Доклади на Българската академия на науките
Comptes rendus de l'Académie bulgare des Sciences

Tome 70, No 12, 2017

MATHEMATIQUES
Algèbre

GREEN'S RELATIONS ON A SEMIGROUP OF SETS OF TRANSFORMATIONS WITH RESTRICTED RANGE

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(*Submitted by Academician V. Drensky on October 20, 2017*)

Abstract

A non-deterministic transformation on a finite set with two-element target can be regarded as a set of transformations with restricted two-element range. For a semigroup (of sets of transformations) having these non-deterministic transformations as subsemigroup, we determine the Green's relations. Moreover, for each of the Green's relations, we provide the greatest included congruence.

Key words: transformation semigroup with restricted range, non-deterministic transformations, Green's relations

2010 Mathematics Subject Classification: 20M14, 20M18, 20M20

1. Introduction. In algebra and also in other fields of Mathematics, associative operations on a given set of mathematical objects are defined and the resulting structures, which are called semigroups, are studied. Let us mention for example the following sets of mathematical objects: transformations on a given set (transformation semigroups); partitions (diagram monoids); matrices of given type (matrices semigroups); Boolean operations on a given set; but also linear transformations in a Hilbert space. In particular, the set on which the associative operation is defined can be the power set of a given set. For example, such semigroups were already studied in the case of tree languages [1] and in the case of Boolean operations on finite set [2]. In the present paper, we consider a semigroup whose universe consists of sets of full transformations on a finite set, where the image is in a fixed two-element subset of this set. The authors of the present paper have already characterized the maximal idempotent as well as the maximal regular subsemigroups of the semigroup. The purpose of this present paper is a description of the algebraic structure of this semigroup. Let X be a finite set, let $Y := \{y_1, y_2\}$ be a two-element subset of X, and denote by $T(X,Y)$ the semigroup (under composition) of all full transformations on the set X with image in Y. This semigroup is called semigroup of transformations with restricted range Y [3] and is studied by several authors [4–8]. It is interesting to note that several subsets of $T(X,Y)$ have an important interpretation in the Automata theory and thus in the wider sense also in the Theoretical Computer Sciences, namely as non-deterministic transformations. A non-deterministic transformation from X in Y is a mapping α^{nd} from X in the set $\{Y, \{y_1\}, \{y_2\}\}$ of all non-empty subsets of Y. In an algebraic setting, α^{nd} can be regarded as a set of transformations $\{\alpha \in T(X,Y) : x\alpha \in x\alpha^{nd} \text{ for all } x \in X\}$, i.e. as an element of the set $T_P(X,Y)$ of all non-empty subsets of $T(X,Y)$. In a canonical way, one can define an associative operation $\cdot$ on the set $T_P(X,Y)$ by

$$A \cdot B := \{\alpha\beta \mid \alpha \in A, \beta \in B\},$$

i.e. $T_P(X,Y)$ forms a semigroup under the operation $\cdot$. We will write AB rather than $A \cdot B$. It is interesting to note that the set of non-deterministic transformations from X to Y forms a subsemigroup of $T_P(X,Y)$. In 2016, the authors of this paper determined the maximal idempotent subsemigroups and the maximal regular subsemigroups of $T_P(X,Y)$ [10]. The semigroup $T_P(X,Y)$ has been already studied by Susanti [2] several years ago, but from different point of view, namely as the multiplicative reduction of the semiring associated to the semigroup of Boolean operations on a finite set (under a canonical associative operation $+$). In particular, she determined the k-regular elements of $T_P(X,Y)$ in [2].

For the study of $T_P(X,Y)$, the structure of the monoid $T(X,Y)$ is central. Therefore, we summarize several important facts about $T(X,Y)$. Since $|Y| = 2$, it is obvious to define a unary operation $*$ on $T(X,Y)$ by

$$x\alpha^* := \begin{cases} y_1 & \text{if} \quad x\alpha = y_2; \\ y_2 & \text{if} \quad x\alpha = y_1. \end{cases}$$

Clearly, $(\alpha^*)^* = \alpha$ for any $\alpha \in T(X,Y)$ and we put $A^* := \{\alpha^* \mid \alpha \in A\}$ for $A \in T_P(X,Y)$. Moreover, α restricted to Y is one of the four full transformations on Y: $\begin{pmatrix} y_1 & y_2 \\ y_1 & y_2 \end{pmatrix}$, $\begin{pmatrix} y_1 & y_2 \\ y_2 & y_1 \end{pmatrix}$, $\begin{pmatrix} \overline{y_1 y_2} \\ y_1 \end{pmatrix}$, and $\begin{pmatrix} \overline{y_1 y_2} \\ y_2 \end{pmatrix}$. We can decompose $T(X,Y)$ in four sets, namely in

$\mathbf{T}_1 := \{\alpha \in T(X,Y) \mid \ y_i\alpha = y_i \text{ for } i = 1,2\}$,
$\mathbf{T}_2 := \mathbf{T}_1^*$,
$\mathbf{T}_3 := \{\alpha \in T(X,Y) : y_1\alpha = y_2\alpha = y_1\}$, and

$$\mathbf{T}_4 := \mathbf{T}_3^* = \{\alpha \in T(X,Y) : y_1\alpha = y_2\alpha = y_2\}.$$

Recall that $T(X,Y)$ is a so-called 4-part semigroup [11]. Analogously, one can decompose any $A \in T_P(X,Y)$ in four (also possible empty) sets:

$$A_i := A \cap \mathbf{T}_i \text{ for } i = 1,2,3,4.$$

It is easy to verify that $\alpha\beta = \alpha$ and $\alpha\beta^* = \alpha^*$ for any $\alpha \in T(X,Y)$ and $\beta \in \mathbf{T}_1$. In particular, $\mathbf{T}_1$ operates as a right-identity element, more precisely, for any $A, B \in T_P(X,Y)$ with $B_1 \neq \varnothing$, it holds $A \subseteq AB$ and $A = AB$, whenever $B_1 = B$. This becomes clear by the fact that $\mathbf{T}_1$ consists of all idempotents in $T(X,Y)$, except of the constant mappings. Note that $T(X,Y)$ contains exactly two constant mappings, denoted by c_1 and c_2, with the image y_1 and y_2, respectively. Clearly, $c_1^* = c_2$ and $AB = \{c_i\}$ for any $A \in T_P(X,Y)$ and $B \subseteq \mathbf{T}_{2+i}$, $i \in \{1,2\}$. This shows that $T_P(X,Y)$ has no identity element and it is not a monoid, but becomes a monoid $T_P(X,Y)^1$ adding an identity element, denoted by $\mathbf{1}$.

The structure of a semigroup can be described very well by its Green's relations. Hence, the present paper is devoted to the Green's relations on $T_P(X,Y)$. Let $\mathcal{L}, \mathcal{R}$, and $\mathcal{J}$ $(= \mathcal{D})$ be the Green's relations on $T_P(X,Y)$, in other words

$$A\mathcal{L}B \Leftrightarrow \exists P, Q \in T_P(X,Y)^1\big(PA = B, QB = A\big),$$
$$A\mathcal{R}B \Leftrightarrow \exists P, Q \in T_P(X,Y)^1\big(AP = B, BQ = A\big), \text{ and}$$
$$A\mathcal{J}B \Leftrightarrow \exists P_1, P_2, Q_1, Q_2 \in T_P(X,Y)^1\big(P_1AQ_1 = B, P_2BQ_2 = A\big).$$

Since $\mathcal{J} = \mathcal{D}$, we have $\mathcal{J} = \mathcal{L} \circ \mathcal{R} = \mathcal{R} \circ \mathcal{L} = \{(A,B) \mid \text{there is } D \in T_P(X,Y) \text{ with } (A,D) \in \mathcal{L} \text{ and } (D,B) \in \mathcal{R}\}$. The purpose of this paper is the characterization of these Green's relations. Sets containing only constant mappings have particular properties. It is routine to verify that the set of the non-empty subsets of the set $\mathbb{C} := \{c_1, c_2\}$ of constant mappings forms an $\mathcal{R}$-class as well as a $\mathcal{J}$-class. But each non-empty subset $A \subseteq \mathbb{C}$ forms a singleton $\mathcal{L}$-class $\{A\}$. Therefore, the main work will be the description of the ρ-classes, for $\rho \in \{\mathcal{L}, \mathcal{R}, \mathcal{J}\}$, whose representatives A can not be contained in $\mathbb{C}$, i.e. $A_{\mathbb{C}} := A \cap \mathbb{C} \neq A$.

Notice that neither the right-congruence $\mathcal{L}$ nor the left-congruence $\mathcal{R}$ nor the relation $\mathcal{J}$ is a congruence. But we determine the greatest congruence $\rho^{\bigcirc}$ contained in ρ, i.e.

$$\rho^{\bigcirc} := \big\{(A,B) \in \rho \mid \forall P, Q \in T_P(X,Y)^1\big((PAQ, PBQ) \in \rho\big)\big\}$$

for $\rho \in \{\mathcal{L}, \mathcal{R}, \mathcal{J}\}$. The given description will be based on the characterization of the respective Green's relation.

2. Main results. We begin with the characterization of the relation $\mathcal{R}$ and of the greatest congruence $\mathcal{R}^{\bigcirc}$ contained in $\mathcal{R}$. After that, we will do the same for the $\mathcal{L}$-relation and the $\mathcal{J}$-relation. We want to note that the reader can verify the propositions below by calculations which use the properties of the operation on the semigroup $T_P(X,Y)$ which we have given in the previous section. In particular, for the characterization of the $\mathcal{R}$-relation, we need the fact that $A, B \subseteq \mathbb{C}$ or $|A_{\mathbb{C}}| = |B_{\mathbb{C}}|$, whenever $A, B \in T_P(X,Y)$ with $A\mathcal{R}B$.

Proposition 2.1. *Let $A, B \in T_P(X,Y)$ with $A \neq B$. Then $A\mathcal{R}B$ if and only if $A, B \subseteq \mathbb{C}$ or $B = A^*$.*

An immediate consequence of Proposition 2.1 is the fact that the $\mathcal{R}$-classes different from the $\mathcal{R}$-class $\{\mathbb{C}, \{c_1\}, \{c_2\}\}$ are of the form $\{A, A^*\}$ for all $A \in T_P(X,Y)$ with $A \neq A_{\mathbb{C}}$. Obviously, $\mathcal{R}$ is not a congruence. But we can verify that $\mathcal{R}^{\bigcirc}$ consists of all non-singleton classes $\{A, A^*\}$ with $\mathbb{C} \subseteq A \in T_P(X,Y)$, together with $\{\mathbb{C}, \{c_1\}, \{c_2\}\}$, as the following proposition will show.

Proposition 2.2. $\mathcal{R}^{\bigcirc} = \{(A,A) \mid A \in T_P(X,Y)\} \cup \{(A,B) \mid \varnothing \neq A, B \subseteq \mathbb{C}\} \cup \{(A,A^*) \mid \mathbb{C} \subseteq A \in T_P(X,Y)\}$.

Let us now characterize the Green's relation $\mathcal{L}$.

Proposition 2.3. *Let $A, B \in T_P(X,Y)$ with $A \neq B$. Then $A\mathcal{L}B$ if and only if the following four statements are satisfied:*

(i) $A_{i+2} \neq \varnothing \Longleftrightarrow B_{i+2} \neq \varnothing \Longleftrightarrow c_i \in A \cap B$ *for* $i \in \{1,2\}$;

(ii) $A_1 \cup A_2 \neq \varnothing$ *and* $B_1 \cup B_2 \neq \varnothing$;

(iii) $A_1 = \varnothing$ *or* $A_2 = \varnothing$ *if and only if* $B_1 = \varnothing$ *or* $B_2 = \varnothing$;

(iv) if $A_1, A_2 \neq \varnothing$, *then* $B \setminus B_{\mathbb{C}} = (B \setminus B_{\mathbb{C}})^*$.

As a straightforward consequence of Proposition 2.3, we can realize that there are exactly eight $\mathcal{L}$-classes of size greater than one. Only the sets $A \in T_P(X,Y)$ with either $A \subseteq \mathbf{T}_3 \cup \mathbf{T}_4$ or $A_1, A_2 \neq \varnothing$ and $A \setminus A_{\mathbb{C}} \neq (A \setminus A_{\mathbb{C}})^*$ form singleton $\mathcal{L}$-classes. Splitting four of the eight non-singleton $\mathcal{L}$-classes, one obtains a congruence.

Proposition 2.4. *$\mathcal{L}^{\bigcirc}$ is the set of all $(A,B) \in T_P(X,Y)^2$ such that $A = B$ or the following four properties are satisfied:*

(i) $A_{i+2} \neq \varnothing \Longleftrightarrow B_{i+2} \neq \varnothing \Longleftrightarrow c_i \in A \cap B$ *for* $i \in \{1,2\}$;

(ii) $A_1 \cup A_2 \neq \varnothing$ *and* $B_1 \cup B_2 \neq \varnothing$;

(iii) $A_i = \varnothing$ *if and only if* $B_i = \varnothing$ *for* $i \in \{1,2\}$;

(iv) if $A_1, A_2 \neq \varnothing$, *then* $B \setminus B_{\mathbb{C}} = (B \setminus B_{\mathbb{C}})^*$.

The $\mathcal{J}$-relation has exactly seven equivalence classes with more than two elements. One of them is the three element set $\{\mathbb{C}, \{c_1\}, \{c_2\}\}$.

Proposition 2.5. *Let $A \neq B \in T_P(X,Y)$ and $A, B \nsubseteq \mathbb{C}$. Then $A\mathcal{J}B$ if and only if $A^* = B$ or the following four statements are satisfied:*

(i) $|A_{\mathbb{C}}| = |B_{\mathbb{C}}|$ *and* $A_{i+2} \neq \varnothing \Leftrightarrow c_i \in A$ *and* $B_{i+2} \neq \varnothing \Leftrightarrow c_i \in B$ *for* $i \in \{1,2\}$;

(ii) $A_1 \cup A_2 \neq \varnothing$ *and* $B_1 \cup B_2 \neq \varnothing$;

(iii) $A_1 = \varnothing$ *or* $A_2 = \varnothing$ *if and only if* $B_1 = \varnothing$ *or* $B_2 = \varnothing$;

(*iv*) *if* $A_1, A_2 \neq \varnothing$*, then* $B \setminus B_{\mathbb{C}} = (B \setminus B_{\mathbb{C}})^*$.

Notice that representatives of the equivalence classes with more than two elements are $\mathbf{T}_1$, $\mathbf{T}_1 \cup \mathbf{T}_2$, $\mathbf{T}_1 \cup \mathbf{T}_3$, $\mathbf{T}_1 \cup \mathbf{T}_2 \cup \mathbf{T}_3$, $\mathbf{T}_1 \cup \mathbf{T}_3 \cup \mathbf{T}_4$, X and $\mathbb{C}$. Finally, we determine the largest congruence in $\mathcal{J}$.

Proposition 2.6. $\mathcal{J}^{\bigcirc}$ *is the set of all* $(A, B) \in \mathcal{J}$ *such that* $A = B$ *or the following both statements are true*

(*i*) $\mathbb{C} \subseteq A, B$*, whenever* $A = B^* \neq B$ *or either* $A_1 = \varnothing$ *or* $B_1 = \varnothing$*;*

(*ii*) $A_{\mathbb{C}} = B_{\mathbb{C}}$*, whenever* $A \neq A_{\mathbb{C}}$.

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