UNIVERSAL TENSOR CATEGORIES AND "ALGEBRAIC FUNCTIONAL ANALYSIS"

JOINT WORK WITH A. CHIRVASITU

BACKGROUND: V and V_* COUNTABLE-DIMENSIONAL VECTOR SPACES WITH NON-DEGENERATE PAIRING

$$V \otimes V_* \longrightarrow \mathbb{C}$$

G. MACKEY: $\exists$ A BASE $v_1, v_2, \ldots$ OF V SUCH THAT

$$\operatorname{span}\{v_1^*, v_2^*, \ldots\} = V_* \subset V^*$$

Lie ALGEBRA $\mathfrak{gl}(\infty) = V_* \otimes V$, V AND V_* are $\mathfrak{g} = \mathfrak{gl}(\infty)$-MODULES. Also $\mathfrak{sl}(\infty) \subset \mathfrak{gl}(\infty)$

CATEGORY $\mathbb{T}_{\mathfrak{sl}(\infty)}$

- SUBQUOTIENTS OF $V^{\otimes m} \otimes (V_*)^{\otimes n}$ FOR SOME m, n
- ABSOLUTE WEIGHT MODULES OF FINITE LENGTH
- WEIGHT MODULES INVARIANT UNDER

$$G = \{\varphi \in \operatorname{Aut} V \mid \varphi^*|_{V_*} \text{ is an automorph. of } V_*\}$$

AND HAVING FINITE LENGTH.

$\mathbb{T}_{\mathfrak{sl}(\infty)}$ is AN ANALOG OF THE CATEGORY OF FIN. DIM. $\mathfrak{gl}(n)$ or $\mathfrak{sl}(n)$-MODULES.

$\mathbb{T}_{\mathfrak{sl}(\infty)}$ IS NOT SEMISIMPLE

$$0 \to \mathfrak{sl}(\infty) \hookrightarrow V \otimes V_* \xrightarrow{tr} \mathbb{C} \to 0$$

SEQUENCE DOES NOT SPLIT

SIMPLE MODULES $\quad V_{\lambda,\mu} = \mathrm{soc}(V_\lambda \otimes (V_*)_\mu)$

$V_{\tiny\begin{array}{|c|}\hline \\ \hline \\ \hline\end{array}} = S^2(V)$

$V_{\tiny\begin{array}{|c|c|}\hline & \\ \hline\end{array}} = \Lambda^2(V)$

INJECTIVE HULLS : $V_\lambda \otimes (V_*)_\mu$ IS THE INJ. HULL OF $V_{\lambda,\mu}$.

UNIVERSALITY PROPERTY (SAM & SNOWDEN)
LET T be A SYMMETRIC (NON-RIGID) MONOIDAL LINEAR (OVER $\mathbb{C}$) ABELIAN CATEGORY WITH TWO OBJECTS X AND Y WITH A PAIRING $X \otimes Y \to \mathbb{1}$.
THEN THERE IS A UNIQUE (UP TO ISO) LEFT EXACT TENSOR FUNCTOR $\mathbb{T}_{sl(\infty)} \to T$ SENDING V TO X, V_* TO Y AND THE PAIRING $V \otimes V_* \to \mathbb{C}$ TO $X \otimes Y \to \mathbb{1}$,

$\boxed{\text{The TRIVIAL } sl(\infty)\text{-MODULE IS INJECTIVE IN } \mathbb{T}_{sl(\infty)}.}$

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THE LIE ALGEBRA $gl^M(V, V_*)$

$gl^M(V,V_*) = \{ \varphi \in \mathrm{End}\, V \mid \varphi^*_{|V_*}(V_*) \subset V_* \}$

"LIE ALGEBRA" OF G

$\mathbb{T}_{sl(\infty)} \simeq \mathbb{T}_{gl^M(V)}$ — SUBQUOTIENTS OF $V^{\otimes m} \otimes (V_*)^{\otimes n}$

FACT! V^* AND $(V_*)^*$ HAVE LENGTH 2 OVER $gl^M(V)$:

$$0 \to V_* \to V^* \to \overset{\text{simple}}{V^*/V_*} \to 0$$

CONSIDER NOW TENSOR PRODUCTS $(V^*)^{\otimes m} \otimes ((V_*)^*)^{\otimes n}$

THEY ALSO HAVE FINITE LENGTH OVER $gl^M(V)$.

LET $\mathbb{T}'$ BE THE FINITE-LENGTH ABELIAN TENSOR CATEGORY OF SUBQUOTIENTS OF $(V^*)^{\otimes n} \otimes ((V_*)^*)^{\otimes m}$.

SIMPLE OBJECTS : $V_{\lambda,\mu,\nu,\varkappa} = (W_*)_\lambda \otimes V_{\mu,\nu} \otimes W_\varkappa$

WHERE $W_* = V^*/V_*$, $W = (V_*)^*/V$

"FAT" $(W_*)_\lambda \otimes W_\varkappa$ AND "THIN" $V_{\mu,\nu}$.

<u>NO</u> INJECTIVE HULLS IN $\mathbb{T}'$ (POINTED OUT BY SERGANOVA)

$$V^* \otimes V \xrightarrow{p} \mathbb{C} \qquad V_* \otimes (V_*)^* \xrightarrow{p_*} \mathbb{C}.$$

$$0 \to \mathbb{C} \to Q = V^* \otimes V_*^* / (\ker p + \ker p_*) \to \underset{\underset{F}{\shortparallel}}{W_* \otimes W} \to 0.$$

NON SPLITTING EXACT SEQ.

$$\begin{array}{ccc} S^k Q & \hookrightarrow & S^{k+1} Q \\ \shortparallel & & \uparrow \text{mult.} \\ (S^k Q) \otimes \mathbb{C} & \hookrightarrow & (S^k Q) \otimes Q \end{array}$$

$$0 \to S^k Q \hookrightarrow S^{k+1} Q \longrightarrow S^{k+1} F \longrightarrow 0.$$

$$I = \varinjlim S^k Q.$$

$\mathbb{T}$ – SMALLEST GROTHENDIECK TENSOR CATEGORY, (CLOSED UNDER SUBQUOTIENTS AND) CONTAINING

I AND $V_{\lambda,\mu,\nu,\varkappa}$

IN GENERAL, OBJECTS OF $\mathbb{T}$ HAVE INFINITE LENGTH.

$\mathbb{T}$ HAS ENOUGH INJECTIVES:

$$V_{\lambda,\mu,\nu,\varkappa} = \mathrm{soc}\big(\underbrace{I \otimes (W_*)_\lambda \otimes (V^*)_\mu \otimes (V_*)^*_\nu \otimes W_\varkappa}_{I_{\lambda,\mu,\nu,\varkappa}}\big)$$

AND $I_{\lambda,\mu,\nu,\varkappa}$ IS INJECTIVE IN $\mathbb{T}$.

IN ADDITION, $\mathbb{T}$ IS A KOSZUL CATEGORY.

WE OBSERVE NOW THAT I HAS THE STRUCTURE OF A COMMUTATIVE ALGEBRA:

$$I \simeq S^{\bullet}Q / (a - 1_{S^{\bullet}Q})$$

WHERE a IS THE DISTINGUISHED ELEMENT $1 \in \mathbb{C} \subset Q$.

THEREFORE IT MAKES SENSE TO CONSIDER THE CATEGORY OF I-SUBMODULES IN $\mathbb{T}$, WITH MORPHISMS BEING MORPHISMS OF $(\mathfrak{g}, I)$-MODULES.

DENOTE THIS CATEGORY BY $_I\mathbb{T}$

THE MONOIDAL UNIT OF ${}_I\mathbb{T}$ IS I AND WE HAVE A PAIRING

$$(*) \qquad (I \otimes V^*) \otimes_I (I \otimes (V_*)^*) = I \otimes V^* \otimes (V_*)^* \to I$$

SINCE WE HAVE A WELL DEFINED SURJECTIVE MORPHISM

$$V^* \otimes (V_*)^* \longrightarrow Q \subset I \text{ AND } I \otimes Q = I \oplus I \otimes W \otimes W_*$$

${}_I\mathbb{T}_{fin}$ — SUBCATEGORY OF FINITE-LENGTH OBJECTS IN ${}_I\mathbb{T}$

<u>MAIN THEOREM.</u> LET T BE A SYMMETRIC MONOIDAL LINEAR (OVER $\mathbb{C}$) ABELIAN CATEGORY WITH TWO OBJECTS X AND Y WITH FIXED SUBOBJECTS $X' \subset X$, $Y' \subset Y$, AND A PAIRING $X \otimes Y \to \mathbb{1}$. THEN THERE EXISTS A UNIQUE (UP TO ISOMORPHISM) LEFT EXACT MONOIDAL FUNCTOR ${}_I\mathbb{T}_{fin} \to \mathsf{T}$ SENDING $I \otimes V^*$ TO X, $I \otimes (V_*)^*$ TO Y, $I \otimes W_*$ TO THE QUOTIENT X/X', $I \otimes W$ TO THE QUOTIENT Y/Y', AND THE PAIRING $(*)$ TO THE PAIRING $X \otimes Y \to \mathbb{1}$.