PEER REVIEW

on the Thesis Submitted for Awarding
the Scientific Degree "Doctor of Sciences"
in the Professional Field 4.5. Mathematics
Scientific Specialty "Algebra and Number Theory"

Author of the Thesis: Peter Vassilev Danchev, Ph.D., Assistant Professor in Department of Algebra and Logic of the Institute of Mathematics and Informatics at the Bulgarian Academy of Sciences.

Title of the Thesis: "Some Classes of Noncommutative Rings and Abelian Groups".

Member of the Scientific Jury: Doctor of Sciences Vesselin Stoyanov Drensky, Professor in the Institute of Mathematics and Informatics at the Bulgarian Academy of Sciences, Full Member of the Bulgarian Academy of Sciences.

The thesis is in two classical parts of algebra – the theory of noncommutative unitary rings and the theory of abelian groups. Although the objects studied in these branches of algebra are quite different, the most important problems are similar and quite often are solved in the same way. There is also another relation which is explored essentially in the thesis. Many properties of the abelian group depend on the properties of rings related with the group (e.g. the endomorphism ring of the group). The thesis is devoted to the following main problem – the description (often up to an isomorphism) of important classes of rings and abelian groups and their natural generalizations. The thesis is written in English on 230 pages. It contains an abstract of the contributions, a short introduction on 5 pages, a background and conventions, two chapters with two sections each which contain the main contributions, a list of open problems, a list of references used in the text, a list of publications of the author in the thesis and a list of citations of these papers.

1. **Actuality of the problems studied in the thesis.** The theory of noncommutative rings and the theory of abelian groups are important branches of modern algebra with numerous relations with other parts of mathematics (both in algebra and out of it, as number theory, mathematical logic, combinatorics and set theory) and with other branches of science as mathematical physics, chemistry, crystallography, etc. The presented thesis is motivated by the idea of explicit description of the objects in important classes of rings and groups. The methods of investigation combine techniques of algebra, logic, combinatorics and number theory and in the part on group theory, also homological algebra and set theory. To achieve his results the author uses both already existing methods and creates a number of new techniques overcoming serious principal and technical difficulties.
2. **Scientific contributions.** According to the author, "the main purpose of this dissertation is to promote some new ideas in certain contemporary subjects of algebra as well as to demonstrate a new insight of ideas and methods in some branches which could be of further interest for future developments. This will be subsequently achieved in the next sections and their subsections. Our strategically point of view is in developing of a modern technology which will be approachable in many cases in both ring theory and commutative group theory." I am not convinced that this

purpose is achieved completely but in all the cases the author has an essential success in this direction.

The first group of contributions is in the third chapter and is in noncommutative ring theory. The author studies properties related to presenting the elements of the ring in terms of idempotents, nilpotent elements and invertible elements. Exactly 150 years ago Benjamin Peirce published his book “Linear Associative Algebra” (written calligraphically and published lithographically in 1870 in a small number of copies for distribution among his friends and posthumously in its journal form in 1881 by his son Charles Sanders Peirce, a philosopher, mathematician and logician). In his book Peirce introduces idempotent and nilpotent elements and already 150 years their study is among the main topics in ring theory. For example the search in the database of Mathematical Reviews gives more than 2600 publications with the word “idempotents” in the title. One of the most studied classes of rings related with idempotents and invertible elements is the class of clean rings introduced by Nicholson in 1977. These are rings with the property that every element is a sum of an idempotent and an invertible element. Besides clean rings, the considerations in the thesis include several more classes – exchange rings (with idempotents of a special form), nil-clean (every element is a sum of an idempotent and a nilpotent element), invo-clean (every element is a sum of an idempotent and an involution), JU-rings (rings with Jacobson units when the invertible elements are of the form 1 + an element of the Jacobson radical), UU-rings (the invertible elements are of the form 1+nilpotent element), etc. as well as the weak versions of these classes when one considers both sums and differences of the corresponding elements. We shall especially mention that some of these classes were introduced by the author. It is shown which class in which is contained and on explicit examples it is explained that the corresponding classes are different. In some of the results the condition that the ring is of characteristic 2 is replaced by the weaker condition that 2 belongs to the Jacobson radical. Some of the results show that under natural restrictions important properties can be lifted from the factor ring modulo the Jacobson radical to the whole ring. In many of the cases a complete description up to an isomorphism of the considered classes is obtained. Most of the examples which illustrate the obtained results are subrings of matrix rings over finite rings. For me it would be interesting to see more instructive examples including also infinite rings. In the end of the chapter the author presents applications to group rings. The abstract of the contributions contains 10 important results, 5 of them are in ring theory. I have some critical remarks for the fourth contribution (Theorem 1.93) which concerns the theory of algebras with polynomial identities. Using known results (the theory of Kruse and Lvov for the identities of finite rings from 1973) this theorem can be stated in a much stronger form. From condition (1) of the theorem one obtains immediately that strongly n-torsion clean rings are locally finite, nil of bounded index and have other interesting properties. On the other hand, the arguments in the thesis are sufficiently transparent as a compensation of the missed possibility for a stronger version of the result.

The second group of contributions is in the fourth chapter of the thesis and is in the theory of torsion abelian groups. This chapter is much bigger than the previous and is based on more papers of the applicant. The description of finite abelian groups is well known. In a language different from the modern it was given by Kronecker in 1870 generalizing results on the description of quadratic forms by Gauss in 1801. In its modern form it was stated by Frobenius and Stickenberger in 1878. Contrary, in the theory of infinite abelian groups there are a lot of difficult unsolved problems. Standard arguments show that the investigations may be restricted to reduced (without divisible subgroups) abelian p-groups, where p is a fixed prime integer. One

of the leading threads of the investigations is the classification in terms of cardinal and ordinal numbers in set theory. Making a relation with ring theory, in the larger part of the chapter (in the first section, on around 100 pages) the author introduces and studies different forms of transitivity when elements with close set theoretic properties are mapped to each other by elements of the endomorphism ring or the automorphism group of the abelian group. It has turned out that important roles in the investigations are played by the idempotents and the commutators in the endomorphism ring. The invariant and fully invariant (or characteristic and fully characteristic) subobjects are central objects in the theory of algebraic systems and in particular in group theory. The description of the fully invariant subgroups of reduced abelian *p*-groups is a difficult and still unsolved problem. In the thesis the author considers a weaker version of this problem and studies the socles of fully invariant subgroups. He introduces the class of socle-regular groups which contains the class of fully transitive groups and studies its properties. Then the thesis investigates the relations between different kinds of transitivity and invariance and the differences between the considered classes are demonstrated with explicit examples. The main results show how these properties are transferred to groups naturally related with the given group. The second section of thy chapter deals with presentations of the groups in terms of generators and defining relations. The class of simply presented abelian groups (which have a system of defining relations such that each relation depends on one or two generators only) is an important class in the theory of abelian groups. The thesis contains generalizations of this class and characterizes the groups of these new classes. The author applies methods of homological algebra and set theory. A special attention is paid on the question whether the obtained results depend on the ZFC (the Zermelo-Fraenkel set theory with the axiom of choice included).

As an additional contribution of the author I want to mention that in many places in the text and in the end of the thesis there are stated open problems some of them being a serious challenge for the mathematicians working in this part of algebra.

3. **Analysis of the publications on the thesis.** The thesis is based on 16 papers, 7 of them are on ring theory (published in 2014 – 2019) and 9 are on group theory (published in 2009 – 2015). All papers are published in recognized journals: in J. Math. Tokushima Univ. – 1, Toyama Math. J. – 2, Commun. Korean Math. Soc. – 1, Internat. Electr. J. Algebra – 1, Eurasian Bull. Math. – 1, Contemp. Math. – 1, Archiv Math. (Basel) – 1, J. Commut. Algebra – 1, Mediterr. J. Math. – 1, Houston J. Math. – 1, Results Math. – 1, J. Group Theory – 2, Commun. Algebra – 1, J. Algebra Appl. – 1. Two of the papers on ring theory have SJR factor and all 9 papers on group theory have impact factor. The author has presented a list of 30 citations of his papers included in the thesis in publications of 31 mathematicians. My opinion is that this is a moderate achievement but I should mention that the papers included in the thesis are very recent and for more citations one needs technological time. From the publications 11 are with one coauthor: 4 with Goldsmith from Ireland, 2 with Keef from the USA and with Chekhlov from Russia, 1 with Cîmpean from Rumania, Al-Mallah from Saudi Arabia and Matczuk from Poland. Since the documentation does not contain statements of the coauthors of the joint papers I accept that the coauthors have the same contribution. The joint papers show the capability to work in a team and I think that this is very important for a scientist. Additionally, the joint work increases the coefficient of useful action because the investigations are based on methods from different branches of mathematics. And, to the best of my knowledge, the joint work was performed in distance, without direct personal contacts.

4. **Approbation of the results.** Part of the results included in the thesis have been presented at the seminar of the Department of Algebra and Logic at the Institute of Mathematics and Informatics, at conferences in Austria and Ireland and at a specialized seminar in the USA. It is not clear from the documentation who presented the results abroad – the author or his coauthors. For example, from the home page of the conference in Graz, Austria, one can see that the talk based on a joint publication was presented by Cîmpean. I would suggest to the applicant to present his results more actively at scientific forums in Bulgaria and abroad. Now, as a member of the scientific team of the Institute he has more possibilities for this. It would be also nice to popularize the text of the thesis, for example to post it at arXiv.org and after some revision in the spirit of my remarks below, to think to publish it as a monograph.

5. **Recommendations and critical remarks.** I do not have essential critical remarks on the exposition. The text on ring theory is written sufficiently detailed and in an understandable language and may be used for attracting Ph.D. and Master students to this area of science, which is a positive feature. But the part on group theory is more difficult for reading by a mathematician with a standard background on algebra because requires preliminary knowledge of quite specific notions. The author should add a couple of more pages to introduce these notions and to make easier the life of the reader. Also, one can see that the author has used verbatim texts from the related papers but has not removed expressions like "the authors of the present article" and has left repetitions of several definitions in different sections of the thesis. I think that the author knows very well the literature in the field including the old literature. The list of references in the thesis contains 110 titles, includes publications from 1945 till nowadays and 61 of these publications are before 2000.

6. **The abstract of the thesis and the abstract of the contributions** are written sufficiently detailed and give clear and adequate information for the contents and the main results of the thesis. I would advise the author to prepare with more care the Bulgarian version of the abstract of the thesis. There are bad translations in Bulgarian of some of the terminology, the Chairman of the Scientific Jury is different from the person given in the abstract. All these small details spoil the very good impression due to the obtained results.

Conclusion. The presented thesis is in an actively developed area of science. It is at very high scientific level and satisfies all requirements to a thesis in the area of mathematics. I recommend to the respectable Scientific Jury to award Assistant Professor Ph.D. Peter Vassilev Danchev with the scientific degree "Doctor of Sciences" in the Professional Direction 4.5. Mathematics.

Sofia, June 16, 2020. Signature:

(V. Drensky)