Bornologies in Metrizable spaces

1. Introduction

Notation and terminology within a metric space

Let \( \langle X, d \rangle \) be a metric space. We write \( B_d(x, \varepsilon) \) for the open ball with center \( x \in X \) and radius \( \varepsilon > 0 \). For \( A \subseteq X \), we write \( B_d(A, \varepsilon) \) for the enlargement \( \cup_{a \in A} B_d(a, \varepsilon) \).

We denote the nonempty subsets of \( X \) by \( \mathcal{P}_0(X) \) and the nonempty finite subsets of \( X \) by \( \mathcal{F}_0(X) \). The nonempty \( d \)-bounded subsets of \( X \) will be denoted by \( \mathcal{B}_d(X) \), and the nonempty \( d \)-totally bounded subsets by \( \mathcal{T B}_d(X) \). All of these families are bornologies.

**Definition 1.1.** Let \( \langle X, d \rangle \) be a metric space. By a bornology \( \mathcal{B} \) on \( X \), we mean a family of nonempty subsets of \( X \) that forms a cover, is stable under taking finite unions, and is stable under taking nonempty subsets.

Remark. Given a cover \( \mathcal{A} \) of \( X \) there is a smallest bornology containing \( \mathcal{A} \), namely the family of nonempty subsets of finite unions of members of \( \mathcal{A} \).

Remark. A bornology on a metrizable space that agrees with \( \mathcal{B}_d(X) \) for some metric \( d \) compatible with the topology is called a metric bornology.

Some other bornologies of interest:

1. the nonempty subsets \( \mathcal{K}_0(X) \) of \( X \) with compact closure; these form a metric bornology iff \( X \) is locally compact and separable.
2. In the case \( X' = X \), the nonempty nowhere dense subsets of \( X \).
3. the nonempty subsets of \( X \) that are functionally bounded with respect to each member of a family of functions defined on \( X \) with values in a second metric space \( \langle Y, d \rangle \).
(4) Given a sequence $\langle f_n \rangle$ of functions on $X$ with values in $\langle Y, d \rangle$ and another function $f : X \to Y$ to which $\langle f_n \rangle$ is pointwise convergent, the family of nonempty subsets of $X$ on which the convergence is uniform.

By a base $B_0$ for a bornology $B$, we mean a subfamily from which $B$ can be recovered by taking nonempty subsets. Clearly,

- $B_d(X)$ has both an open base and a closed base;
- $\mathcal{K}_0(X)$ always has a closed base and has an open base iff $X$ is locally compact.

A bornology $B$ is called local if each $x \in X$ has a neighborhood in $B$. The bornology is called stable under small enlargements if $\forall B \in B, \exists \varepsilon > 0$ such that $B_d(B, \varepsilon) \in B$.

$$\text{stable under small enlargements} \implies \text{open base} \implies \text{local}$$

An important condition for a bornology that is weaker than stability under small enlargements is the following [7].

**Definition 1.2.** A bornology $B$ is called shielded from closed sets if and only if for each $A \in B$, there exists a superset $B \in B$ such that each neighborhood of $B$ contains $B_d(A, \varepsilon)$ for some $\varepsilon > 0$.

**Example 1.3.** In any metric space, the bornology of subsets with compact closure is shielded from closed sets.

**Example 1.4.** Let $X = \{(x, y) \in \mathbb{R}^2 : \text{either } x \neq 0 \text{ or } y \leq 0 \}$ equipped with the Euclidean metric. Let $B_n = \{(x, y) : y \leq n|x|\}$; then the bornology $B$ having as a base $\{B_n : n \in \mathbb{N}\}$ is shielded from closed sets.
Given metrizable spaces $X$ and $Y$ equipped with bornologies $\mathcal{B}_X$ and $\mathcal{B}_Y$ and a function $f : X \to Y$, the property

$$(\Diamond) \quad \forall B \in \mathcal{B}_Y, \ f^{-1}(B) \in \mathcal{B}_X$$

corresponds in the case of metric bornologies to coercivity as it usually understood. On the other hand, the property

$$(\spadeclub) \quad \forall A \in \mathcal{B}_X, \ f(A) \in \mathcal{B}_Y$$

is enjoyed by Lipschitz functions with respect to the metric bornologies.

### 2. Bornologies of bounded/totally bounded sets

The systematic study of bornologies in topological spaces starts with a paper of S.-T. Hu that appeared 64 years ago [19]. The main result of that paper characterizes bornologies that are metric bornologies

**Theorem 2.1.** Let $X$ be a metrizable topological space and let $\mathcal{B}$ be a bornology on $X$. The following conditions are equivalent:

1. $\mathcal{B}$ is a metric bornology;
2. $\mathcal{B}$ has a countable base, a closed base and an open base;
3. $\mathcal{B}$ has a countable base, and $\forall A \in \mathcal{B}, \exists B \in \mathcal{B}$ with $A \subseteq \text{int}(B)$.

**Example 2.2.** On $\mathbb{R}^2$ the bornology with base $\{B_n : n \in \mathbb{N}\}$ where $B_n = \{(x, y) : \text{either } |x| \leq n \text{ or } |y| \leq n\}$ is metrizable.

**Example 2.3.** On $\{ (x, y) \in \mathbb{R}^2 : \text{either } x \neq 0 \text{ or } y \leq 0\}$, the bornology with base $\{B_n : n \in \mathbb{N}\}$ where $B_n = \{(x, y) : y \leq n|x|\}$ is not metrizable.
Related facts [3] :

- If $X$ is metrizable and noncompact, then there are uncountably many distinct metric bornologies;

- If $d$ is a compatible metric for $X$ and $B$ is a bornology on $X$, then there is a metric $\rho$ uniformly equivalent to $d$ such that $B = B_\rho(X)$ if and only if $B$ has a countable base and is uniformly stable under small enlargements: for some $\delta > 0$ and each $A \in B$, $B_d(A, \delta) \in B$.

Remark. Given a countable family of metric spaces $\{\langle X_n, d_n \rangle : n \in \mathbb{N}\}$, their product equipped with the product topology is metrizable. While the box bornology in general fails to have a countable base, the product bornology generated by $\{\pi_n^{-1}(B_n) : B_n \text{ is } d_n \text{ - bounded}\}$ is a metric bornology which respects the bornologies of the factors [6].

In finite dimensional Euclidean space, bounded sets are totally bounded. So one may ask: when is a metric bornology a bornology of totally bounded sets with respect to a remetrization? The following results were obtain by Beer, Levi and Costantini [8].

**Theorem 2.4.** Let $\langle X, d \rangle$ be a metric space. The following conditions are equivalent:

(1) There exists an equivalent metric $\rho$ such that $\mathcal{B}_d(X) = \mathcal{T} \mathcal{B}_\rho(X)$;

(2) $\langle X, d \rangle$ is separable;

(3) There exists an equivalent metric $\rho$ with $\mathcal{B}_d(X) = \mathcal{T} \mathcal{B}_\rho(X) = \mathcal{B}_\rho(X)$.

For a general result that describes when a bornology on a metrizable space is a bornology of totally bounded subsets, we give this result.

**Theorem 2.5.** Let $X$ be a metrizable space and let $\mathcal{B}$ be a bornology on $X$. The following conditions are equivalent:
(1) $\mathcal{B} = \mathcal{T}\mathcal{B}_d(X)$ for some compatible metric $d$;

(2) there exists an embedding $\psi$ of $X$ into a completely metrizable space $Y$ with the following property:

$$\mathcal{B} = \{ E \in \mathcal{P}_0(X) : \psi(E) \text{ is relatively compact in } Y \};$$

(3) there is a star-development $\langle \mathcal{U}_n \rangle$ for $X$ such that

$$\mathcal{B} = \{ E \in \mathcal{P}_0(X) : \forall n \in \mathbb{N}, \mathcal{U}_n \text{ admits a finite subcover of } E \}.$$

3. Bornological convergence of nets of closed sets

We denote the closed subsets of $\langle X, d \rangle$ by $\mathcal{C}(X)$. We begin with the definition of bornological convergence of a net of closed sets with respect to a prescribed bornology.

**Definition 3.1.** Let $\mathcal{B}$ be a bornology on a metric space $\langle X, d \rangle$. A net $\langle C_\lambda \rangle$ of closed subsets of $X$ is declared ($\mathcal{B}, d$)-convergent to $C \in \mathcal{C}(X)$ provided whenever $B \in \mathcal{B}$ and $\varepsilon > 0$, then eventually both

$$C \cap B \subseteq B_d(C_\lambda, \varepsilon) \text{ and } C_\lambda \cap B \subseteq B_d(C, \varepsilon).$$

When this occurs, we write $C = (\mathcal{B}, d) - \lim C_\lambda$

Remark: $\mathcal{B}$ can be replaced by any base for $\mathcal{B}$ without changing the convergence.

Remark: Following the seminal paper of Lechicki, Levi, and Spakowski on the subject [20], specialists consider bornological convergence more generally in three senses: (1) general nets of subsets are considered; (2) the bornology is replaced by an ideal of subsets; (3) the convergence is split into upper and lower halves.

**Example 3.2.** When $\mathcal{B} = \mathcal{B}_d(X)$ we get *Attouch-Wets convergence* [1, 2, 14], also known as *bounded Hausdorff convergence* [21]. In the context of
normed linear spaces, this convergence is stable with respect to duality [2] and operator norm convergence of a sequence of continuous linear transformations means AW-convergence of the associated sequence of graphs [21].

**Example 3.3.** When $\mathcal{B} = \mathcal{P}_0(X)$ we get convergence in the *Hausdorff metric topology* because $\{X\}$ is a base for the bornology.

**Example 3.4.** When $\mathcal{B} = \mathcal{K}_0(X)$ we get convergence in the *Fell topology* [2] having as a subbase all families of the form

$$\left\{ A \in \mathcal{C}(X) : A \cap V \neq \emptyset \right\} \quad (V \text{ an open subset of } X),$$

and

$$\left\{ A \in \mathcal{C}(X) : A \cap K = \emptyset \right\} \quad (K \text{ a nonempty compact subset of } X).$$

**Facts:**

- Bornological convergence is in general admissible;
- Bornological limits are unique in $\mathcal{C}(X)$ if and only if the bornology is local; [20]
- Given two bornologies $\mathcal{A}$ and $\mathcal{B}$ on a metrizable space $X$ with two compatible metrics $d$ and $\rho$ for the topology of $X$, then $(\mathcal{A}, \rho)$-convergence ensures $(\mathcal{B}, d)$-convergence if and only if $\mathcal{B} \subseteq \mathcal{A}$ and $\forall B \in \mathcal{B}$ and $\varepsilon > 0$, $\exists \delta > 0$ such that $B_\rho(B, \delta) \subseteq B_d(B, \varepsilon)$ [11].
- $(\mathcal{B}, d)$-convergence is topological on $\mathcal{C}(X)$ if and only if $\mathcal{B}$ is shielded from closed sets.
Given a bornology $\mathcal{B}$ on $\langle X, d \rangle$, there is a natural "pre-uniform" structure associated with $(\mathcal{B}, d)$-convergence, consisting of supersets of all "basic entourages" in $\mathcal{C}(X) \times \mathcal{C}(X)$ of the form

$$[B, \varepsilon] := \{(A, C) : A \cap B \subseteq B_d(C, \varepsilon) \text{ and } C \cap B \subseteq B_d(A, \varepsilon)\},$$

where $B \in \mathcal{B}$ and $\varepsilon > 0$.

- All sets of the form $[B, \varepsilon]$ form a base for a uniformity compatible with $(\mathcal{B}, d)$-convergence if and only if $\mathcal{B}$ is stable under small enlargements [11, 20];

- $(\mathcal{B}, d)$-convergence on $\mathcal{C}(X)$ is metrizable if and only if $\mathcal{B}$ is stable under small enlargements and has a countable base; in this case bornological convergence is actually Attouch-Wets with respect to an equivalent metric [11].

Remark: Attouch-Wets convergence of nonempty subsets means uniform convergence of distance functions on bounded subsets of $X$. While this fails more generally, necessary and sufficient conditions for equality have been identified by Beer, Naimpally and Rodriguez-Lopez [13].

Remark: Uniform convergence of a sequence of linear transformations between normed spaces on members of a bornology for the domain space $X$ can expressed in terms of bornological convergence of graphs provided the bornology has a base of norm bounded sets each starshaped with respect to the origin $0_X$ [4].

4. APPROXIMATION

By definition, a nonempty subset $E$ of a metric space $\langle X, d \rangle$ is $d$-totally bounded provided $\forall \varepsilon > 0$, $\exists F \in \mathcal{F}_0(X)$ with $E \subseteq B_d(F, \varepsilon)$. 
As is well-known, we can choose the finite set $F$ such that

$$F \subseteq E \subseteq B_d(F, \varepsilon).$$

**Definition 4.1.** Let $\mathcal{A}$ be a family of nonempty subsets of $X$. We call $E$ **weakly $\mathcal{A}$-totally bounded** if $\forall \varepsilon > 0$, $\exists A \in \mathcal{A}$ with

$$E \subseteq B_d(A, \varepsilon),$$

and **$\mathcal{A}$-totally bounded** if for each $\varepsilon > 0$, the approximating set from $\mathcal{A}$ can be chosen inside $E$.

We denote the weakly totally bounded subsets determined by $\mathcal{A}$ by $\mathcal{A}^*$ and the totally bounded subsets by $\mathcal{A}_*$. Evidently, $\mathcal{A}_* \subseteq \mathcal{A}^*$ and the operators $\mathcal{A} \mapsto \mathcal{A}^*$ and $\mathcal{A} \mapsto \mathcal{A}_*$ are idempotent and monotone.

**Proposition 4.2.** Let $\mathcal{B}$ be a bornology; then

1. $\mathcal{B}^*$ is the closure of $\mathcal{B}$ with respect to the Hausdorff pseudometric.
2. $\mathcal{B}^*$ is a bornology;
3. $\mathcal{B}_*$ contains each finite set and is stable under finite unions;
4. $\mathcal{B}^* = \{ E \in \mathcal{P}_0(X) : \exists W \in \mathcal{B}_* \text{ with } E \subseteq W \}$.

From the last proposition it easily follows $\mathcal{B}^* = \mathcal{B}_*$ if and only $\mathcal{B}_*$ is hereditary, i.e., is a bornology [10].

We give an example from [20] showing that $\mathcal{B}_*$ need not be hereditary.

**Example 4.3.** On $\mathbb{R}^2$ let $\mathcal{B}$ be bornology generated by the family of vertical lines in the plane. While $[0, 1] \times (-\infty, \infty)$ is $\mathcal{B}$-totally bounded its subset $\{(x, 1/x) : 0 < x \leq 1\}$ is not.
Our last result also from [10] answers the question: which families of subsets are the totally bounded subsets determined by a bornology?

**Theorem 4.4.** Let $\mathcal{A}$ be a family of nonempty subsets of $\langle X, d \rangle$. Then $\mathcal{A} = \mathcal{B}_*$ for some bornology $\mathcal{B}$ if and only if $\mathcal{A}$ contains the finite subsets, is stable under finite unions and

$$\mathcal{A} = \{ A \in \mathcal{A} : \mathcal{P}_0(A) \subseteq \mathcal{A} \}^*.$$ 

**References**


