_i\gamma_n \in \langle G_n \rangle$;

1.6) $\delta = \begin{pmatrix} 1 & \cdots & i-1 & i & i+1 & \cdots & n \\ n & \cdots & n-i+2 & & n-i & \cdots & 1 \end{pmatrix} = \alpha_i^2\gamma_n \in \langle G_n \rangle$;

1.7) $\delta = \begin{pmatrix} 1 & \cdots & i-1 & i & i+1 & \cdots & n \\ n-i+2 & \cdots & n & & n-i & \cdots & 1 \end{pmatrix} = \gamma_n\alpha_{n-i+1} \in \langle G_n \rangle$;

1.8) $\delta = \begin{pmatrix} 1 & \cdots & i-1 & i & i+1 & \cdots & n \\ n-i+2 & \cdots & n & & 1 & \cdots & n-i \end{pmatrix} = (\gamma_n\alpha_{n-i+1})(\gamma_n\alpha_i\gamma_n) \in \langle G_n \rangle$ (a composition of cases 1.7 and 1.3).

Case 2: Let i be odd. Then $\alpha_i = \begin{pmatrix} 1 & \dots & i-1 & i & i+1 & \dots & n \\ 1 & \dots & i-1 & & i+1 & \dots & n \end{pmatrix} \in G_n$ for $i \le \frac{n+1}{2}$. Let $i > \frac{n+1}{2}$. Then $n-i+1 = 2(\frac{n+1}{2}) - i < 2(\frac{n+1}{2}) - \frac{n+1}{2} = \frac{n+1}{2}$, i.e. $\alpha_{n-i+1} \in G_n$. So, we have $\alpha_i = \gamma_n\alpha_{n-i+1}\gamma_n \in \langle G_n \rangle$. Further, there is one more type for $\delta \in J_n$:

$$\delta = \begin{pmatrix} 1 & \dots & i-1 & i & i+1 & \dots & n \\ n & \dots & n-i+2 & & n-i & \dots & 1 \end{pmatrix} = \alpha_i\gamma_n \in \langle G_n \rangle.$$

So, we have shown that $\delta \in \langle G_n \rangle$, whenever $\operatorname{rank}\delta \ge n-1$. It remains to show that $\delta \in \langle G_n \rangle$, whenever $\operatorname{rank}\delta = n-2$.

Let i be even. Then we have $\beta_i^{odd} = \begin{pmatrix} 1 & 2 & 3 & \dots & i & i+1 & i+2 & \dots & n \\ i & & 1 & \dots & i-2 & & i+2 & \dots & n \end{pmatrix} \in G_n$ and $\beta_i^{even} = \begin{pmatrix} 1 & \dots & i-2 & i-1 & i & i+1 & i+2 & \dots & n \\ 3 & \dots & i & & 1 & & i+2 & \dots & n \end{pmatrix} \in G_n$ for $i \le \frac{n+1}{2}$ is even. Let $i > \frac{n+1}{2}$ be even. Then $n-i+1 = 2(\frac{n+1}{2}) - i < 2(\frac{n+1}{2}) - \frac{n+1}{2} = \frac{n+1}{2}$, i.e. $\beta_{n-i+1}^{even}, \beta_{n-i+1}^{odd} \in G_n$. So, we have $\beta_i^{odd} = \alpha_2\beta_{n-i+1}^{odd}\gamma_n \in \langle G_n \rangle$ and $\beta_i^{even} = \gamma_n\beta_{n-i+1}^{even}\alpha_2 \in \langle G_n \rangle$.

Let now $i, j \in \overline{n} \setminus \operatorname{dom}\delta$ with $i < j$.

Suppose that i and j have the same parity with $4 \le j - i < n-1$. Then

$$\alpha_{i,j} = \begin{pmatrix} 1 & \ldots & i-1 & i & i+1 & \ldots & j-1 & j & j+1 & \ldots & n \\ 1 & \ldots & i-1 & & j-1 & \ldots & i+1 & & j+1 & \ldots & n \end{pmatrix} \in G_n$$

for $i \leq n-j+1$. Let $i > n-j+1$. Then $n-j+1 < i = n-(n-i+1)+1$, i.e. $\alpha_{n-j+1,n-i+1} \in G_n$, and we have $\alpha_{i,j} = \gamma_n \alpha_{n-j+1,n-i+1} \gamma_n \in \langle G_n \rangle$.

After these preliminary remarks, we discuss all possible types of δ.

Case 3: Let $i = 1$ and $j = 2$. Then we have 4 types for δ:

3.1) $\delta = \begin{pmatrix} 1 & 2 & 3 & \ldots & n \\ & & 3 & \ldots & n \end{pmatrix} = \alpha_1 \alpha_2^2 \in \langle G_n \rangle$;

3.2) $\delta = \begin{pmatrix} 1 & 2 & 3 & \ldots & n \\ & & n & \ldots & 3 \end{pmatrix} = \alpha_1 \alpha_2 \in \langle G_n \rangle$;

3.3) $\delta = \begin{pmatrix} 1 & 2 & 3 & \ldots & n \\ & & 1 & \ldots & n-2 \end{pmatrix} = \alpha_1 \alpha_2 \gamma_n \in \langle G_n \rangle$;

3.4) $\delta = \begin{pmatrix} 1 & 2 & 3 & \ldots & n \\ & & n-2 & \ldots & 1 \end{pmatrix} = \alpha_1 \alpha_2^2 \gamma_n \in \langle G_n \rangle$.

Case 4: Let $i = 1$ and $j = n$. Then we have two types for δ:

$\delta = \begin{pmatrix} 1 & 2 & \ldots & n-1 & n \\ \ldots & 2 & \ldots & n-1 & \end{pmatrix} = \alpha_1 \alpha_n \in \langle G_n \rangle$ and

$\delta = \begin{pmatrix} 1 & 2 & \ldots & n-1 & n \\ \ldots & n-1 & \ldots & 2 & \end{pmatrix} = \alpha_1 \alpha_n \gamma_n \in \langle G_n \rangle$.

Case 5: Let $i = 1$ and let $j \geq 4$ be even. Then we have 8 types for δ, which are the compositions of α_1 and the eight types in Case 1:

5.1) $\delta = \begin{pmatrix} 1 & 2 & \ldots & j-1 & j & j+1 & \ldots & n \\ & n-1 & \ldots & n-j+2 & & 1 & \ldots & n-j \end{pmatrix} = \alpha_1 \alpha_j \gamma_n \in \langle G_n \rangle$;

5.2) $\delta = \begin{pmatrix} 1 & 2 & \ldots & j-1 & j & j+1 & \ldots & n \\ & n-j+3 & \ldots & n & & 1 & \ldots & n-j \end{pmatrix} = \alpha_1 (\gamma_n \alpha_{n-j+1})(\gamma_n \alpha_j \gamma_n) \in \langle G_n \rangle$;

5.3) $\delta = \begin{pmatrix} 1 & 2 & \ldots & j-1 & j & j+1 & \ldots & n \\ & n-j+3 & \ldots & n & & n-j & \ldots & 1 \end{pmatrix} = \alpha_1 \gamma_n \alpha_{n-j+1} \in \langle G_n \rangle$;

5.4) $\delta = \begin{pmatrix} 1 & 2 & \ldots & j-1 & j & j+1 & \ldots & n \\ & n-1 & \ldots & n-j+2 & & n-j & \ldots & 1 \end{pmatrix} = \alpha_1 \alpha_j^2 \gamma_n \in \langle G_n \rangle$;

5.5) $\delta = \begin{pmatrix} 1 & 2 & \ldots & j-1 & j & j+1 & \ldots & n \\ & j-2 & \ldots & 1 & & n & \ldots & j+1 \end{pmatrix} = \alpha_1 \gamma_n \alpha_{n-j+1} \gamma_n \alpha_j \in \langle G_n \rangle$;

5.6) $\delta = \begin{pmatrix} 1 & 2 & \ldots & j-1 & j & j+1 & \ldots & n \\ & j-2 & \ldots & 1 & & j+1 & \ldots & n \end{pmatrix} = \alpha_1 \gamma_n \alpha_{n-j+1} \gamma_n \in \langle G_n \rangle$;

5.7) $\delta = \begin{pmatrix} 1 & 2 & \ldots & j-1 & j & j+1 & \ldots & n \\ & 2 & \ldots & j-1 & & j+1 & \ldots & n \end{pmatrix} = \alpha_1 \alpha_j^2 \in \langle G_n \rangle$;

5.8) $\delta = \begin{pmatrix} 1 & 2 & \ldots & j-1 & j & j+1 & \ldots & n \\ & 2 & \ldots & j-1 & & n & \ldots & j+1 \end{pmatrix} = \alpha_1 \alpha_j \in \langle G_n \rangle$.

Case 6: Let $i = 1$ and $j = 3$. Then we have 6 types for δ:

6.1) $\delta = \begin{pmatrix} 1 & 2 & 3 & 4 & \ldots & n \\ & 2 & & 4 & \ldots & n \end{pmatrix} = \alpha_1 \alpha_3 \in \langle G_n \rangle$;

6.2) $\delta = \begin{pmatrix} 1 & 2 & 3 & 4 & \ldots & n \\ & n-1 & & n-3 & \ldots & 1 \end{pmatrix} = \alpha_1 \alpha_3 \gamma_n \in \langle G_n \rangle$;

6.3) $\delta = \begin{pmatrix} 1 & 2 & 3 & 4 & \ldots & n \\ & 1 & & 4 & \ldots & n \end{pmatrix} = \beta_2^{even} \in G_n$;

6.4) $\delta = \begin{pmatrix} 1 & 2 & 3 & 4 & \ldots & n \\ & n & & n-3 & \ldots & 1 \end{pmatrix} = \beta_2^{even} \gamma_n \in \langle G_n \rangle$;

6.5) $\delta = \begin{pmatrix} 1 & 2 & 3 & 4 & \cdots & n \\ & 1 & & n-1 & \cdots & 3 \end{pmatrix} = \beta_2^{even}\alpha_2 \in \langle G_n \rangle;$

6.6) $\delta = \begin{pmatrix} 1 & 2 & 3 & 4 & \cdots & n \\ & n & & 2 & \cdots & n-2 \end{pmatrix} = \beta_2^{even}\alpha_2\gamma_n \in \langle G_n \rangle.$

Case 7: Let $i = 1$ and let $j > 4$ be odd. Then we have 4 types for δ :

7.1) $\delta = \begin{pmatrix} 1 & 2 & \dots & j-1 & j & j+1 & \dots & n \\ & n-1 & \dots & n-j+2 & & n-j & \dots & 1 \end{pmatrix} = \alpha_1\alpha_j\gamma_n \in \langle G_n \rangle;$

7.2) $\delta = \begin{pmatrix} 1 & 2 & \dots & j-1 & j & j+1 & \dots & n \\ & n-j+2 & \dots & n-1 & & n-j & \dots & 1 \end{pmatrix} = \alpha_{1,j}\gamma_n \in \langle G_n \rangle;$

7.3) $\delta = \begin{pmatrix} 1 & 2 & \dots & j-1 & j & j+1 & \dots & n \\ & 2 & \dots & j-1 & & j+1 & \dots & n \end{pmatrix} = \alpha_1\alpha_j \in \langle G_n \rangle;$

7.4) $\delta = \begin{pmatrix} 1 & 2 & \dots & j-1 & j & j+1 & \dots & n \\ & j-1 & \dots & 2 & & j+1 & \dots & n \end{pmatrix} = \alpha_{1,j} \in \langle G_n \rangle.$

Case 8: Let $i = n$. Then we can show that $\delta \in \langle G_n \rangle$ dually as in the cases 3 until 7.

Case 9: Let $i = 2$ and $j = 3$. Then we have 6 types for δ :

9.1) $\delta = \begin{pmatrix} 1 & 2 & 3 & 4 & \cdots & n \\ 2 & & & 4 & \cdots & n \end{pmatrix} = \beta_2^{odd} \in G_n;$

9.2) $\delta = \begin{pmatrix} 1 & 2 & 3 & 4 & \cdots & n \\ n-1 & & & n-3 & \cdots & 1 \end{pmatrix} = \beta_2^{odd}\gamma_n \in \langle G_n \rangle;$

9.3) $\delta = \begin{pmatrix} 1 & 2 & 3 & 4 & \cdots & n \\ 1 & & & 4 & \cdots & n \end{pmatrix} = \alpha_2^2\alpha_3 \in \langle G_n \rangle;$

9.4) $\delta = \begin{pmatrix} 1 & 2 & 3 & 4 & \cdots & n \\ n & & & n-3 & \cdots & 1 \end{pmatrix} = \alpha_2^2\alpha_3\gamma_n \in \langle G_n \rangle;$

9.5) $\delta = \begin{pmatrix} 1 & 2 & 3 & 4 & \cdots & n \\ 1 & & & n-1 & \cdots & 3 \end{pmatrix} = \alpha_2\alpha_3 \in \langle G_n \rangle;$

9.6) $\delta = \begin{pmatrix} 1 & 2 & 3 & 4 & \cdots & n \\ n & & & 2 & \cdots & n-2 \end{pmatrix} = \alpha_2\alpha_3\gamma_n \in \langle G_n \rangle.$

Case 10: Let $i \geq 4$ be even and $j = i + 1$, i.e.

$$\delta = \begin{pmatrix} 1 & \cdots & i-1 & i & i+1 & i+2 & \cdots & n \\ 1\delta & \cdots & (i-1)\delta & & & (i+2)\delta & \cdots & n\delta \end{pmatrix}.$$

In Case 8, we have argued that all transformations α in with $\operatorname{rank}\alpha = n - 2$ and $n \notin \operatorname{dom}\alpha$ are generated by G_n. So, we have

$$\begin{pmatrix} 1 & \cdots & i-1 & i & i+1 & \cdots & i+2 & n \\ 1\delta & \cdots & (i-1)\delta & & n\delta & \cdots & (i+2)\delta & \end{pmatrix} \in \langle G_n \rangle$$

and

$$\delta = \alpha_i \begin{pmatrix} 1 & \cdots & i-1 & i & i+1 & \cdots & i+2 & n \\ 1\delta & \cdots & (i-1)\delta & & n\delta & \cdots & (i+2)\delta & \end{pmatrix} \in \langle G_n \rangle.$$

Case 11: Let $i \geq 3$ be odd and let $j = i + 1$. Then j is even and we can dually show that $\delta \in \langle G_n \rangle$ as in in the previous Case 10.

Case 12: Let j be even and let $i \geq 2$ with $i+1<j$. We observe that $\operatorname{im}\delta$ is determined by the order of the elements 1δ, $(i+1)\delta$, and $n\delta$. So, we have to consider 6 cases and define a mapping $\delta_0 \in \langle G_n\rangle$ as follows:

12a) If $1\delta < (i+1)\delta < n\delta$ then let

$$\delta_0 := \alpha_i^2\alpha_j^2 = \begin{pmatrix} 1 & \cdots & i-1 & i & i+1 & \cdots & j-1 & j & j+1 & \cdots & n \\ 1 & \cdots & i-1 & & i+1 & \cdots & j-1 & & j+1 & \cdots & n \end{pmatrix};$$

12b) Let $1\delta < n\delta < (i+1)\delta$. Note that if i is odd then we have to have that $j=n-1$. We put

$$\delta_0 := \begin{cases} \alpha_i\alpha_j^2 = \begin{pmatrix} 1 & \cdots & i-1 & i & i+1 & \cdots & j-1 & j & j+1 & \cdots & n \\ 1 & \cdots & i-1 & & n & \cdots & n+i-j+2 & & n+i-j & \cdots & i+1 \end{pmatrix} & \text{if } i \text{ is even} \\ \gamma_n\beta_{n-i}^{odd}\gamma_n = \begin{pmatrix} 1 & \cdots & i-1 & i & i+1 & \cdots & n-2 & n-1 & n \\ 1 & \cdots & i-1 & & i+3 & \cdots & n & & i+1 \end{pmatrix} & \text{if } i \text{ is odd;} \end{cases}$$

12c) If $(i+1)\delta < 1\delta < n\delta$ then let

$$\delta_0 := \alpha_i^2\gamma_n\alpha_{n-j+1}\gamma_n = \begin{pmatrix} 1 & \cdots & i-1 & i & i+1 & \cdots & j-1 & j & j+1 & \cdots & n \\ j-1 & \cdots & j-i+1 & & j-i-1 & \cdots & 1 & & j+1 & \cdots & n \end{pmatrix};$$

12d) If $n\delta < 1\delta < (i+1)\delta$ then let

$$\delta_0 := \alpha_i^2\gamma_n\alpha_{n-j+1} = \begin{pmatrix} 1 & \cdots & i-1 & i & i+1 & \cdots & j-1 & j & j+1 & \cdots & n \\ n-j+2 & \cdots & n-j+i & & n-j+i+2 & \cdots & n & & n-j & \cdots & 1 \end{pmatrix};$$

12e) Let $(i+1)\delta < n\delta < 1\delta$. Note that if i is odd then we have to have that $j=n-1$. We put

$$\delta_0 := \begin{cases} \alpha_j^2\alpha_i\gamma_n = \begin{pmatrix} 1 & \cdots & i-1 & i & i+1 & \cdots & j-1 & j & j+1 & \cdots & n \\ n & \cdots & n-i+2 & & 1 & \cdots & j-i-1 & & j-i+1 & \cdots & n-i \end{pmatrix} & \text{if } i \text{ is even} \\ \gamma_n\beta_{n-i}^{odd} = \begin{pmatrix} 1 & \cdots & i-1 & i & i+1 & \cdots & n-2 & n-1 & n \\ n & \cdots & n-i+2 & & j-i-1 & \cdots & 1 & & n-i \end{pmatrix} & \text{if } i \text{ is odd;} \end{cases}$$

12f) If $n\delta < (i+1)\delta < 1\delta$ then let

$$\delta_0 := \alpha_i^2\alpha_j^2\gamma_n = \begin{pmatrix} 1 & \cdots & i-1 & i & i+1 & \cdots & j-1 & j & j+1 & \cdots & n \\ n & \cdots & n-i+2 & & n-i & \cdots & n-j+2 & & n-j & \cdots & 1 \end{pmatrix}.$$

Now, we define transformations δ_1, δ_2, and δ_3 in $\langle G_n\rangle$ as follows:

$\delta_1 := \begin{pmatrix} 1 & \cdots & i-1 & i & i+1 & \cdots & n \\ i-1 & & 1 & & i+1 & \cdots & n \end{pmatrix} = \gamma_n\alpha_{n-i+1}\gamma_n$ if $1\delta-(i-1)\delta \neq 1\delta_0-(i-1)\delta_0$;

$\delta_2 := \begin{pmatrix} 1 & \cdots & i-1 & i & i+1 & \cdots & j-1 & j & j+1 & \cdots & n \\ 1 & & i-1 & & j-1 & \cdots & i+1 & & j+1 & \cdots & n \end{pmatrix} = \alpha_{i,j}$ if $i(i+1)\delta-(j-1)\delta \neq (i+1)\delta_0-(j-1)\delta_0$;

$\delta_3 := \begin{pmatrix} 1 & \cdots & i-1 & i & i+1 & \cdots & n \\ 1 & & i-1 & & n & \cdots & i+1 \end{pmatrix} = \alpha_i$ if $(i+1)\delta-n\delta \neq (i+1)\delta_0-n\delta_0$;

and $\delta_i := \mathrm{id}_{\overline{n}}$, for $i \in \{1,2,3\}$, otherwise. Then $\delta = \delta_1\delta_2\delta_3\delta_0 \in \langle G_n\rangle$.

Case 13: Let i be even and let $j \leq n-1$ with $j-1 > i$. The we obtain dually as in Case 12 that $\delta \in \langle G_n \rangle$.

Case 14: Let $i, j \in \{3, \dots, n-2\}$ be odd. We observe that $\operatorname{im} \delta$ is determined by the order of the elements 1δ, $(i+1)\delta$, and $n\delta$. One observation more is that $j - i = 2$, whenever $(i+1)\delta$ is the least or greatest element of these three elements 1δ, $(i+1)\delta$, and $n\delta$. Moreover, only δ restricted to $\{i+1, \dots j-1\}$ can be orientation-reversing, namely in the case $j - i \geq 4$. Thus, we have 8 types of δ :

14.1) $\delta = \left(\begin{array}{ccccccccccc} 1 & \cdots & i-1 & i & i+1 & \cdots & j-1 & j & j+1 & \cdots & n \\ 1 & \cdots & i-1 & & i+1 & \cdots & j-1 & & j+1 & \cdots & n \end{array}\right) = \alpha_{i,j}^2 \in \langle G_n \rangle$

14.2) $\delta = \left(\begin{array}{ccccccccccc} 1 & \cdots & i-1 & i & i+1 & \cdots & j-1 & j & j+1 & \cdots & n \\ 1 & \cdots & i-1 & & j-1 & \cdots & i+1 & & j+1 & \cdots & n \end{array}\right) = \alpha_{i,j} \in G_n$;

14.3) $\delta = \left(\begin{array}{ccccccccccc} 1 & \cdots & i-1 & i & i+1 & \cdots & j-1 & j & j+1 & \cdots & n \\ n & \cdots & n-i+2 & & n-i & \cdots & n-j+2 & & n-j & \cdots & 1 \end{array}\right) = \alpha_{i,j}^2 \gamma_n \in \langle G_n \rangle$:

14.4) $\delta = \left(\begin{array}{ccccccccccc} 1 & \cdots & i-1 & i & i+1 & \cdots & j-1 & j & j+1 & \cdots & n \\ n & \cdots & n-i+2 & & n-j+2 & \cdots & n-i & & n-j & \cdots & 1 \end{array}\right) = \alpha_{i,j} \gamma_n \in \langle G_n \rangle$;

14.5) $\delta = \left(\begin{array}{ccccccccc} 1 & \cdots & i-1 & i & i+1 & j & j+1 & \cdots & n \\ 3 & \cdots & i+1 & & 1 & & j+1 & \cdots & n \end{array}\right) = \beta_{i+1}^{even} \in G_n$;

14.6) $\delta = \left(\begin{array}{ccccccccc} 1 & \cdots & i-1 & i & i+1 & j & j+1 & \cdots & n \\ n & \cdots & n-i+2 & & 1 & & n-j+2 & \cdots & 3 \end{array}\right) = \beta_{i+1}^{even} \alpha_2 \in \langle G_n \rangle$;

14.7) $\delta = \left(\begin{array}{ccccccccc} 1 & \cdots & i-1 & i & i+1 & j & j+1 & \cdots & n \\ 1 & \cdots & i-1 & & n & & i+1 & \cdots & n-2 \end{array}\right) = \beta_{i+1}^{even} \alpha_2 \gamma_n \in \langle G_n \rangle$;

14.8) $\delta = \left(\begin{array}{ccccccccc} 1 & \cdots & i-1 & i & i+1 & j & j+1 & \cdots & n \\ n-2 & \cdots & n-j+2 & & n & & n-j & \cdots & 1 \end{array}\right) = \beta_{i+1}^{even} \gamma_n \in \langle G_n \rangle$.

Now we have considered all possibilities for i and j. Consequently, $\delta \in \langle G_n \rangle$, whenever $\operatorname{rank} \delta = n-2$. $\square$

Lemma 5 shows $\langle J_n \rangle = \langle G_n \rangle$. Thus, Proposition 1 provides that G_n is a generating set for $\mathscr{F}\mathscr{I}_n$.

Corollary 1 $\mathscr{F}\mathscr{I}_n = \langle G_n \rangle$.

The following example gives a minimal generating set for $n = 5$. It should help to understand the generating transformations of $\mathscr{F}\mathscr{I}_n$ for any n.

Example 1 $\{\gamma_5, \alpha_1, \alpha_2, \alpha_3, \beta_2^{odd}, \beta_2^{even}\}$ is a generating set for $\mathscr{F}\mathscr{I}_5$ where $\left(\begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \\ 5 & 4 & 3 & 2 & 1 \end{array}\right) = \gamma_5$, $\left(\begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \\ & 2 & 3 & 4 & 5 \end{array}\right) = \alpha_1$, $\left(\begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & & 4 & 5 \end{array}\right) = \alpha_3$, $\left(\begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \\ 1 & & 5 & 4 & 3 \end{array}\right) = \alpha_2$, $\left(\begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \\ 2 & & & 4 & 5 \end{array}\right) = \beta_2^{odd}$, and $\left(\begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \\ & 1 & & 4 & 5 \end{array}\right) = \beta_2^{even}$.

It remains to show that G_n is a generating set of minimal size. It is useful to classify the partial injections in $\mathscr{F}\mathscr{I}_n$ with rank $n-1$. For $1 \leq i \leq \frac{n+1}{2}$, let $R_i := \{\alpha \in \mathscr{F}\mathscr{I}_n : \operatorname{dom} \alpha = \overline{n} \backslash \{i\}$ or $\operatorname{dom} \alpha = \overline{n} \backslash \{n-i+1\}\}$. Clearly, $\bigcup \{R_i : 1 \leq i \leq \frac{n+1}{2}\} = \{\alpha \in \mathscr{F}\mathscr{I}_n : \operatorname{rank} \alpha = n-1\}$. Moreover, any generating set of $\mathscr{F}\mathscr{I}_n$ contains elements from each R_i, $1 \leq i \leq \frac{n+1}{2}$, as the following lemma will show.

Lemma 6 *Let* $G \subseteq \mathscr{FI}_n$ *with* $\mathscr{FI}_n = \langle G \rangle$. *Then* $G \cap R_i \neq \emptyset$ *for all* $i \in \{1, 2, \dots, \frac{n+1}{2}\}$.

Proof Assume there is an $i \in \{1, 2, \dots, \frac{n+1}{2}\}$ with $G \cap R_i = \emptyset$. Then $\alpha_i \notin G$ and there are $g_1, \dots, g_s \in G \backslash \{\mathrm{id}_{\bar{n}}\}$ such that $\alpha_i = g_1 \cdots g_s$, where $g_k g_{k+1} \neq \mathrm{id}_{\bar{n}}$ for $1 \leq k < s$. This implies $\operatorname{rank} g_1 = n$, i.e. $g_1 = \gamma_n$, or $g_1 \in R_i$. The latter one is not possible. But $g_1 = \gamma_n$ provides $\operatorname{rank} g_2 = n$, i.e. $g_2 = \gamma_n$, and thus $g_1 g_2 = \mathrm{id}_{\bar{n}}$, or $g_2 \in R_i$. Both are not possible. □

Now, we are able to state the minimal size of a generating set of $\mathscr{FI}_n$. It will coincide with the size of G_n, which gives us the rank of $\mathscr{FI}_n$.

Proposition 2 *Let G be a generating set for $\mathscr{FI}_n$. Then $|G| \geq 5$ if $n = 3$ and $|G| \geq \frac{n-5}{2} + \left\lfloor \frac{n+6}{4} \right\rfloor \left\lfloor \frac{n+7}{4} \right\rfloor$ if $n \geq 5$.*

Proof Since γ_n and $\mathrm{id}_n = \gamma_n \gamma_n$ are the only transformations in $\mathscr{FI}_n$ with rank n, we can conclude that we have at least one transformation with rank n in G, namely γ_n.

Lemma 6 provides that there are at least $\left\lceil \frac{n}{4} \right\rceil$ transformations in $G \cap \bigcup \{R_i : 1 \leq i \leq \frac{n+1}{2}, i \text{ is odd}\}$, where $\left\lceil \frac{n}{4} \right\rceil = \left| \{m \in \{1, \dots, \frac{n+1}{2}\} : m \text{ is odd}\} \right|$. Moreover, there is at least one element in $G \cap R_2$, by Lemma 6. If $n = 3$ then we have at least $\left\lceil \frac{n}{4} \right\rceil + 1 = 2$ elements in G with rank $n - 1$.

Suppose that $n \geq 5$. If $\frac{n+1}{2}$ is even then there is at least one element in $G \cap R_{\frac{n+1}{2}}$, by Lemma 6, too. If $n = 5$ then we have at least $\left\lceil \frac{n}{4} \right\rceil + 1 = \left\lceil \frac{n}{4} \right\rceil + \frac{n-3}{2}$ elements in G with rank $n - 1$. If $n = 7$ then we have at least $\left\lceil \frac{n}{4} \right\rceil + 1 + 1 = \left\lceil \frac{n}{4} \right\rceil + \frac{n-3}{2}$ elements in G with rank $n - 1$.

Let now $n \geq 9$. As already mentioned, we have $|G \cap R_2| \geq 1$ and $\left|G \cap R_{\frac{n+1}{2}}\right| \geq 1$, whenever $\frac{n+1}{2}$ is even. Let now $i \in \{4, \dots, \frac{n-1}{2}\}$ be even. By Lemma 6, we can conclude that $|G \cap R_i| \geq 1$. Assume that $|G \cap R_i| = 1$. Then there is $\alpha \in \mathscr{FI}_n$ with $G \cap R_i = \{\alpha\}$. Note that any transformation in R_j ($j \in \{1, \dots, \frac{n+1}{2}\}$) has a domain as well as an image of the form $\bar{n} \backslash \{i\}$ or $\bar{n} \backslash \{n - i + 1\}$. Then we conclude that if $\beta_1, \beta_2 \in \mathscr{FI}_n$ with $\operatorname{rank}(\beta_1 \beta_2) \geq n - 1$ then $\beta_1, \beta_2, \beta_1 \beta_2 \in R_j \cup \{\mathrm{id}_{\bar{n}}, \gamma_n\}$ for some $j \in \{1, \dots, \frac{n+1}{2}\}$. Hence, any element in R_i is generated by $\mathrm{id}_{\bar{n}}, \gamma_n$, and α. We have two cases:

- If $\operatorname{im} \alpha = \operatorname{dom} \alpha$ then $\alpha^3 = \alpha$ and $\operatorname{rank}(\alpha \gamma_n \alpha) < n - 1$. Thus

$$\langle \mathrm{id}_{\bar{n}}, \gamma_n, \alpha \rangle \cup \{\theta \in \mathscr{FI}_n \mid \operatorname{rank} \theta \geq n - 1\} =$$
$$\{\mathrm{id}_{\bar{n}}, \gamma_n, \alpha, \alpha^2, \gamma_n \alpha, \gamma_n \alpha^2, \alpha \gamma_n, \alpha^2 \gamma_n, \gamma_n \alpha \gamma_n, \gamma_n \alpha^2 \gamma_n\}.$$

- If $\operatorname{im} \alpha \neq \operatorname{dom} \alpha$ then $\operatorname{im} \alpha \gamma_n = \operatorname{dom} \alpha$, $\operatorname{im} \alpha = \operatorname{dom} \gamma_n \alpha$, $(\alpha \gamma_n)^3 = \alpha \gamma_n$, and $(\gamma_n \alpha)^3 = \gamma_n \alpha$. Thus

$$\langle \mathrm{id}_{\bar{n}}, \gamma_n, \alpha \rangle \cup \{\theta \in \mathscr{FI}_n \mid \operatorname{rank}\theta \geq n-1\} =$$
$$\{\mathrm{id}_{\bar{n}}, \gamma_n, \alpha, \alpha\gamma_n\alpha, \gamma_n\alpha, \gamma_n\alpha\gamma_n\alpha, \alpha\gamma_n, \alpha\gamma_n\alpha\gamma_n, \gamma_n\alpha\gamma_n, \gamma_n\alpha\gamma_n\alpha\gamma_n\}.$$

This shows that $|\langle G\rangle \cap R_i| = 8$. On the other hand, the elements in R_i have pairwise different image or different domain, or different image of 1 or different image of n. This mean that R_i contains 16 elements, namely

$$R_i = \{\alpha_i, \alpha_i\gamma_n, \alpha_i^2, \alpha_i^2\gamma_n, \gamma_n\alpha_{n-i+1}\gamma_n, \alpha_i\gamma_n\alpha_{n-i+1}\gamma_n, \alpha_i\gamma_n\alpha_{n-i+1}, \alpha_{n-i+1},$$
$$\alpha_{n-i+1}\gamma_n, \alpha_{n-i+1}^2, \alpha_{n-i+1}^2\gamma_n\}.$$

Thus, $\langle G\rangle \cap R_i \subsetneq R_i$, a contradiction. If $\frac{n+1}{2}$ is even then $\frac{n-7}{4} = |\{m \in \{4, \dots, \frac{n-1}{2}\} : m \text{ is even}\}|$, i.e. we have at least $1 + 1 + 2\left(\frac{n-7}{4}\right) = \frac{n-3}{2}$ elements in $G \cap \bigcup\{R_i : 1 \leq i \leq \frac{n+1}{2}, i \text{ is even}\}$. If $\frac{n+1}{2}$ is odd then $\frac{n-5}{4} = \left|\{m \in \{4, \dots, \frac{n-1}{2}\} : m \text{ is even}\}\right|$, i.e. we have at least $1 + 2\left(\frac{n-5}{4}\right) = \frac{n-3}{2}$ elements in $G \cap \bigcup\{R_i : 1 \leq i \leq \frac{n+1}{2}, i \text{ is even}\}$. Altogether, we have at least $\left\lceil \frac{n}{4} \right\rceil + \frac{n-3}{2}$ elements in G with rank $n-1$, whenever $n \geq 5$.

Assume that $|G \cap Par_n| < 2\left\lfloor \frac{n+1}{4} \right\rfloor$, where $\left\lfloor \frac{n+1}{4} \right\rfloor = |\{m \in \{2, \dots, \frac{n+1}{2}\} : m \text{ is even }\}|$. Note, there exists exactly one $x \in \operatorname{dom}\alpha$ with x and $x\alpha$ have different parity, whenever $\alpha \in J_n \cap Par_n$, by Lemma 4. This provides that there is an even $j \in \{2, \dots, \frac{n+1}{2}\}$ such that $j\alpha, (n-j+1)\alpha \notin \{1, n\}$ for all $\alpha \in G$ or $1\alpha, n\alpha \notin \{j, n-j+1\}$ for all $\alpha \in G$.

Suppose that $1\alpha, n\alpha \notin \{j, n-j+1\}$ for all $\alpha \in G$. In particular, this implies that $\beta_j^{odd} \notin G$. Hence, there are $g_1, \dots, g_s \in G \backslash \{\mathrm{id}_{\bar{n}}\}$ such that $\beta_j^{odd} = g_1 \cdots g_s$. Since $1\beta_j^{odd}$ and 1 have different parity, we conclude that there is $k \in \{1, \dots, s\}$ such that $g_k \in Par_n$. Without loss of generality, we can assume that $g_r \notin Par_n$ for $k < r \leq s$. By Lemma 4, then there is an even $m \in \overline{n}$ such that $1g_k = m$ or $ng_k = m$. Since $1g_k, ng_k \notin \{j, n-j+1\}$, we conclude that $m \notin \{j, n-j+1\}$. Note that $\operatorname{im}\beta_j^{odd} = \overline{n}\backslash\{j-1, j+1\}$. If $k = s$ then $\operatorname{im}\beta_j^{odd} = \operatorname{im}(g_1 \cdots g_s) = \operatorname{im} g_k = \overline{n}\backslash\{m-1, m+1\} \neq \overline{n}\backslash\{j-1, j+1\}$, a contradiction. If $k < s$ then we put

$$g := g_{k+1} \cdots g_s.$$

If $g = \mathrm{id}_n$ then we get a contradiction by the previous arguments. If $g = \gamma_n$ then $\operatorname{im}\beta_j^{odd} = \operatorname{im}(g_1 \cdots g_k g) = \{n-m, n-m+2\} \neq \overline{n}\backslash\{j-1, j+1\}$ since $m \neq n-j+1$, a contradiction. If $\operatorname{rank} g = n-1$ then $\operatorname{im} g = \overline{n}\backslash\{j-1\}$ or $\operatorname{im} g = \overline{n}\backslash\{j+1\}$. First, we consider the case $\operatorname{im} g = \overline{n}\backslash\{j-1\}$. Since $j-1$ is odd, we can conclude that $g \in \{\alpha_{j-1}, \gamma_n\alpha_{j-1}\}$. Suppose that $g = \alpha_{j-1}$. Since $\operatorname{rank}(g_1 \cdots g_k g) = \operatorname{rank} g_k = n-2$, we have $\operatorname{im} g_k \subseteq \operatorname{dom} g$. This implies $j-1 \in \{m-1, m+1\}$. Because of $j \neq m$, we have $j-1 = m+1$. This provides $m-1 = j-3$, i.e. $j-3 \notin \operatorname{im} g_k = \operatorname{im}(g_1 \cdots g_k) = \operatorname{im}(g_1 \cdots g_k\alpha_{j-1}) = \operatorname{im}(g_1 \cdots g_k g)$, a contradiction. Suppose that $g = \gamma_n\alpha_{j-1}$. Clearly, $\operatorname{dom} g = \overline{n}\backslash\{n-j+2\}$. Then $\operatorname{im} g_k \subseteq \operatorname{dom} g$ provides $n-j+2 \in \{m-1, m+1\}$. But $n-j+1 \neq m$ gives $n-j+2 = m-1$. Hence,

$m+1=n-j+4$ and we obtain $j-3=(n-j+4)\gamma_n\alpha_{j-1}$. Thus $j-3\notin \operatorname{im}(g_1\cdots g_k g)$, a contradiction. Dually, we can treat the case $\operatorname{im} g=\overline{n}\backslash\{j+1\}$. If $\operatorname{rank} g=n-2$ then $\operatorname{im} g=\overline{n}\backslash\{j-1,j+1\}$. Since both $j-1$ and $j+1$ are odd, we can conclude that $\operatorname{dom} g=\overline{n}\backslash\{j-1,j+1\}$ or $\operatorname{dom} g=\overline{n}\backslash\{n-j,n-j+2\}$, since $g=\alpha_{j-1}\alpha_{j+1}$ or $g=\gamma_n\alpha_{j-1}\alpha_{j+1}$. But because of $m\neq n-j+1$ and $m\neq j$, we have $m-1\neq n-j$ and $m-1\neq j-1$, respectively. This implies $\operatorname{im} g_k=\{m-1,m+1\}\neq \operatorname{dom} g$, a contradiction.

Suppose that $j\alpha,(n-j+1)\alpha\notin\{1,n\}$ for all $\alpha\in G$. Then we conclude dually that there are $\delta_1,\ldots,\delta_t\in G\backslash\{\operatorname{id}_{\bar n}\}$ such that $\beta_j^{even}=\delta_1\cdots\delta_t$ but $\operatorname{dom}(\delta_1\cdots\delta_t)\neq \operatorname{dom}\beta_j^{even}$, i.e. we obtain a contradiction, too.

Suppose that $n\geq 5$. By straightforward combinatorial calculations, one obtains that there are exactly $\left\lfloor\frac{n}{4}\right\rfloor\left\lfloor\frac{n+2}{4}\right\rfloor-1$ pairs $(i,\ j)$ of odd numbers $i,j\in\overline{n}$ with $4\leq j-i<n-1$ and $i\leq n-j+1$. Assume that there are less than $\left\lfloor\frac{n}{4}\right\rfloor\left\lfloor\frac{n+2}{4}\right\rfloor-1$ elements in $G\backslash Par_n$ with rank $n-2$. Note that for odd numbers $i,j\in\overline{n}$ with $4\leq j-i<n-1$ and $i\leq n-j+1$, it holds $4\leq(n-i+1)-(n-j+1)<n-1$ but $(n-j+1)>n-(n-i+1)+1$, whenever $i\neq n-j+1$. In particular, $i=n-j+1$ implies $j=n-i+1$. This justifies the existence of a pair $(i,\ j)$ of odd numbers $i,j\in\overline{n}$ with $4\leq j-i<n-1$ and $i\leq n-j+1$ such that $\operatorname{dom}\alpha\neq\overline{n}\backslash\{i,j\}$ and $\operatorname{dom}\alpha\neq\overline{n}\backslash\{n-i+1,n-j+1\}$ for all $\alpha\in G$. In particular, $\alpha_{i,j}\notin G$ and there are $h_1,\ldots,h_u\in G\backslash\{\operatorname{id}_{\bar n}\}$ such that $\alpha_{i,j}=h_1\cdots h_u$. Note that each $\alpha\in\mathscr{F}\mathscr{I}_n$ with $\operatorname{dom}\alpha=\overline{n}\backslash\{v\}$ for some odd $v\in\overline{n}$ is either order-preserving or order-reversing. Assume that $\operatorname{rank} h_q\geq n-1$ for all $q\in\{1,\ldots,u\}$. Since $\alpha_{i,j}$ is neither order-preserving nor order-reversing, there is an even $v\in\overline{n}$ with $v\notin \operatorname{dom}(h_1\cdots h_u)$, a contradiction. This implies the existence of a $k\in\{1,\ldots,u\}$ with $\operatorname{rank} h_k=n-2$. Without loss of generality, we can assume that $\operatorname{rank} h_q\geq n-1$ for $1\leq q<k$. In particular, an odd number is missing in $\operatorname{dom} h_q$, for $q\in\{1,\ldots,k\}$. Let $\operatorname{dom} h_k=\{s,t\}$. Since $h_k\in G$, we have $\{s,t\}\neq\{i,j\}$ as well as $\{s,t\}\neq\{n-i+1,n-j+1\}$, i.e. $\{n-s+1,n-t+1\}\neq\{i,j\}$. Clearly, $\operatorname{dom} h_q\subseteq \operatorname{im} h_{q-1}$ for $1<q\leq k$. Therefore and since h_q is order-preserving or order-reversing for $1\leq q<k$, there is $h\in\{\operatorname{id}_n,\gamma_n\}$ such that $h_1\cdots h_{k-1}h_k=hh_k$. Hence, $\operatorname{dom}(h_1\cdots h_k)=\{s,t\}$ or $\operatorname{dom}(h_1\cdots h_k)=\{n-s+1,n-t+1\}$. Since $\operatorname{rank}(h_1\cdots h_k)=2=\operatorname{rank}(h_1\cdots h_u)$, we can conclude that $\operatorname{dom}\alpha_{i,j}=\operatorname{dom}(h_1\cdots h_u)=\operatorname{dom}(h_1\cdots h_k)\neq\{i,j\}$, a contradiction.

Altogether, we have shown that $|G|\geq 1+2+2\left\lfloor\frac{3+1}{4}\right\rfloor=5$ if $n=3$ and $|G|\geq 1+\left\lceil\frac{n}{4}\right\rceil+\frac{n-3}{2}+2\left\lfloor\frac{n+1}{4}\right\rfloor+\left\lfloor\frac{n}{4}\right\rfloor\left\lfloor\frac{n+2}{4}\right\rfloor-1$, whenever $n\geq 5$. It is easy to verify that $\left\lceil\frac{n}{4}\right\rceil+2\left\lfloor\frac{n+1}{4}\right\rfloor=n-\left\lfloor\frac{n}{4}\right\rfloor$ and $n-\left\lfloor\frac{n}{4}\right\rfloor+\left\lfloor\frac{n}{4}\right\rfloor\left\lfloor\frac{n+2}{4}\right\rfloor=\left\lfloor\frac{n+6}{4}\right\rfloor\left\lfloor\frac{n+7}{4}\right\rfloor-1$. Hence, $|G|\geq\left\lfloor\frac{n+6}{4}\right\rfloor\left\lfloor\frac{n+7}{4}\right\rfloor-1+\frac{n-3}{2}=\frac{n-5}{2}+\left\lfloor\frac{n+6}{4}\right\rfloor\left\lfloor\frac{n+7}{4}\right\rfloor$, whenever $n\geq 5$. ☐

It is easy to calculate that $\left|G_n\right|=1+\left\lceil\frac{n}{4}\right\rceil+\frac{n-3}{2}+2\left\lfloor\frac{n+1}{4}\right\rfloor+\left\lfloor\frac{n}{4}\right\rfloor\left\lfloor\frac{n+2}{4}\right\rfloor-1=\frac{n-5}{2}+\left\lfloor\frac{n+6}{4}\right\rfloor\left\lfloor\frac{n+7}{4}\right\rfloor$, whenever $n\geq 5$, and $\left|G_3\right|=5$. Together with the calculations for the even case in [4], we obtain the following ranks:

Theorem 1 $\operatorname{rank} \mathscr{F}\mathscr{I}_n = \begin{cases} 5 & \text{if } n = 3 \\ n+1 & \text{if } n \text{ is even} \\ \frac{n-5}{2} + \left\lfloor \frac{n+6}{4} \right\rfloor \left\lfloor \frac{n+7}{4} \right\rfloor & \text{if } n \text{ is odd.} \end{cases}$

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