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Binary symmetry groups of polytopes as quotients of braid groups

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The symmetry group of a regular n-polytope is a finite Coxeter group of rank n. Let $\mathfrak{g} \subset SO(n)$ be the normal subgroup of orientation-preserving symmetries of the polytope. A natural object to study is the group G of homotopy classes of paths $R : [0, 1] \to SO(n)$ with $R(0) = Id$ and $R(1) \in \mathfrak{g}$. This group is a finite analog of the topological universal covering space of $SO(n)$. As a group we expect it to be a nontrivial central extension by $\mathbb{Z}_2$ of $\mathfrak{g}$. G can be generated by a finite set of simple "generating paths" R_i. Words giving (explicitly) contractible closed paths must be set to the identity and this provides a set of relations among the generators. After simplifications it turns out that in a number of cases the generators R_i satisfy Artin's relations for the braid group plus one additional relation. Thus, for example the braid group B_n factorized by the normal closure of the element $R_1 R_2^2 R_1 R_2^{-2}$ is the so-called binary hyperoctahedral group – the group of rotational symmetries of the hyperoctahedron (also called cross-polytope) in dimension n. This provides a uniform way to present all double covers of the orientation-preserving subgroups of the Coxeter B-series.

The binary icosahedral group is obtained as B_3/N with N being the normal closure of $R_1R_2^2R_1R_2^{-3}$. (The full icosahedral group is the exceptional Coxeter group H_3). The binary tetrahedral group can be obtained as B_3/N with N the normal closure of $R_1R_2^2R_1R_2^{-1}$. The double covers of the Coxeter groups of the A-series (which are the symmetric groups S_n) are obtained by imposing on B_n the additional relation $R_1R_2^2R_1 = Id$.

Some finite factors of braid groups appear in classical works by Coxeter and Moser ([1, 2]). In particular the presentation for the binary octahedral group ($n = 3$) appears in [1]. In [2] the author studies the groups obtained by imposing the additional condition $R_1^n = Id$, $n > 2$. However it seems that the connection between braid groups and Coxeter groups has not been explored thoroughly and our work is a step in that direction.

As a bi-product, constructing explicitly the binary hyperoctahedral group G and studying it's subgroup of homotopy classes of closed paths we have been able to provide a (new) proof (using just vector analysis and algebra) of the well-known fact that $\pi_1(SO(n)) = \mathbb{Z}_2$ for all $n \geq 3$ [3].

References

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