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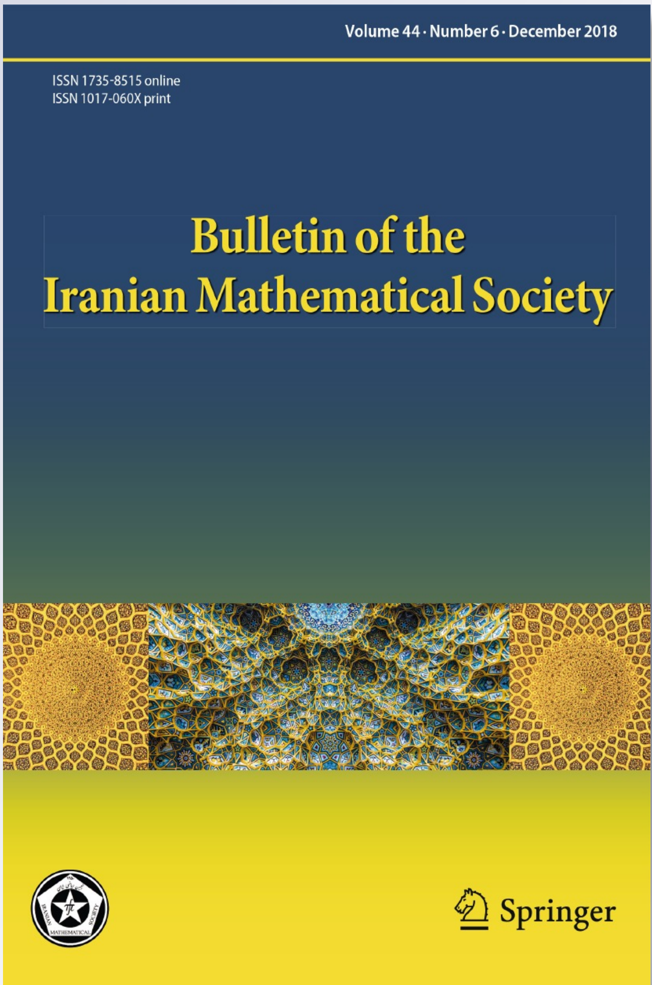

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ORIGINAL PAPER

Rings Whose Elements are Sums of Three or Differences of Two Commuting Idempotents

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Abstract
We define and examine a new class of rings whose elements are the sum of three commuting idempotents or the difference of two commuting idempotents. We fully describe them up to an isomorphism and our obtained results considerably extend some well-known achievements due to Hirano and Tominaga (Bull Aust Math Soc 37:161–164, 1988), to Ying et al. (Can Math Bull 59:661–672, 2016) and to Tang et al. (Lin Multilin Algebra, 2018).

Keywords Boolean rings · Units · Idempotents · Nilpotents · Jacobson radical

Mathematics Subject Classification 16D 60 · 16S 34 · 16U 60

1 Introduction and Fundamentals

Everywhere in the text all our rings are assumed to be associative, containing the identity element 1 which, in general, differs from the zero element 0. Our terminology and notations are mainly in agreement with [8]. For instance, for such a ring R, $U(R)$ will always denote the unit group of R, $J(R)$ the Jacobson radical of R, $\mathrm{Nil}(R)$ the set of all nilpotents in R, and $\mathrm{Id}(R)$ the set of all idempotents in R.

The classical famous concept of a *Boolean* ring states that each of its elements is an idempotent, that is, each element satisfies the equality $x^2 = x$. We shall say such a ring is of the type C^{1+}. It is well known that these rings are subdirect products of copies of the field $\mathbb{Z}_2$, and thus they are commutative of characteristic 2. Moreover, the definition is obviously equivalent to the condition that every element is equal to minus an idempotent, i.e., the ring classes of types C^{1+} and C^{1-} do coincide. As a

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common generalization of these two types, it is rather natural to consider those rings whose elements are either idempotents or minus idempotents, calling them weakly Boolean. Using the terms above, we shall say that these rings are of the type $C^{1\pm}$. We have determined their structure in [4, Theorem 1.13] by proving for such a ring R that it is commutative, satisfying the equality $x^2 = x$ or $x^2 = -x$ for all its elements, and so it is isomorphic to either a Boolean ring B or to $\mathbb{Z}_3$ or to $B \times \mathbb{Z}_3$.

Generalizing Boolean rings, in [6], were examined rings whose elements are the sum of two commuting idempotents. These rings are established there to be commutative satisfying the equation $x^3 = x$ and, specifically, they are a subdirect product of copies of the fields $\mathbb{Z}_2$ and $\mathbb{Z}_3$. Besides, this is obviously tantamount to the condition that any element is the difference of two commuting idempotents or that any element is minus the sum of two commuting idempotents and, that is why, [10, Proposition 2.2] is self-evident and its proof is superfluous. In fact, if r is an arbitrary element of a ring R, then $r + 1$ is still in R and hence $r + 1 = e + f$, where e, f are idempotents with $ef = fe$. Thus, $r = e-(1-f)$, as required. Reciprocally, if $r \in R$ and $r-1 = g-h$, where g, h are idempotents with $gh = hg$, then $r = g+(1-h)$, as claimed. Likewise, it follows that the identity $(g-h)^3 = g-h$ holds always. Calling these two classes of rings to be of types C^{2+} and C^{2-}, respectively, by what we have jut shown one sees that they will coincide as well. Further, to expand the above considerations due to [6], in [10] were investigated those rings whose elements are either a sum or a difference of two commuting idempotents. To keep the new terminology introduced above, we shall say that such a ring R is of the type $C^{2\pm}$.

In [10, Theorem 4.4] the following interesting result was proved:

- *A ring R is of the type $C^{2\pm}$ if, and only if, $R \cong R_1 \times R_2$, where $R_1 = \{0\}$ or R_1 is an abelian ring such that $R_1/J(R_1)$ is a Boolean ring with $J(R_1) = \{0\}$ or $J(R_1) = \{0, 2\}$, and $R_2 = \{0\}$ or R_2 is a subdirect product of copies of the field $\mathbb{Z}_3$.*

Although in [10] no concrete example was given realizing this necessary and sufficient condition, simple calculations show that the ring $\mathbb{Z}_4$ satisfies its requirements. However, one may check that the direct product $\mathbb{Z}_4 \times \mathbb{Z}_4$ does not retain this property further. As it will be shown below, any ring R of the type $C^{2\pm}$ (and, more generally, also of the type C^{3+}) has to be even commutative.

Our motivation is then to discover what can be said for the elements of $\mathbb{Z}_4 \times \mathbb{Z}_4$ in terms of its idempotents. A routine verification shows that any element from that product is the sum of three idempotents.

Therefore, we come to our basic tool, which parallels [5].

Definition 1.1 A ring R is said to be of type C^{3+} if all its elements are sums of three commuting idempotents.

For subsequent applications we shall slightly reformulate this definition as follows: a ring R is of the type C^{3+} if, and only if, for every $r \in R$ there exist three commuting idempotents e_1, e_2, e_3 such that $r = e_1 + e_2 - e_3$. In fact, as noticed above, we may use the trick with the elements $r + 1$ and $r - 1$, respectively.

Also, since one writes that $r = e_1 + e_2(1 - e_3) - e_3(1 - e_2)$ as the latter two elements are idempotents with zero product, we may without loss of generality assume additionally that $e_2e_3 = 0 = e_3e_2$.

Thus, it is now clear to detect that any ring of type $C^{2\pm}$ is also of the type C^{3+}, itself.

2 Main Results

We are now focussing on the classification of rings whose elements depend only on idempotents. In what follows, we shall completely characterize their structure as well as we simplify some arguments and methods for proof developed in [6] and [10], respectively. Nevertheless, the situation for C^{3+} rings is rather more complicated than that for $C^{2\pm}$ rings.

First, we need the following technicality:

Lemma 2.1 *Suppose R is a ring and $q \in \mathrm{Nil}(R)$. The following two items are true:*

1. *If $q = e - f$, where e and f are commuting idempotents, then $q = 0$.*
2. *If $q = e + f$ with $q^2 = 0$ and either $3 \in \mathrm{Nil}(R)$ or $5 \in \mathrm{Nil}(R)$, where e and f are commuting idempotents, then $q = 0$.*

Proof 1. Assuming $q^t = 0$ for some $t \in \mathbb{N}$, it follows that $q^{3^t} = 0$. Hence $0 = q^{3^t} = (e-f)^{3^t} = e - f = q$, because it is not too hard to check that $(e-f)^3 = e - f$.

2. First of all, let 3 be a nilpotent. By squaring the equality $q = e + f$, we deduce that $e + 2ef + f = 0$ and thus $e(1-f) = 0$, that is, $e = ef$. Analogically, one obtains that $f(1-e) = 0$, that is, $f = fe$. Since $ef = fe$, we may infer that $e = f$. We, furthermore, write that $q = 2e = 3e - e$ and thus $e = 3e - q \in \mathrm{Nil}(R)$ because both q and e commute. Finally, $e = 0$ giving that $q = 0$, as stated.

The same trick also successfully works in the case when 5 is a nilpotent. In fact, as above $e = f$ gives that $q = 2e$ and thus by squaring $4e = 0$. This yields that $e = 5e \in \mathrm{Nil}(R)$ which means that $e = 0 = q$, as expected. □

We now arrive at our central result here, established independently also in [9, Proposition 3.8], which states as follows:

Theorem 2.2 *Let R be a ring. Then R is of the type C^{3+} if, and only if, $R \cong R_1 \times R_2$, where $R_1 = \{0\}$ or R_1 is a commutative ring for which $4 = 0$ such that $R_1/J(R_1)$ is a Boolean ring with $J(R_1) = \{0\}$ or $J(R_1) = 2\mathrm{Id}(R_1)$, and $R_2 = \{0\}$ or R_2 is a subdirect product of copies of the field $\mathbb{Z}_3$.*

Proof "**Necessity**" We will show first that 6 is a nilpotent in R. To that aim, writing $3 = e_1 + e_2 - e_3$ for some commuting idempotents e_1, e_2, e_3 and assuming the latter two ones are orthogonal. Therefore, $3e_2 = e_1e_2 + e_2$, i.e., $2e_2 = e_1e_2$. Now, $2(e_1e_2 + e_2) = 6e_2 = 9e_2 - 3e_2 = (3e_2)^2 - 3e_2 = (e_1e_2 + e_2)^2 - (e_1e_2 + e_2) = 2e_1e_2$, that is, $2e_2 = 0$ and thus $e_1e_2 = 0$.

On the other hand, multiplying subsequently the relation $3 = e_1 + e_2 - e_3$ by $2(1 - e_2)$ and $1 - e_2$, and taking into account that $e_1e_2 = e_2e_3 = 0$, we derive that $2e_1 - 2e_3 = 6(1-e_2) = 9(1-e_2) - 3(1-e_2) = (3(1-e_2))^2 - 3(1-e_2) = (e_1 - e_3)^2 - (e_1 - e_3) = -2e_1e_3 + 2e_3$. Comparing both sides, we detect that $2e_1 = -2e_1e_3 + 4e_3$. Multiplying both sides by e_1, we obtain after all that $2e_1 = 2e_1e_3$.

Moreover, since $e_1e_2 = e_3e_2 = 0$, one sees that $3(1 - e_2) = e_1 - e_3$ and squaring we deduce that $9(1-e_2) = e_1 - 2e_1e_3 + e_3 = e_1 - 2e_1 + e_3 = e_3 - e_1 = -3(1-e_2)$. Finally, $12(1 - e_2) = 0$. But as we have inferred above $2e_2 = 0$, so that $12e_2 = 0$ and thereby $12 = 0$. The last implies that $12.3 = 6^2 = 0$ which substantiates our claim.

Furthermore, an appeal to the Chinese Remainder Theorem gives that $R \cong R_1 \times R_2$, where R_1 is a ring in which 2 is a nilpotent (with index of nilpotence ≤ 2) and R_2 is a ring in which 3 is a nilpotent. Since a direct factor of a C^{3+} ring is obviously again C^{3+}, both R_1, R_2 are C^{3+} too. We foremost will examine R_1. But $2 \in \mathrm{Nil}(R_1)$ ensures that $3 \in U(R_1)$ and thus the equality $12 = 0$, which also holds in R_1, gives that $4 = 0$. Since $2 \in J(R_1)$ being a central nilpotent and since the quotient $R_1/J(R_1)$ is a C^{3+} ring of characteristic 2, it is quite elementary to verify that this factor-ring is necessarily Boolean. Consequently, $U(R_1)/(1 + J(R_1)) \cong U(R_1/J(R_1)) = \overline{1}$ which yields that $U(R_1) = 1 + J(R_1)$. That is why, $1 + \mathrm{Nil}(R_1) \subseteq U(R_1)$ which means that $\mathrm{Nil}(R_1) \subseteq J(R_1)$. We shall now derive that $J(R_1) = 2\mathrm{Id}(R_1)$ which, in view of $2 \in \mathrm{Nil}(R_1)$, will guarantee that $J(R_1) \subseteq \mathrm{Nil}(R_1)$ and so will allow us to conclude that $J(R_1) = \mathrm{Nil}(R_1)$, as desired. To prove the wanted equality for $J(R_1)$, take an arbitrary $y \in J(R_1)$ and write $y = g_1 + g_2 - g_3$ for some three commuting idempotents g_1, g_2, g_3 where the last two ones are orthogonal. Multiplying both sides on the right by g_2, we infer that $yg_2 = g_1g_2 + g_2$. Since g_1g_2 remains an idempotent, this assures that $(yg_2 - g_2)^2 = yg_2 - g_2$ which is tantamount to $y^2g_2 - 3yg_2 + 2g_2 = 0$ or, equivalently, $(yg_2 - 2g_2)(y - 1) = 0$. As $y - 1$ lies in $U(R_1)$, it follows at once that $yg_2 = 2g_2$. Next, again multiplying both sides of the initial equality by $1 - g_2$ on the right, we have that $y(1 - g_2) = g_1(1 - g_2) - g_3$. Since $g_1(1 - g_2)$ is still an idempotent, by what we have commented above, it follows that $[y(1 - g_2)]^3 = [g_1(1 - g_2) - g_3]^3 = g_1(1 - g_2) - g_3 = y(1 - g_2)$, i.e, $y^3(1 - g_2) = y(1 - g_2)$ guaranteeing that $y(1 - g^2)[y^2 - 1] = 0$. As again $y^2 - 1 \in U(R_1)$, we detect that $y(1 - g_2) = 0$. We finally conclude that $y = yg_2 = 2g_2$, as required.

We are now ready to prove that the ring R_1 is abelian and thus commutative. To that goal, we consider the Pierce's decomposition of R_1 represented like this:

$$R_1 = \begin{pmatrix} eR_1e & eR_1\,(1-e) \\ (1-e)\,R_1e & (1-e)\,R_1\,(1-e) \end{pmatrix},$$

where e is an arbitrary idempotent in R_1. We intend to obtain that $eR_1(1 - e) = (1 - e)R_1e = \{0\}$ which forces at once that all idempotents in R_1 are central. To do that, given any $z \in eR_1(1 - e)$, we deduce that $z^2 = 0$ and consequently $z \in \mathrm{Nil}(R_1) = J(R_1) = 2\mathrm{Id}(R_1)$ by what we have already shown above. We, therefore, write that $\begin{pmatrix} 0 & z \\ 0 & 0 \end{pmatrix} = 2\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 2a & 2b \\ 2c & 2d \end{pmatrix}$, where $a \in eR_1e$, $b \in eR_1(1 - e)$, $c \in (1 - e)R_1e$ and $d \in (1 - e)R_1(1 - e)$ with $\begin{pmatrix} a & b \\ c & d \end{pmatrix}^2 = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Furthermore, one sees that $z = 2b = 2(ab + bd) = 2ab + 2bd = 0$, because $2a = 2d = 0$, as required. By a way of similarity, the other pursued relationship $(1 - e)R_1e = \{0\}$ is true as well. Thus R_1 is really abelian and, because its elements are generated by idempotents only, it must be commutative itself.

As for R_2, we shall first establish that $\mathrm{Nil}(R_2) = \{0\}$ which enables us that $3 = 0$; actually that the equality $3 = 0$ holds in R_2 is an immediate consequence of the above fact that $12 = 0$ in R and $4 \in U(R_2)$ because $3 \in \mathrm{Nil}(R_2)$. In fact, choosing an arbitrary nilpotent $q \in R_2$ of index 2, we write that $q = h_1 + h_2 - h_3$ for some commuting idempotents h_1, h_2, h_3 with $h_2.h_3 = h_3.h_2 = 0$. Consequently, $qh_2 = h_1h_2 + h_2$ is a sum of two commuting idempotents and remains a nilpotent of index 2 as q and h_2 commutes. Since $3 \in \mathrm{Nil}(R_2)$, we employ Lemma 2.1 (2) to get that $qh_2 = 0$. But $q(1 - h_2) = h_1(1 - h_2) - h_3(1 - h_2) = h_1(1 - h_2) - h_3$ is a difference of two commuting idempotents, whereas $q(1 - h_2)$ remains a nilpotent. Hereafter, we just apply Lemma 2.1 (1) to get that $q(1 - h_2) = 0$. Resultantly, $q = qh_2 = 0$ after all, which substantiates the claim that R_2 is reduced, indeed. We further observe that for any element $r \in R_2$, written as $r = f_1 + f_2 - f_3$ in the standard manner of an element in a C^{3+} ring, it must be that $r^3 = r$ and therefore, as it is well-known in virtue of the comments quoted above, R_2 has to be a subdirect product of copies of the field $\mathbb{Z}_3$, as promised.

"**Sufficiency**" Since it is an easy exercise to establish that the direct product of two rings of the type C^{3+} is again a ring of the type C^{3+}, and since the nonzero ring R_2 is known to be of the type C^{2+} (and thus of the type C^{3+}), it is enough to show only that the ring R_1 is of the type C^{3+}. To this purpose, given $x \in R_1$, it must be that $x + J(R_1) \in \mathrm{Id}(R/J(R_1))$ and so $x^2 - x \in J(R_1) = 2\mathrm{Id}(R_1)$. But $J(R_1)$ is nil and hence there exists $h \in \mathrm{Id}(R_1)$ with the property $h - x \in 2\mathrm{Id}(R_1)$. Finally, it is mandatory for x to be the sum of three commuting idempotents, bearing in mind that R_1 is commutative. $\square$

The following sheds some light on the concrete exhibition of sorts of C^{3+} rings.

Remark 2.3 As an alternative simpler or at least different proof, without the usage of Pierce matrix representation of rings, we may process as follows: writing $4 = e + f + g$ for some three commuting idempotents e, f, g from a ring R we get $4e = e + ef + eg$, so that $3e = ef + eg$ and hence $3ef = ef + efg$, so that $2ef = efg$ is an idempotent. This means that $4ef = 2ef$, whence $2ef = 0$. Similarly, $2eg = 2fg = 0$. Furthermore, by what we have shown so far, $4^2 = (e + f + g)^2 = e + f + g = 4$ leading to $12 = 0$. Therefore, we may decompose R into its 2 and 3 primary components like this: $R = I \oplus J$, where $I = \{r \in R \mid 4r = 0\}$ and $J = \{r \in R \mid 3r = 0\}$. Therefore, we may assume either $4R = \{0\}$ or $3R = \{0\}$.

Assume $3R = \{0\}$. Then, for any $x \in R$, we write $x = e + f + g$ as usual and deduce $x^3 = x$. Thus, it is principally known that R has to be commutative and reduced. Now, if P is a prime ideal in R, then the quotient R/P only has trivial idempotents. Since each element of R is a sum of three idempotents, the factor-ring R/P must be the field of order 3. Consequently, the natural map $R \to \prod_P R/P$, with P running over all prime ideals of R, has kernel equal to the intersection of all prime ideals, which is $\mathrm{Nil}(R) = \{0\}$, and hence embeds R into a direct product of fields of order 3, as required.

Now, assume that $4R = \{0\}$. Put $T = 2R$. Thus, a simple check shows that $B = R/T$ is a Boolean ring. Fix an idempotent e and consider the element $x \in eR(1 - e)$. Then, because of the commutativity of B, x maps to 0 in B, i.e., $x \in T$ so that $x = 2y$ for some $y \in R$. We write the standard decomposition of the element y thus

$$y = eye + ey(1 - e) + (1 - e)ye + (1 - e)y(1 - e).$$

One observes that both $ey(1-e)$, $(1-e)ye$ map to 0 in B as it is commutative, and hence they belong to $2R$. That is why, the elements $2ey(1-e)$, $2(1-e)ye$ are in $4R$ and thus they are zero. Therefore, we have $x = 2y = 2eye + 2(1-e)y(1-e)$ and hence $x = ex(1-e) = 0$ meaning that $eR(1-e) = \{0\}$ and, by analogy, so also $(1-e)Re = \{0\}$. Finally, $R = eRe + (1-e)R(1-e)$ and so e is central. Since R is generated by idempotents only, it has to be commutative, as asserted.

Returning again to the original proof alluded to above, it is also worthwhile noticing that, since $J(R_1)$ is nil and $R_1/J(R_1)$ is Boolean, it follows from [3] (see, [7], too) that the ring R_1 is strongly nil-clean.

On the other vein, we once again show now that the ring $R = \mathbb{Z}_4 \times \mathbb{Z}_4$ satisfies the conditions in Theorem 2.2. Indeed, $J(\mathbb{Z}_4) = \mathrm{Nil}(\mathbb{Z}_4) = 2\mathbb{Z}_4 = 2\mathrm{Id}(\mathbb{Z}_4) = \{0, 2\}$ and hence immediately $J(R) = \mathrm{Nil}(R) = J(\mathbb{Z}_4) \times J(\mathbb{Z}_4) = 2\mathrm{Id}(R)$. Likewise, $\mathbb{Z}_4/J(\mathbb{Z}_4) = \mathbb{Z}_4/2\mathbb{Z}_4 \cong \mathbb{Z}_2$, so that the isomorphism $R/J(R) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ is directly valid.

We can now slightly enlarge Definition 1.1 in the following way:

Definition 2.4 A ring R is called *weakly* C^{3+} if, for each $r \in R$, there exist three idempotents e_1, e_2, e_3 such that e_2 commutes with both e_1, e_3 and $r = e_1 + e_2 + e_3$.

As shown above, this amounts to the condition that for every $r \in R$, there are three idempotents e_1, e_2, e_3 such that e_2 commutes with both e_1, e_3 and $r = e_1 + e_2 - e_3$. Certainly, as we have already illustrated, we may with no harm of generality assume that e_2 and e_3 are orthogonal.

To simplify the terminology, we shall say that a ring is of the type *weakly* C^{2+}, provided that all its elements are sums of two idempotents. This is exactly Definition 2.4 with the extra limitation $e_2 = 0$.

The next significantly strengthens [10, Proposition 6.1] as well as it somewhat naturally continues Theorem 2.2.

Proposition 2.5 *The ring R is weakly C^{3+} if, and only if, $R \cong R_1 \times R_2$, where R_1 is either $\{0\}$ or otherwise is a ring of even characteristic ≤ 4 for which the quotient $R_1/J(R_1)$ is of the type weakly C^{2+}, and R_2 is either $\{0\}$ or is a ring which is a subdirect product of copies of the field $\mathbb{Z}_3$ otherwise.*

Proof Writing $3 = e_1 + e_2 - e_3$, where e_1, e_2, e_3 are idempotents as stated above, one observes that e_1 and e_3 also commute, and so we just appeal the method used in Theorem 2.2 to conclude that 6 is a nilpotent and even that $12 = 0$ in R. Furthermore, the Chinese Remainder Theorem works to get the desired decomposition for R into the direct product of two rings R_1 and R_2 again of the type weakly C^{3+} such that $2 \in \mathrm{Nil}(R_1)$ and $3 \in \mathrm{Nil}(R_2)$.

Concerning R_1, it is clear that $R_1/J(R_1)$ remains of the type weakly C^{3+} having characteristic 2. Thus, it plainly follows then that this factor-ring is necessarily weakly C^{2+}, because the sum of two orthogonal idempotents is again an idempotent as well as the sum of two commuting idempotents in a ring of characteristic 2 is an idempotent too.

As for R_2, we once again may apply the scheme for proof from Theorem 2.2 to get that $3 = 0$. We now intend to prove that R_2 is a reduced ring and hence it will be of necessity an abelian one. But then it has to be commutative being (weakly) C^{3+}. This will imply, in turn, that R_2 is a C^{2+} ring obviously satisfying the equation $x^3 = x$. And so, given $q \in \mathrm{Nil}(R_2)$ with $q^2 = 0$, we write that $q-1 = e_1+e_2-e_3$, where $e_1, e_2, e_3 \in \mathrm{Id}(R_2)$ such that e_2 commutes with e_1 and e_3 along with e_2 and e_3 being orthogonal. Therefore, $q = (e_1+1)+e_2-e_3$ and so $[q-(e_1+1)]^3 = q-(e_1+1)$. Standard computations show that the last is tantamount to the equality $e_1qe_1-qe_1q+qe_1+e_1q = q$. Multiplying by q on the right, we detect that $e_1qe_1q = -qe_1q$ and multiplying this on the left by e_1, we deduce that $e_1qe_1q = 0 = qe_1q$. Thus, the initial equality takes the form $e_1qe_1 + qe_1 + e_1q = q$. Again multiplying by e_1 on the left, it follows that $e_1qe_1 = 0$. Finally, the explored equality is of the kind $q = qe_1 + e_1q$.

On the other side, $[q-(e_1+1)]^2 = (e_2-e_3)^2$ which is pretty equal to $-q(e_1+1)-(e_1+1)q+1 = e_2+e_3$. Summing this with $q-(e_1+1) = e_2-e_3$, and taking into account that $q = qe_1+e_1q$, which relation was derived above, we infer that $q = e_1-e_2$. Consequently, $0 = q^3 = q$ which is a guarantor that $\mathrm{Nil}(R_2) = \{0\}$, as asked for. □

Remark 2.6 As previously indicated, the last statement is a substantial strengthening of Proposition 6.1 from [10] by taking $e_2 = 0$. Interestingly, a note concerning its proof, is that if we directly choose $q = e_1 + e_2 - e_3$, where q and these three idempotents are as above, it will follow that $e_2 = 0$ and so $q = e_1 - e_3$. But since these two idempotents may not commute, it is hereafter difficult to approach. That is why, the idea to consider $q - 1$ was successful.

The next reduction statement is somewhat surprising.

Proposition 2.7 *Every C^{3+} ring in which $6 = 0$ is a C^{2+} ring.*

Proof For such a ring R and any $r \in R$, it suffices to prove that the identity $r^3 = r$ is fulfilled. And therefore, writing $r = e_1+e_2-e_3$ which all are commuting idempotents as e_2 and e_3 are both orthogonal, we calculate that

$$\begin{aligned} r^3 &= e_1 + 3e_1(e_2-e_3) + 3e_1(e_2-e_3)^2 + (e_2-e_3)^3 \\ &= e_1 + 3e_1(e_2-e_3) + 3e_1(e_2+e_3) + (e_2-e_3) \\ &= e_1 + 3e_1e_2 - 3e_1e_3 + 3e_1e_2 + 3e_1e_3 + e_2 - e_3 = r + 6e_1e_2 = r, \end{aligned}$$

and so the assertion sustained. □

A reasonably adequate question is then to know what can be said for ring R whose elements satisfy one of the equations $x^3 = x$ or $x^3 = -x$? It is readily seen by a direct check that the ring $\mathbb{Z}_4$, which is of the type $C^{2\pm}$ (and so of the type C^{3+}), does not possess that property, however, which is in sharp contrast with our expectation—compared also with the two problems posed at the end of the paper.

We are now classifying when direct products of $C^{2\pm}$ rings are again rings of this type, thus explaining the previously discussed fact why $\mathbb{Z}_4 \times \mathbb{Z}_4$ is *not* a $C^{2\pm}$ ring.

Proposition 2.8 *Suppose $R = R_1 \times R_2$ is a ring. Then R is of the type $C^{2\pm}$ if, and only if, R_1 and R_2 are of the type $C^{2\pm}$ and one of them is of the type C^{2+}.*

Proof "$\Rightarrow$". As above in the case of C^{3+} rings, it is readily checked that the direct factor of a $C^{2\pm}$ ring is again a $C^{2\pm}$ ring. What is sufficient to be proved now is that one of the two factors has to be a C^{2+} ring. To see that, assume the contrary that neither of them need not be of the type C^{2+}. This means that, for example, there are $r_1 \in R_1$ which cannot be written as the sum of two commuting idempotents and $r_2 \in R_2$ which cannot be written as the difference of two commuting idempotents. Letting now $r = (r_1, r_2) \in R = R_1 \times R_2$, it follows immediately that r cannot be written as either the sum or the difference of two commuting idempotents.

"$\Leftarrow$". Given $r \in R$, one writes that $r = (r_1, r_2)$, where $r_1 \in R_1$ and $r_2 \in R_2$. With no harm in generality, we may assume that R_2 is a C^{2+} ring. Thus, since r_2 can be written simultaneously as the sum of two commuting idempotents, as well as the difference of two commuting idempotents, it is an easy technical matter to show that r is either the sum or the difference of two commuting idempotents, because the same property has the element r_1. This manifestly demonstrates that R is a $C^{2\pm}$ ring, as asserted. □

We now intend to considerably extend the listed above Theorem 2.2 and, thereby, the aforementioned [9, Proposition 3.8] as well as [10, Theorem 4.4] to a new point of view by considering certain rings whose elements depend on commuting idempotents only.

Definition 2.9 We shall say that a ring R belongs to the class $\mathcal{K}$ if, for every $r \in R$, there exist commuting $e_1, e_2, e_3 \in \mathrm{Id}(R)$ such that $r = e_1 + e_2 + e_3$ or $r = e_1 - e_2$.

The leitmotif here is to describe the isomorphic structure of all of the rings lying in the class $\mathcal{K}$ since there exist some sorts of rings, e.g., $\mathbb{Z}_5$, which lie in the class $\mathcal{K}$ but are definitely totally different from these lying in the foregoing examined class C^{3+}.

Lemma 2.10 *In a ring $R \in \mathcal{K}$ the containment $30 \in \mathrm{Nil}(R)$ holds.*

Proof Consider the element -2 of R. First, if $-2 = e_1 - e_2$ with $e_1e_2 = e_2e_1$, one checks that $(-2)^3 = -2$ giving up that $6 = 0$. This implies that $30 = 0$ is a nilpotent, as required.

Second, we write that $-2 = e_1 + e_2 + e_3$ with $e_1e_2 = e_2e_1$, $e_2e_3 = e_3e_2$ and $e_1e_3 = e_3e_1$. Hence $-3 = e_1 + e_2 - (1 - e_3) = e_1 + e_2 - e_3'$ putting $e_3' = 1 - e_3$. But $e_2 - e_3' = e_2(1 - e_3') - e_3'(1 - e_2)$ and the latter two idempotents are obviously orthogonal, so that with no harm in generality we shall hereafter assume that $e_2e_3' = 0$, whence $e_3'e_2 = 0$ owing to their commutativity. Furthermore, multiplying the equality $-3 = e_1 + e_2 - e_3'$ by e_2, we deduce that $4e_2 = -e_1e_2$ and multiplying this by e_1, we derive that $5e_1e_2 = 0$. Thus, multiplying $4e_2 = -e_1e_2$ by 5, we infer that $20e_2 = 0$. Now, squaring the equation $-3 = e_1 + e_2 - e_3'$, we detect that $12 = 2e_3' - 2e_1e_3' + 2e_1e_2$ which assures that $12e_1 = 2e_1e_2$ and, consequently, $60e_1 = 0$. Likewise, $12e_1 = 2e_1e_2$ enables us that $e_1e_2 = 6e_1e_2 = 36e_1$ which, in turn, forces that $36e_1e_3' = 0$. That is why, under the multiplication by 18.5 of the relation $12 = 2e_3' - 2e_1e_3' + 2e_1e_2$, we get that $180e_3' = 60.18$. We, therefore, multiply $-3 = e_1 + e_2 - e_3'$ by 180 to conclude that $3.180 = 0$. However, this immediately allows us to write that $(30)^3 = 3.180.50 = 0$ which means that 30 is a nilpotent, as expected. □

The next technicality is useful (see, [1], for more details).

Proposition 2.11 *Let R be a ring of characteristic 5 whose elements satisfy the equations $x^3 = x$ or $x^3 = -x$. Then $R \cong \mathbb{Z}_5$.*

We now have all the ingredients to prove our next chief result which considerably improved the listed above Theorem 2.2 (and thus Proposition 3.8 from [9]) as well as [10, Theorem 4.4].

Theorem 2.12 *A non-zero ring R is from the class $\mathcal{K}$ if, and only if, $R \cong R_1 \times R_2 \times R_3$, where R_1, R_2, R_3 are rings satisfying the next three conditions:*

1. *$R_1 = \{0\}$, or otherwise R_1 is a commutative ring in which $4 = 0$, $R_1/J(R_1)$ is Boolean and either $J(R_1) = \{0\}$ or $J(R_1) = 2\mathrm{Id}(R_1)$.*
2. *$R_2 = \{0\}$, or R_2 is a subdirect product of family of copies of the field $\mathbb{Z}_3$ otherwise.*
3. *$R_3 = \{0\}$ (which is mandatory when $J(R_1)$ is non-zero), or R_3 is isomorphic to the field $\mathbb{Z}_5$ otherwise.*

Proof "**Necessity**". With the aid of Lemma 2.10, the Chinese Remainder Theorem allows us to decompose R as $R_1 \times R_2 \times R_3$, where $2 \in \mathrm{Nil}(R_1)$, $3 \in \mathrm{Nil}(R_2)$ and $5 \in \mathrm{Nil}(R_3)$. Certainly, R_1, R_2, R_3 remain rings from the class $\mathcal{K}$.

First, we consider the ring R_1: Suppose that $x = e_1+e_2+e_3$ or $x = e_1-e_2$ for some commuting idempotents e_1, e_2, e_3 from R_1. In the first case, by squaring we obtain that $x^2-x \in J(R_1)$ as $2 \in J(R_1)$ which can be written as $(x+J(R_1))^2 = x+J(R_1)$. In the second case, one writes that $x = e_1+e_2-2e_2$ and so $x+J(R_1) = e_1+e_2+J(R_1) = [e_1 + J(R_1)] + [e_2 + J(R_1)] \in \mathrm{Id}(R_1/J(R_1))$ because $2e_1e_2 \in J(R_1)$. Finally, in both cases, it must be that $R_1/J(R_1)$ is a Boolean quotient. Next, in virtue of the proof of Lemma 2.10, we know that $3 \times 180 = 4 \times 135 = 0$. Since $(4, 135) = 1$ and $2 \in J(R_1)$ yield together that $135 \in 1 + J(R_1) \leq U(R_1)$, we derive that $4 = 0$, as stated. The further facts that either $J(R_1) = \{0\}$ or $J(R_1) = 2\mathrm{Id}(R_1)$ follow in the same manner as in Theorem 2.2. Indeed, if $z = e_1 - e_2$ for any $z \in J(R_1)$, it follows that $z^3 - z = z(z^2 - 1) = 0$ implying that $z = 0$ as $z^2 - 1$ inverts in R_1. When z is a sum of three commuting idempotents, the technique used in Theorem 2.2 or Remark 2.3 could be successfully adapted.

Second, we consider the ring R_2: We claim that $\mathrm{Nil}(R_2) = \{0\}$ and hence $3 = 0$. In fact, given a nilpotent q in R_2 with $q^2 = 0$, we write $q + 1 = e + f + g$ or $q+1 = e-f$ for some commuting idempotents e, f, g of R_2. In the latter possibility, we have $-q = (1-e)+f$ and with Lemma 2.1 (2) at hand we deduce that $q = 0$, as expected. In the first possibility, we have $q = e + f - (1-g)$. So, $qg = eg + fg$ and again Lemma 2.1 (2) implies that $qg = 0$. Since $q = e + fg - (1 - f)(1 - g)$, we deduce that $q(1-g) = e(1-g) - (1-f)(1-g)$ is a nilpotent which is a difference of two commuting idempotents. Therefore, it is easily verified invoking Lemma 2.1 (1) that $q(1 - g) = 0$ and, finally, $q = qg = 0$, as claimed. Thus, it is now obvious that all elements in R_2 satisfy the equation $x^3 = x$ and, in accordance with [6], we can conclude that R_2 is a subdirect product of isomorphic copies of the field $\mathbb{Z}_3$.

Third, we consider the ring R_3: We assert that $\mathrm{Nil}(R_3) = \{0\}$ whence $5 = 0$. But with Lemma 2.1 (2) in hand, we may copy the idea presented above to infer the wanted reduced property of R_3.

Now, given a, b, c are three commuting idempotents, we write $a+b-c = b(1-c) + a(1-c(1-b)) - c(1-b)(1-a)$, where all $b(1-c)$, $a(1-c(1-b))$ and $c(1-b)(1-a)$ retain the idempotent property as the first and the second ones are both orthogonal with the third one, so that, without loss of generality, we may assume that $ac = bc = 0$. That is why, $(a+b-c)^3 = a+b-c+d$ for some idempotent d. Consequently, the equality $[(a+b-c)^3 - (a+b-c)]^2 = (a+b-c)^3 - (a+b-c)$ is fulfilled.

With this simple but helpful observation in mind, choose an arbitrary $x \in R_3$. Hence $x = e - f$ for some two commuting idempotents e, f of R_3 and thus it is easily checked that $x^3 = x$. However, if $x = g + h + j$ for some three commuting idempotents g, h, j of R_3, we can write $x - 1 = g + h - (1 - j)$ and, by what we have shown in the preceding paragraph, we deduce that

$$\left[(x-1)^3 - (x-1)\right]^2 = (x-1)^3 - (x-1),$$

which amounts to $x^4 - x^3 + x^2 - x = 0$, taking into account that $5 = 0$ and so $x^5 = x$.

Let us now consider the element $-x$. If $-x$ is a difference of two commuting idempotents, then again $x^3 = x$ holds since $(-x)^3 = -x$. If, however, $-x$ is a sum of three commuting idempotents, then in the same manner as above, replacing x by $-x$ in $x^4 - x^3 + x^2 - x = 0$, we get that $x^4 + x^3 + x^2 + x = 0$. Combining them, we extract that $2x^3 + 2x = 0$ implying that $6x^3 = -6x$ and that $x^3 = -x$, as required, because $5 = 0$. Finally, consulting with Proposition 2.11, we get that R_3 has exactly five elements, as desired.

"**Sufficiency**". A direct consultation with [6] enables us that every element of R_2 is a sum of two idempotents. Since it is pretty easy that each element in $\mathbb{Z}_5$ is a sum of three idempotents (e.g., the elements 0, 1, 2 and 3) or a difference of two idempotents (e.g., the elements 0, 1 and 4), what remains to prove is that any element from R_1 is a sum of three idempotents. It is, really, well known that if $2 = 0$ in R_1 it must have a sum of two idempotents or even just a single idempotent. To that purpose, taking an arbitrary $r \in R_1$, we may write that $r + J(R_1)$ is an idempotent and thus $r - r^2 \in J(R_1) = 2\mathrm{Id}(R_1)$. But $J(R_1)$ is nil with $J^2(R_1) = \{0\}$ (as $4 = 0$) and hence there exists an idempotent $g \in R_1$ with $r - g \in 2\mathrm{Id}(R_1)$. This containment allows us to write that $r = g + 2h = g + h + h$ for some $h \in \mathrm{Id}(R_1)$, as required. $\square$

The following constructions shed some more light on the formulation of the preceding theorem and, especially, on point (3).

Example 2.13 Various examples of rings satisfying the statement of the previous theorem are $\mathbb{Z}_2$, $\mathbb{Z}_3$, $\mathbb{Z}_4$ and $\mathbb{Z}_5$ as well as the direct products $\mathbb{Z}_2 \times \mathbb{Z}_3$, $\mathbb{Z}_2 \times \mathbb{Z}_5$, $\mathbb{Z}_3 \times \mathbb{Z}_5$ and $\mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_5$. In fact, in $\mathbb{Z}_2$ the signs "+" and "-" are tantamount, in $\mathbb{Z}_3$ we have that $0 = 0 - 0 = 0 + 0$, $1 = 1 + 0 = 1 - 0$ and $2 = 1 + 1 = 0 - 1$, whereas in $\mathbb{Z}_4$ it is impossible to represent all elements simultaneously as a sum of three idempotents and a difference of two idempotents; e.g., the element 2 which is a nilpotent of order 2 cannot be presented in this aspect, while the other elements can, because $3 = 0 - 1$. That is why, contrasting with the above, the ring $\mathbb{Z}_4 \times \mathbb{Z}_5$ does not satisfy it. In fact, by what we have already commented, the element $(2, 4)$ will work, since $4 = 0 - 1$ in $\mathbb{Z}_5$.

We finish off the work with a brief discussion and two questions of interest. First, we pose the following:

Problem 2.14 Let $n \in \mathbb{N}$. Describe all rings whose elements satisfy one of the equalities $x^n = x$ or $x^n = -x$.

It is worthwhile noticing that the cases $n = 2$, $n = 3$ and $n = 4$ were successfully settled in [4], [1] and [2], respectively.

Our comments alluded to above, and especially that all elements of the rings of the type $C^{1\pm}$ are solutions of one of the equations $x^2 = x$ or $x^2 = -x$, lead rather logically that the rings of the type $C^{2\pm}$ should have satisfied the equations $x^3 = x$ or $x^3 = -x$. But, however, this is *not* so, since it is rather easily verified that the ring $\mathbb{Z}_4$ does not inherited this property, while the rings $\mathbb{Z}_2$, $\mathbb{Z}_3$ and $\mathbb{Z}_5$ do. In fact, it follows that $x^5 = x$ and so such a ring must be commutative regular (and hence reduced), whence it is a subdirect product of fields. Two questions which immediately arise are to find the kind of these fields as well as to compute their number (of repetations) being finite or infinite. Nevertheless, the determination of the exact ones in this decomposition will be the theme of some other subsequent research article.

In closing, we state:

Problem 2.15 Describe C^{n+} rings for each $n \in \mathbb{N}$.

It is worthwhile noticing that in [5] these rings were somewhat called n-thin.

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