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Rings whose elements are sums or minus sums of two commuting idempotents

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Abstract

We completely characterize up to an isomorphism those rings whose elements x have the property that x or $-x$ is a sum of two commuting idempotents. This enlarges well-known results in the subject due to Hirano and Tominaga (Bull Austral Math Soc 37:161–164, 1988) and Ying et al. (Can Math Bull 59(3):661–672, 2016).

Keywords Rings · Idempotents · Boolean rings · Jacobson radical

Mathematics Subject Classification 16D 60 · 16S 34 · 16U 60

1 Introduction and background

Everywhere in the text of the present paper, all our rings R are assumed to be associative, containing the identity element 1, which in general differs from the zero element 0. Our terminology and notations are mainly in agreement with [7]. For instance, $J(R)$ stands for the Jacobson radical of R, $Nil(R)$ stands for the set of all nilpotents in R and $Id(R)$ stands for the set of all idempotents in R.

Our starting point of view here is the following one:

Definition 1.1 We shall say that a ring R belongs to the class $\mathcal{C}$ if, for every $r \in R$, there exist two commuting $e_1, e_2 \in Id(R)$ such that $r = e_1 + e_2$ or $r = -e_1 - e_2$.

This amounts to the fact that, for any $r \in R$, we have r or $-r$ is a sum of two commuting idempotents. Obvious examples of such rings are $\mathbb{Z}_2$, $\mathbb{Z}_3$, $\mathbb{Z}_4$ and, in contrast to [8, Theorem 4.4], the field $\mathbb{Z}_5$ as well.

The leitmotif of the current article is to describe the isomorphic structure of the above defined rings which lie in the class $\mathcal{C}$. Our motivation is based on the following principally known facts:

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In [1,2,5], the authors independently described those rings R whose elements are idempotents or minus idempotents. In fact, either $R \cong B$ or $R \cong \mathbb{Z}_3$ or $R \cong B \times \mathbb{Z}_3$, where B is a Boolean ring. Thus these rings are necessarily commutative.

Enlarging this, in [6] the authors isomorphically characterized rings whose elements are sums of two idempotents. Specifically, such a ring is a subdirect product of copies of the fields $\mathbb{Z}_2$ and $\mathbb{Z}_3$. Thereby these rings also remain commutative.

This was substantially improved in [8] to the class of rings whose elements are sums or differences of two commuting idempotents. It was proved there that such a ring is decomposed as $R_1 \times R_2$, where R_1 is a zero ring or a ring for which the quotient $R_1/J(R_1)$ is Boolean with either $J(R_1) = \{0\}$ or $J(R_1) = \{0, 2\}$, and R_2 is either a zero ring or a ring that is a subdirect product of copies of the field $\mathbb{Z}_3$. Certainly, one observes that R_1 is either zero, or $\mathbb{Z}_4$, or a Boolean ring B (and thus a subdirect product of copies of the field $\mathbb{Z}_2$), or $\mathbb{Z}_4 \times B$. Thus these rings are commutative too.

The last achievement was recently extended in [3] to rings whose elements are sums of three commuting idempotents. It was established there that such a ring is decomposed as $R_1 \times R_2$, where either $R_1 = \{0\}$ or R_1 is a ring for which the factor-ring $R_1/J(R_1)$ is Boolean with $J(R_1) = 2Id(R_1)$, and either $R_2 = \{0\}$ or R_2 is a ring that is a subdirect product of copies of the field $\mathbb{Z}_3$. So these rings are commutative as well.

Our assertions, which somewhat strengthen the quoted above corresponding ones, are listed in the next section.

2 Main results

We begin here with the following technicality.

Lemma 2.1 *In a ring $R \in \mathcal{C}$ the relation $30 \in Nil(R)$ is valid.*

Proof Writing $3 = e_1 + e_2$ and squaring this, we deduce that $2e_1e_2 = 6$. Thus $36 = 12$, i.e., $24 = 0 = 24.9 = 6^3$ giving up that 6 is a nilpotent.

Let now we write $3 = -e_1 - e_2$ and again square. We thereby obtain that $2e_1e_2 = 12$. So, $144 = 24$, that is, $120 = 0 = 120.225 = (30)^3 = 0$ ensuring that 30 is a nilpotent.

Finally, in both cases, one concludes that 30 is a central nilpotent of order not exceeding 3, as asserted. □

The next assertion is useful (see [4] for more details).

Proposition 2.2 *Let R be a ring of characteristic 5 whose elements satisfy the equations $x^3 = x$ or $x^3 = -x$. Then $R \cong \mathbb{Z}_5$.*

Proof Let P be the subring of R generated by 1, and thus note that $P \cong \mathbb{Z}_5$. We claim that $P = R$, so we assume in a way of contradiction that there exists $b \in R \setminus P$. With no loss of generality, we shall also assume that $b^3 = b$ since $b^3 = -b$ obviously implies that $(2b)^3 = 2b$ as $5 = 0$ and $b \notin P \iff 2b \notin P$.

Let us now $(1 + b)^3 = -(1 + b)$. Hence $b = b^3$ along with $5 = 0$ enable us that $b^2 = 1$. This allows us to conclude that $(1+2b)^3 \neq \pm(1+2b)$, however. In fact, if $(1+2b)^3 = 1+2b$, then one deduces that $2b = 3 \in P$ (and so $b = -1 = 4 \in P$) which is, certainly, manifestly untrue. If now $(1 + 2b)^3 = -1 - 2b$, then one infers that $2b = 2 \in P$ which is, of course, obviously false as well. That is why, $(1 + b)^3 = 1 + b$ must hold. This, in turn, guarantees that $b^2 = -b$. Moreover, $b^3 = b$ is equivalent to $(-b)^3 = -b$ and, by what we have proved

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so far applied to $-b \notin P$, it follows that $-b = b^2 = (-b)^2 = -(-b) = b$. Consequently, $2b = 0 = 6b = b \in P$ because $5 = 0$, which is the wanted contradiction. We thus conclude that $P = R$, as expected. $\square$

It is plainly checked that both $x^3 = x$ and $x^3 = -x$ imply $x^5 = x$ and along with $5 = 0$ these two conditions guaranteed that R is either zero or a subdirect product of copies of the field $\mathbb{Z}_5$ (see, e.g., [7, Exercise 12.11, p. 200]). The above assertion can be somewhat considered as a refinement to this fact.

Lemma 2.3 *Let R be a ring in which either 3 or 5 is a nilpotent such that $q = e + f$ or $q = -e - f$ for some two commuting idempotents $e, f \in R$ and $q^2 = 0$. Then $q = 0$.*

Proof Assume that $5 \in Nil(R)$. Squaring the equality $q = e + f$, we obtain that $q = -2ef$. Therefore, again by squaring, we have $4ef = 0$ and so $ef = 5ef \in Id(R) \cap Nil(R) = \{0\}$ which forces at once that $q = 0$, as promised. Similarly, by squaring, the equality $q = -e - f$ enables us that $q = 2ef$ and that $4ef = 0$, as needed to get $q = 0$.

The same trick also works in the case when $3 \in Nil(R)$. $\square$

We have now all the ingredients necessary to establish our chief result, which substantially extends the corresponding ones from [6] and [8], respectively.

Theorem 2.4 *A non-zero ring R is from the class $\mathcal{C}$ if, and only if, $R \cong R_1 \times R_2 \times R_3$, where R_1, R_2, R_3 are rings satisfying the next three conditions:*

(1) *$R_1 = \{0\}$, or otherwise R_1 is a commutative ring in which $4 = 0$, $R_1/J(R_1)$ is Boolean and either $J(R_1) = \{0\}$ or $J(R_1) = \{0, 2\}$. Actually, $R_1 = \{0\}$, or $R_1 \cong B$ is Boolean (and so a subdirect product of copies of the field $\mathbb{Z}_2$), or $R_1 \cong \mathbb{Z}_4$, or $R_1 \cong B \times \mathbb{Z}_4$.*

(2) *$R_2 = \{0\}$, or R_2 is a subdirect product of isomorphic copies of the field $\mathbb{Z}_3$ otherwise.*

(3) *$R_3 = \{0\}$ (which is mandatory when $J(R_1)$ is non-zero), or R_3 is isomorphic to the field $\mathbb{Z}_5$ otherwise.*

Proof (*Necessity*) The Chinese Remainder Theorem allows us to decompose R as $R_1 \times R_2 \times R_3$, where $2 \in Nil(R_1)$, $3 \in Nil(R_2)$ and $5 \in Nil(R_3)$. Evidently, R_1, R_2, R_3 remain rings from the class $\mathcal{C}$.

Firstly, we consider the ring R_1: Given $x \in R_1$. Suppose that $x = e_1 + e_2$ or $x = -e_1 - e_2$ for some commuting idempotents e_1, e_2 from R_1. In the first case, $x + J(R_1) = (e_1 + J(R_1)) + (e_2 + J(R_1)) \in Id(R_1/J(R_1))$ as $2 \in J(R_1)$ which can also be written as $(x + J(R_1))^2 = x + J(R_1)$. In the second case, one writes that $x = e_1 + e_2 - 2(e_1 + e_2) \in e_1 + e_2 + J(R_1)$ and so the previous trick applies to get that each element in $R_1/J(R_1)$ is an idempotent, as formulated. Next, we assert $4 = 0$. Write $3 = e + f$ or $-3 = e + f$ for two commuting idempotents e, f in R_1. Squaring the first record, we have $6 = 2ef$. Also, $2f = ef$ and thus $2ef = ef = 0$. Hence $6 = 0$ and so $2 = 0$ as $3 \in U(R_1)$. By the same token, the second record insures that $12 = 2ef$. Likewise, $-4f = ef$ and, again by squaring, $16f = ef = -4f$, whence $20f = 0$, i.e., $4f = ef = 0$ since $5 \in U(R_1)$. Hence $12 = 0$ and, therefore, $4 = 0$ as $3 \in U(R_1)$. This sustained the assertion. Moreover, one asserts that $J(R_1) = 2Id(R_1)$. To that aim, given $z \in J(R_1)$, we have $z = e_1 + e_2$ is a sum of two commuting idempotents, and hence $z(1 - e_1) = e_2(1 - e_1) \in J(R_1) \cap Id(R_1) = \{0\}$. Thus $e_2 = e_2e_1$. Similarly, by symmetry, $e_1e_2 = e_1$. Therefore, $e_1 = e_2$ and so $z = 2e_1 \in 2Id(R_1)$. Analogously, the same trick works for the other equality $z = -e_1 - e_2$ to get that $J(R_1) \subseteq 2Id(R_1)$ as $4 = 0$. But since $2 \in J(R_1)$, the converse containment also holds, as desired. That is why, the asserted equality is true. But, by what we have already shown,

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$R_1/J(R_1)$ being Boolean yields that $U(R_1/J(R_1)) \cong U(R_1)/(1+J(R_1))$ is the identity, whence $U(R_1) = 1 + J(R_1) = 1 + 2Id(R_1)$. We thereby may write that $-1 = 1 + 2f$ for some $f \in Id(R_1)$. This means that $-2 = 2f$ and, by multiplying both sides subsequently with f and with 2, we derive once again that $4 = -4f = 0$. We furthermore claim that either $J(R_1) = \{0\}$ or $J(R_1) = \{0, 2\}$, i.e., for every $e \in Id(R_1)$ we will have $2e = 0$ or $2e = 2$ (the latter amounts to $2(1-e) = 0$). To this purpose, consider $eR_1(1-e)$ and $(1-e)R_1e$. Since $eR_1(1-e) \subseteq Nil(R_1) \subseteq J(R_1)$ and $(1-e)R_1e \subseteq Nil(R_1) \subseteq J(R_1)$, it follows that $eR_1(1-e) + (1-e)R_1e \subseteq J(R_1) = 2Id(R_1)$. For any $r \in R_1$ one sees that $2er(1-e) = 2(1-e)re = 0$ and thus $2er = 2ere = 2re$ which allows us to conclude that $J(R_1) = 2Id(R_1)$ is commutative and even much more—its elements commute with these of R_1. We shall show now that even R_1 is abelian and so commutative, that is, $eR_1(1-e) = (1-e)R_1e = \{0\}$. In fact, for each $r \in R_1$ we have the representation $r = ere+(1-e)r(1-e)+er(1-e)+(1-e)re = ere+(1-e)r(1-e)+2h$ for some $h \in Id(R_1)$. Consequently, $(r-2h)^2 = [ere+(1-e)r(1-e)]^2$ and $r^2 = erere+(1-e)r(1-e)r(1-e)$ giving that $er^2 = erere = r^2e$. But $r^2 - r \in J(R_1)$ for any $r \in R_1$ and, finally, one deduces that $er = re$, as expected, because as we have just observed the ideal $J(R_1)$ commutes with the elements of R_1. And so, e being a central idempotent implies the direct decomposition for $R_1 = K \times L$, where $K = R_1e$ and $L = R_1(1-e)$. Let now we assume that $2e \neq 0$ and $2(1-e) \neq 0$; otherwise $2e = 0$ or $2e = 2$. We therefore can find $x \in K$ which is not a sum of two commuting idempotents (otherwise $J(K) = \{0\}$ which is impossible because $0 \neq 2e \in J(K)$) as well as $y \in L$ such that $-y$ is not a sum of two commuting idempotents (otherwise $J(L) = \{0\}$ which is impossible because $0 \neq 2(1-e) \in J(L)$). But then the two component vector (x, y), being an element in R_1, is neither of the type a sum of two commuting idempotents nor minus a sum of two commuting idempotents, which is an obvious contradiction. So, our assumption is wrong guaranteeing that at least one of $2e = 0$ or $2(1-e) = 0$ is valid, and thus we are set after all.

Secondly, we consider the ring R_2: We claim that $Nil(R_2) = \{0\}$ and hence $3 = 0$. But, with Lemma 2.3 at hand, this follows immediately because any element of R_2 is either the sum or minus the sum of two commuting idempotents and because 3 is nilpotent in R_2. Thus, it is now obvious that all elements in R_2 satisfy the equation $x^3 = x$ and, in accordance with the main result from [6], we can conclude that R_2 is a subdirect product of family of copies of the field $\mathbb{Z}_3$ (see [8, Proposition 3.9] too).

Thirdly, we consider the ring R_3: We assert that $Nil(R_3) = \{0\}$ whence $5 = 0$. But, as above in the second case, with Lemma 2.3 in hand, this follows immediately. Now, given $x \in R_3$. If $x - 1 = -e_1 - e_2$, then $x = (1-e_1) - e_2$ and thus it is pretty easy to check that $x^3 = x$. However, if now $x - 1 = e_1 + e_2$, then $x - 2 = e_1 - (1-e_2)$ and we again easily verify that $(x-2)^3 = x - 2$ amounting to $x^3 - x^2 + x - 1 = 0$. On the other side, let $y \in R_3$ with $y^3 \neq y$, that is, $(-y)^3 \neq -y$. Hence one deduces as above that $y^3 - y^2 + y - 1 = 0$ yielding, applied to $-y$, that $-y^3 - y^2 - y - 1 = 0$. Comparing the last two central equalities for y, we infer that $2y^3 + 2y = 0$ and hence $6y^3 = -6y$ assuring that $y^3 = -y$ as $5 = 0$. Finally, all requirements in Proposition 2.2 are manifestly fulfilled to get that R_3 is the five element field, as stated.

(*Sufficiency*) It is pretty easy to establish that a (finite) direct product of rings from the class $\mathcal{C}$ is also a ring from the class $\mathcal{C}$. A direct consultation with [6] enables us that every element of R_2 is a sum of two idempotents. Since it is pretty easy that each element in $\mathbb{Z}_5$ is a sum of two idempotents (e.g., 0, 1, 2) or minus a sum of two idempotents (e.g., 0, 3 and 4), what remains to prove is that any element from R_1 has the same property. It is, really, very obvious that if $2 = 0$ in R_1, then each its element is a sum of two idempotents. To that purpose, taking an arbitrary $r \in R_1$, we may write that $r + J(R_1)$ is an idempotent and thus

$r - r^2 \in J(R_1) = 2\{0, 1\}$. But $J(R_1)$ is nil with $J(R_1)^2 = \{0\}$ (as $4 = 0$) and hence there exists an idempotent $g \in R_1$ with $r - g \in 2\{0, 1\}$. This containment allows us to write that $r = g$ or that $r = 2 + g = -1 - (1 - g)$ (as $2 = -2$), as required. $\square$

The next general comments could be helpful.

Remark 2.5 In the case when $2e = 0$ or $2e = 2 = -2$ holds for any idempotent e of a ring R, the conditions from our Definition are equivalent to these from the corresponding result from [8] cited above. In fact, $x = e_1 - e_2 = -e_1 - e_2 = e_1 + e_2$ provided $2e_1 = 0 = 2e_2$, and $x = e_1 - e_2 = -(1 - e_1) - (1 - e_2) = (1 - e_1) + (1 - e_2)$ provided $2e_1 = 2 = -2 = 2e_2$. Moreover, our presented above proof of Theorem 2.4 is rather more conceptual and easy than that from [8, Theorem 4.4] and gives a new strategy for further useful generalizations. Indeed, the Pierce's matrix representation is not used here instead of [8].

Besides, [8, Proposition 2.2] is self-evident and its very complicated proof is superfluous. In fact, if r is an arbitrary element of a ring R, then $r + 1$ is still in R and hence $r + 1 = e + f$, where e, f are idempotents with $ef = fe$. Thus $r = e - (1 - f)$ is a difference of two commuting idempotents, as required. Reciprocally, if $r \in R$ and $r - 1 = g - h$, where g, h are idempotents with $gh = hg$, then $r = g + (1 - h)$ is a sum of two commuting idempotents, as needed.

Example 2.6 Surprisingly, the rings $\mathbb{Z}_4 \times \mathbb{Z}_4$ and $\mathbb{Z}_4 \times \mathbb{Z}_5$ do not meet the condition in Theorem 2.4. Indeed, for the first direct product, the element $(1, 3)$ cannot be presented as a sum of two idempotents nor as minus a sum of two idempotents. As for the second direct product, we may consider the element $(1, 4)$ and use similar arguments to get our claim.

We end our work with the following three problems of some interest and importance:

Problem 2.7 Find the equations satisfied by the elements of rings from the class $\mathcal{C}$.

For rings of characteristic 5 we may consult with Proposition 2.2.

Problem 2.8 Describe the structure of those rings R for which there exist four commuting idempotents e, f, g, h such that $r = e + f - g - h$ holds for any $r \in R$.

Problem 2.9 Describe the structure of those rings R for which there exist four commuting idempotents e, f, g, h such that $r = e + f + g + h$ holds for any $r \in R$.

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