R E P O R T

on a Thesis for awarding the degree "Doctor"

Scientific field: 4. Natural sciences, mathematics and informatics

Professional field: 4.6. Informatics and Computer Sciences

Title: Codes and Designs in Polynomial Metric Spaces

Author: Konstantin Vasilev Delchev

Overview

The presented thesis deals with the investigation of codes and designs in polynomial metric spaces. The main problems that are considered are related to existence questions and characterizations of various designs and codes, as well as to deriving bounds on the parameters of such codes.

The thesis is based on the reserch carried out by the author in the past few years.The thesis is focused on the following problems:

1) Derivation of new bounds on the cardinality of antipodal spherical codes with a small number of different distances.

2) Derivation of upper bounds for the energies of spherical designs close to the Delsarte-Goethals-Seidel bound.

3) Proof of asymptotical bounds on the energy of spherical designs.

4) Investigation of block codes with two distances that are close; characterization results for such codes.

5) Derivation of bounds and constructions for two-weight (nonlinear) codes over different alphabets.

State of the current research

My general impression is that the author is well acquainted with the state of the art and the most recent results in research on spherical codes and designs, as well as in the classical coding theory. A good deal of the treated problems are considered in the field as important

theoretically, as well as for the applications. The author demonstrates deep knowledge of his field of research and capacity to apply his knowledge to the solution of important problems.

Methods

In his investigations the author uses algebraic and combinatorial methods and other techniques that are typical for the field. The research relies also on serious computer searches.

Brief description of the thesis

The presented thesis amounts 84 pages of text and consists of an introduction, four chapters, and a list of references including 100 items. In what follows, I shall give a short description of the topics covered in this dissertation.

Chapter 1 is introductory and contains a brief description of the more significant definitions and theoretical facts needed for the thesis. The main focus is set on results from the theory of spherical codes and designs. The author introduces the Gegenbauer polynomials, as well as various bounds on codes, designs and the energy of codes related to the linear programming bound. This chapter does not contain original results.

In chapter 2 the author investigates antipodal codes with a small number of distances. He considers in some detail antipodal codes with scalar products $-1, \pm s$ (section 2.3), scalar products $-1, 0, \pm s$ (section 2.4), and scalar products $-1, \pm s_1, \pm s_2$ (section 2.5). In section 2.3 the author gives a shorter proof of a result by Barg and Yu (Teorem 2.4), which provides an uper bound on the cardinality of an antipodal code with three scalar products: $1, \pm s$.

In section 2.4 the author considers the case where an additional scalar product of 0 is being added. In this case, an upper bound on the cardinality of the corresponding code is proved (Theorem 2.5), as well as a result saying that in the case of equality the number s is rational. Theorem 2.7 is an analogue of Theorem 2.4 in the case when one has four scalar products $-1, 0, \pm s$ and gives a bound for the cardinality of a code which is a spherical 3-design.

The investigations are carried on for the case of antipodal codes with scalar products $-1, \pm s_1, \pm s_2$. Once again a bound on the cardinality of an antipodal spherical code is proved under some condition on the numbers s_1 and s_2 (Theorem 2.8). As in the previous two sections the author proves a bound on the cardinality of a spherical code with scalar products $-1, \pm s_1, \pm s_2$, which is a 5-design (Theorem 2.9).

In chapter 3 the investigation is focused on bounds on the energies of spherical designs of small cardinality. The original results are contained in section 3.3. The contain bounds on the function $\mathcal{U}(n, M, \tau; h)$ defined as the supremum of the energy of a code in $\mathbb{S}^{n-1}$ of cardinality M, which is a τ-design. The most important facts proved here are Theorems 3.4 and 3.5. At the end of the chapter the author presents tables containing these bounds $n = 3, 4, 5$ and Newton energy.

Chapter 4 deals with the investigation of classical (non-linear) block codes over various alphabets with two distances that differ by 1. Codes with this property can be viewed as a generalization of the constant weight codes. There are interesting connections with results by

Jay Wood on constant weight codes over various structures. It turns out that for linear codes with this property the characterization is straightforward.

The contributions made by the author are contained in sections 4.3, 4.5, and 4.6. The results in section 4.3 amount to upper linear programming bounds for codes with two distances that differ by 1 (Theorems 4.6 and 4.8).

In section 4.5 the author presents various constructions for codes with two distances – d and $d+1$. They make use of various combinatorial structures like combinatorial designs, latin squares, difference sets. The author uses a nice connection of these codes with the spherical codes to prove an upper bound on the cardinality of a q-ary code with two distances d and $d+1$ (Theorem 4.13).

In section 4.6 all the results are used to compile tables with exact values and bounds on the cardinality of a q-ary code with two distances differing by 1.

In section 5 the author investigates codes with two different distances (called two-weight codes with a slight abuse of language). Such codes are extensively studied in coding theory in the linear case, but almost nothing is known in the non-linear case. The section contains bounds on the cardinality of such codes, as well as various construction. A central result here is Theorem 5.4, which contains a relatively simple bound on the cardinality of a q-ary code of length n and distances d and $d+\delta$. Further on the function $A_q(n, \{d, d+\delta\})$ is investigated and some exact values are found.

In section 5.2 the author presents tables with exact values and bounds on the cardinalities of two-weight codes over a fixed alphabet with q letters, $q = 2, 3, 4$, small length n, and two distances d and $d+\delta$. A part of the obtained codes is obtained by extensive computer search.

Main results

The main contributions of this thesis are the following:

(1) Bounds are derived on the cardinality of antipodal codes with a small number of scalar products.

(2) Rationality results of Lloyd-type are proved.

(3) New results on the energies of spherical designs of cardinality closer to the Delsarte-Goethals-Seidel bound are proved.

(4) Asymtotic bounds on the energies of spherical 2-designs are proved.

(5) Bounds on the cardinality of (non-linear) two-weight codes are proved.

(6) Constructions of non-linear codes with two close distances are presented.

(7) Tables are compiled for the cardinality of such codes for relatively small lengths and small alphabets.

Remarks and comments

I have the following remarks, questions and comments related to this thesis:

(1) How tight are the bounds from section 2? Are there codes meeting these bounds, or lying sufficiently close to them?

(2) There is something wrong with the numbering of the sections in chapter 3.

(3) At places there are inconsistencies and technical errors.

(4) The lemmata towards the end of section 5.1 should be in my view stated as theorems. These are in fact important results in their own right.

(5) The thesis is extremely well-written. The considerable efforts to produce a consise and readable text are clearly seen. The very few typos here and there cannot spoil the overall positive impression.

Publications related to the thesis

The results in this thesis are published in six papers. The respective journals are as follows:

- Discrete Mathematics – IF 0.77
- Problems of Information Transmission – IF 0.593
- Discrete and Computational Geometry – IF 0.661
- Electronic Notes in Discrete Mathematics – SJR 0.262
- Proceedings ACCT – XVI
- Proceedings ACCT – XVII

Three of the papers are in journals with an impact factor. The remaining three papers one has SJR and two are published in the volumes of scientific conferenceswith the proceedings of scientific conferences.

The list of publications meets the minimal national criteria required in paragraph 7.

Authorship of the obtained results

The authorship of the six papers on which this thesis is based can be described as follows: one of the presented papers is with one, one is with two and four are with three coauthors.

I have known the author for a long time. Therefore I have no doubt that his contribution in this research is significant.

Citations

The candidate has attached a list of 4 citations of the papers used in this thesis. There is no doubt that this list will increase since all the results are very recent.

Authors summary

The author's summary is made according to the existing regulations and reflects properly the main results and contributions of this thesis.

Conclusion

This thesis is focused on problems from coding theory and algebraic combinatorial theory that are of great importance for the theory and as well as for the applications. They shed light on the problem of how linear programming can be applied to prove bounds on various combinatorial structures. This work does not only answer open problems of principal importance, but also motivates new directions for an ongoing research.

I am deeply convinced that the presented thesis "Codes and Designs in Polynomial Metric Spaces" by Konstantin Vassilev Delchev contains results that are an original contribution to coding theory. The candidate demonstrates deep knowledge of the theory and capacity to develop it in new and important ways. With this, he meets the legal national requirements prescribed by the law, as well as the specific requirements of the Institute of Mathematics and Informatics of the Bulgarian Academy of Sciences for the professional field 4.6 "Informatics and Computer Sciences". I assess **positively** the presented PhD Thesis and recommend to this panel to award **Konstantin Vassilev Delchev** the scientific degree "Doctor" in the scientific field 4. Natural Sciences, Mathematics and Informatics, Professional field 4.6 "Informatics and Computer Sciences".

Sofia, 15.05.2021 Member of the Scientific Panel:

(Prof. DSc Ivan Landjev)