Algebra and Logic Seminar
Institute of Mathematics and Informatics
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Non-finitely based and limit varieties of algebraic systems

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One of the main problems in algebra naively stated

In a recent talk we stated it in the following way:
Describe all groups, semigroups, rings and algebras over a field (both associative and nonassociative).

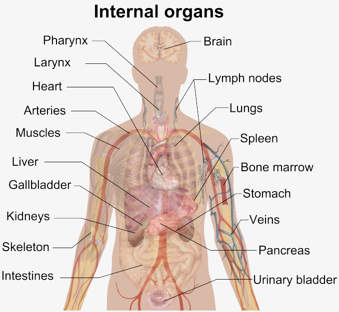
It seems that it is a hopeless problem to list all groups, semigroups, rings and algebras.

We need another approach!

Two possible approaches

- Structure approach;
- Combinatorial approach.

Figuratively speaking, structure theory is like anatomy. (The word "anatomy" comes from the Greek $\alpha\nu\alpha\tau o\mu\acute{\eta}$ which means "dissection". One studies the structure of the body, its organs and the interactions between the organs.)



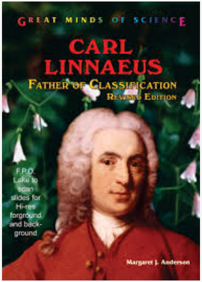
Example: Structure theory of finite-dimensional associative algebras

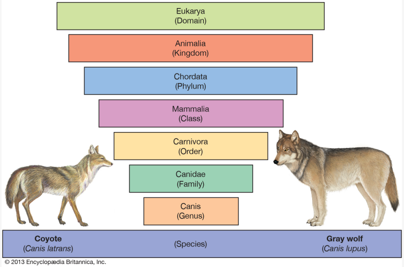
If R is an associative algebra over a field K and $\dim(R) < \infty$, then one defines an ideal $J(R)$ of R called the Jacobson radical such that:

- The factor algebra $R/J(R)$ is semisimple. Every semisimple algebra is a direct sum of simple algebras. The only finite-dimensional simple algebras are the matrix algebras with entries from a finite-dimensional division ring. (division ring = not necessarily commutative field)
- The radical is nilpotent, i.e. there is a positive integer such that $u_1 \cdots u_n = 0$ for all $u_1, \ldots, u_n$ in J which can be written as $J^n = 0$.
- Over a perfect field K the algebra R is a split extension of a semisimple subalgebra by the radical, i.e. $R \cong R/J(R) \oplus J(R)$.

Combinatorial approach to algebraic systems

Combinatorial ring theory is similar to the Organism classification of Carl Linnaeus. As the living creatures are divided in classes (e.g. reptilia, birds, mammals) and each class has subclasses





in the same way the algebraic objects are collected in classes called varieties depending on the identical relations which they satisfy.

Advantage of this approach

We can treat in the same way groups, semigroups, rings and algebras (commutative, associative, Lie, Jordan, etc.).

This idea comes from the works by Garrett Birkhoff on universal algebras and Bernhard Neumann on groups:

- G. Birkhoff, On the structure of abstract algebras, Proc. Cambridge Phil. Soc. 31 (1935), 433-454.
- B.H. Neumann, Identical relations in groups. I, Math. Ann. 114 (1937), 506-525.

Free objects and their universal property

Let $\mathfrak{V}$ be the class of all groups (or all semigroups, all associative rings, all rings, all algebras, etc.) and let X be an arbitrary nonempty set. Then there exists an algebraic system $F(X) \in \mathfrak{V}$ generated by X with the property that for any $R \in \mathfrak{V}$ any map $X \to R$ can be extended uniquely to a homomorphism $F(X) \to R$.
We call $F(X)$ the free algebraic system in $\mathfrak{V}$ freely generated by X.

In the sequel we assume that

$$X = \{x_1, x_2, \ldots\}, \quad X_d = \{x_1, \ldots, x_d\} \subset X,$$

and write $F(\mathfrak{V})$ and $F_d(\mathfrak{V})$ instead of $F(X)$ and $F(X_d)$ if we want to emphasize that we work in the class $\mathfrak{V}$.

Description of $F(\mathfrak{V})$

The free object $F(\mathfrak{V})$ has a similar description for groups, semigroups, associative and nonassociative rings and algebras.

- **The class $\mathfrak{S}$ of all semigroups:** Then $F(\mathfrak{S})$ consists of all nonempty words in the alphabet X:

$$F(\mathfrak{S}) = \langle X \rangle = \{x_{i_1} \cdots x_{i_n} \mid x_{i_k} \in X, n = 1, 2, \ldots\}$$

  with multiplication

$$(x_{i_1} \cdots x_{i_n})(x_{j_1} \cdots x_{j_m}) = x_{i_1} \cdots x_{i_n} x_{j_1} \cdots x_{j_m};$$

- **The calss $\mathfrak{M}$ of all monoids:** monoid = semigroup with 1.
- **The class $\mathfrak{A}$ of all commutative semigroups:**

$$F(\mathfrak{S}) = [X] = \{x_1^{a_1} \cdots x_d^{a_d} \mid a_i \geq 0, d = 1, 2, \ldots, a_1 + \cdots + a_d > 0\}$$

- **The class $\mathfrak{G}$ of all groups:**

$$F(\mathfrak{G}) = \{x_{i_1}^{\pm 1} \cdots x_{i_n}^{\pm 1} \mid x_{i_k} \in X, n = 0, 1, 2, \dots\}$$

  after reduction and deleting the subwords $x_i x_i^{-1}$ and $x_i^{-1} x_i$ if they appear in the word $x_{i_1}^{\pm 1} \cdots x_{i_n}^{\pm 1}$.

- **The class $\mathfrak{A}$ of all abelian groups:**

$$F(\mathfrak{A}) = \{x_1^{a_1} \cdots x_d^{a_d} \mid a_i \in \mathbb{Z}\}.$$

- **The class $\mathfrak{R}$ of all associative rings and algebras over a field K:** The free associative ring and algebra or the ring and algebra of polynomials in noncommuting variables (unitary or nonunitary):

$$F(\mathfrak{R}) = \mathbb{Z}\langle X\rangle = \{\sum \alpha_i x_{i_1}\cdots x_{i_n} \mid \alpha_i \in \mathbb{Z}\},$$

$$F(\mathfrak{R}) = K\langle X\rangle = \{\sum \alpha_i x_{i_1}\cdots x_{i_n} \mid \alpha_i \in K\}.$$

- **The class $\mathfrak{A}$ of all commutative-associative rings and algebras:** The polynomial ring $\mathbb{Z}[X]$ and the polynomial algebra $K[X]$.
- **The class of all nonassociative rings and algebras:** The basis of the free nonassociative ring $\mathbb{Z}\{X\}$ as a free $\mathbb{Z}$-module and of the free nonassociative algebra $K\{X\}$ as a K-vector space consists of all monomials in nonassociative variables (i.e. preserving the parentheses).

Identities and varieties of algebraic systems

We fix $\mathfrak{V}$ to be one of the classes of algebraic systems: semigroups, groups, associative or nonassociative rings and algebras over a field. Let $R \in \mathfrak{V}$ and $u(x_1, \ldots, x_n), v(x_1, \ldots, x_n) \in F(\mathfrak{V})$. Then $u = v$ is an identity for R if

$$u(r_1, \ldots, r_n) = v(r_1, \ldots, r_n) \text{ for all } r_1, \ldots, r_n \in R.$$

If the subclass $\mathfrak{W}$ of $\mathfrak{V}$ consists of all $R \in \mathfrak{V}$ which satisfy a given system of identities $\{u_i = v_i \mid i \in I\}$ it is called the variety defined by these identities.

The identity $u = v$ follows from the identities $\{u_i = v_i \mid i \in I\}$ if it holds for all algebraic systems in $\mathfrak{V}$ which satisfy the identities $u_i = v_i$, $i \in I$.

If the variety $\mathfrak{W}$ has the same identities as an algebra $R \in \mathfrak{V}$, then it is called generated by R. This definition agrees with the following theorem of Bikhoff.

HSP-Theorem (Birkhoff, 1935)

(i) The class $\mathfrak{W}$ of of algebraic systems from $\mathfrak{V}$ is a variety if and only if it is closed under homomorphic objects $(\mathcal{H})$, subobjects $(\mathcal{S})$, and cartesian products $(\mathcal{P})$.

(ii) The variety $\mathfrak{W} = \text{var}(R)$ is generated by R if and only if

$$\mathfrak{W} = \mathcal{HSP}(R).$$

Any set of defining identities of the variety $\mathfrak{W} \subset \mathfrak{V}$ is called a basis of the identities of $\mathfrak{W}$. The variety is finitely based if it can be defined by a finite number of identities. Otherwise it is infinitely (or non-finitely) based.

The finite basis (or Specht) problem

- Is every variety $\mathfrak{W} \subset \mathfrak{V}$ finitely based?
- **Weaker version:** If the variety satisfies an explicitly given identity, does it have a finite basis of identities?
- If non-finitely based varieties exist, give an explicit example.

In the sequel we shall discuss only examples of infinitely based varieties. We shall not consider the problem whether the infinite systems of identities are independent although there are many results in this direction. There are also many positive results showing that important varieties of algebraic systems are finitely based.

Infinitely based varieties of semigroups

First examples:

- A.P. Birjukov, On infinite aggregates of identities in semi-groups (Russian), Algebra Logika 4 (1965), No. 2, 31-32.

$$y(x_1\cdots x_n)z(yx_n\cdots x_1)yz=y(x_1\cdots x_n)z(x_n\cdots x_1)yz, n=1,2,\dots$$

- A.K. Austin, A closed set of laws which is not generated by a finite set of laws, Q. J. Math., Oxf. II. Ser. 17 (1966), 11-13.

$$((x_1\cdots x_n)y)^2=y(x_1\cdots x_n)y(x_n\cdots x_1), n=2,3,\dots$$

- More than 40 examples can be found in L.N. Shevrin, M.V. Volkov, Identities of semigroups (Russian), Izv. Vyssh. Uchebn. Zaved., Mat. (1985), No. 11(282), 3-47. Translation: Sov. Math. 29 (1985), No. 11, 1-64 (1985).

Infinitely based varieties of groups

The problem stated in 1937:

- B.H. Neumann, Identical relations in groups. I, Math. Ann. 114 (1937), 506-525.

First results in 1970:

- A.Yu. Olshanskij, On the problem of a finite basis of identities in groups (Russian), Izv. Akad. Nauk SSSR, Ser. Mat. 34 (1970), 376-384. Translation: Math. USSR, Izv. 4 (1970), 381-389.
  There are continuously many varieties of groups and hence there are infinitely based varieties.

First examples:

- S.I. Adyan, Infinite irreducible systems of group identities (Russian), Izv. Akad. Nauk SSSR, Ser. Mat. 34 (1970), 715-734. Translation: Math. USSR, Izv. 4 (1970), 721-739.
  For every odd $n \geq 4381$ the group identities
  $$u_p = (x^{pn}, y^{pn})^n = 1, \quad p \text{ prime},$$
  are independent (i.e., u_p does not follow from the other identities). Here $(u,v) = u^{-1}v^{-1}uv$ is the group commutator.
- The proof uses methods from the solution of the Burnside problem which can be stated in the language of varieties: Is the variety of groups defined by the identity $x^k = 1$ locally finite?
  P.S. Novikov, S.I. Adyan, Infinite periodic groups. I, II, III (Russian), Izv. Akad. Nauk SSSR, Ser. Mat. 32 (1968), 212-244, 251-524, 709-731. Translation: Math. USSR, Izv. 2 (1968), 209-236, 241-480, 665-685.

- M.R. Vaughan-Lee, Uncountably many varieties of groups, Bull. Lond. Math. Soc. 2 (1970), 280-286.
  Let

$$w_k = ((x, y, z), (x_1, x_2), (x_3, x_4), \dots, (x_{2k-1}, x_{2k}), (x, y, z)),$$

$k = 1, 2, \dots$. Then the identity $w_k = 1$ does not follow from the identities

$$x^{16} = ((x_1, x_2, x_3), (x_4, x_5, x_6), (x_7, x_8)) = 1, w_n = 1, n \neq k.$$

Here the group commutators are left normed:

$$(u_1, \dots, u_{n-1}, u_n) = ((u_1, \dots, u_{n-1}), u_n).$$

The simplest examples:

- Yu.G. Kleiman, On the basis of a product variety of groups. I. (Russian), Izv. Akad. Nauk SSSR, Ser. Mat. 37 (1973), 95-97. Translation: Math. USSR, Izv. 7(1973), 91-94.
- R.M. Bryant, Some infinitely based varieties of groups, J. Aust. Math. Soc. 16 (1973), 29-32.

The variety $\mathfrak{B}_4\mathfrak{A}_2$ is not finitely based. The group G belongs to $\mathfrak{B}_4\mathfrak{A}_2$ if it has a normal subgroup H which is abelian of exponent 2 and the factor group G/H is of exponent 4. The variety has a basis of identities

$$(x_1^2 \cdots x_n^2)^4 = 1, \quad n = 1, 2, \ldots .$$

Infinitely based varieties of nonassociative objects. Easy examples:

The varieties

- of binary magmas (with one nonassociative binary operation, i.e. nonassociative semigroups);
- of nonassociative rings;
- of nonassociative algebras over any field K

defined by the identities

$$u_n = (x_1x_2)(x_3x_4)x_5\cdots x_{n-2}(x_{n-1}x_n) = 0, \quad n = 6, 7, \ldots,$$

are not finitely based.
(The parentheses are left normed, e.g., $xyz = (xy)z$.)

Infinitely based varieties of Lie algebras, char$(K) = 2$:

- M.R. Vaughan-Lee, Varieties of Lie algebras, Q. J. Math., Oxf. II. Ser. 21 (1970), 297-308.

The variety of Lie algebras over a filed of characteristic 2 defined by the identities

$$[[x_1, x_2], [x_3, x_4], x_5] = 0,$$

$$[[x_1, x_2, x_3, \dots, x_n], [x_1, x_2]] = 0, \quad n = 3, 4, \dots,$$

is not finitely based.
(The Lie commutators are left normed, e.g. $[u, v, w] = [[u, v], w]$.)

- I.B. Volichenko, On varieties of center-by-metabelian Lie algebras, Preprint No. 16 (96), Institute of Mathematics, National Academy of Sciences of Belarus, Minsk, 1980.

Volicehnko added to the previous system of identities one more identity to obtain an infinitely based locally finite variety.
(Locally finite = every finitely generated algebra is finite-dimensional.)

If the field K of characteristic 2 is infinite, then the Lie algebra $M_2(K)^{(-)}$ of 2×2 matrices (with operation $[u,v] = uv - vu$) has no finite basis of its polynomial identities.

- M.R. Vaughan-Lee (1970) (partial information on the identities of $M_2(K)^{(-)}$, $\mathrm{char}(K) = 2$, $|K| = \infty$);
- V. Drensky, Solvable Varieties of Lie Algebras, Ph.D. Thesis, Moscow State University, 1979;
- A. Lopatin, Identities for the Lie algebra gl_2 over an infinite field of characteristic two, arXiv:1612.07748v1 [math.RA].

The Lie algebra $M_2(K)^{(-)}$, $\mathrm{char}(K) = 2$, $|K| = \infty$, has the following infinite basis of polynomial identities:

$$[[x_1, x_2], [x_3, x_4], x_5] = 0,$$

$$[[x_1, x_2, x_3, \ldots, x_n], [x_1, x_2]] = 0, \quad n = 3, 4, \ldots,$$

$$[[x_1, x_2, x_5, \ldots, x_n], [x_3, x_4]] + [[x_1, x_3, x_5, \ldots, x_n], [x_4, x_2]]$$
$$+[[x_1, x_4, x_5, \ldots, x_n], [x_2, x_3]] = 0, \quad n = 4, 5, \ldots.$$

- M.R. Vaughan-Lee, Abelian-by-nilpotent varieties of Lie algebras, J. Lond. Math. Soc., II. Ser. 11 (1975), 263-266.

If $\mathrm{char}(K) = 2$, then the variety of Lie algebras defined by the identities

$$[[x_1, x_2, x_3], [x_4, x_5, x_6]] = 0,$$

$$[x_{n+1}, x_{n+2}, x_{n+3}, [x_1, x_2], [x_2, x_3], \ldots, [x_{n-1}, x_n], [x_n, x_1]] = 0,$$

$n = 2, 3, \ldots$, is not finitely based.

Problem

There is a simple tree-dimensional Lie algebra $(K^3, \times)$ with the vector product. If $\mathrm{char}(K) = 2$, $|K| = \infty$, is the basis of its polynomial identities finite?

Infinitely based varieties of Lie algebras, char$(K) = p > 2$:

- V.S. Drensky, On identities in Lie algebras. (Russian), Algebra Logika 13 (1974), 265-290. Translation: Algebra Logic 13 (1974), 150-165.
- Yu.G. Kleiman (unpublished):

If char$(K) = p > 0$, then the variety defined by

$$[[[x_1, x_2], [x_3, x_4]], [[x_5, x_6], [x_7, x_8]]] = 0,$$

$$[[x_1, x_2], \ldots, [x_{2p-1}, x_{2p}], x_{2p+1}] = 0,$$

$$[x_1, x_2, x_3, \ldots, x_n](\mathrm{ad}[x_1, x_2])^{p-1} = 0, \quad n = 3, 4, \ldots,$$

is not finitely based. (Here $u(\mathrm{ad}v) = [u, v]$.)
(Obtained independently by the speaker and Kleiman.)

- V.S. Drensky, On varieties of Lie algebras with infinite basis property (Russian), Serdica 9 (1983), 79-82.

If we add to the previous system of identities the identity

$$x_1(\mathrm{ad}x_2)^{p^2+2} = 0$$

we shall obtain an infinitely based locally finite variety.
(Locally finite $=$ every finitely generated algebra is finite-dimensional.)

- V.S. Drensky, On identities in Lie algebras. (Russian), Algebra Logika 13 (1974), 265-290. Translation: Algebra Logic 13 (1974), 150-165.

If $\mathrm{char}(K) = p > 0$ and $|K| = \infty$, then there exists a finite-dimensional Lie algebra over K with infinite basis of its polynomial identities.

Open problem

Are the varieties of Lie algebras over a field of characteristic 0 finitely based?

The Specht problem for associative algebras

- W. Specht, Gesetze in Ringen. I, Math. Z. 52 (1950), 557-589.

Translated in the modern language Specht asked the problem whether it is possible to define any variety of unitary associative algebras over a field of characteristic 0 by a finite number of polynomial identities.

The influence of the paper by Specht and the activity trying to solve the problem were so strong that now a variety of any algebraic systems is called Spechtian if it and all its subvarieties can be defined by a finite system of identical relations.

The results of Kemer in characteristic 0

In the 1980's Kemer developed structure theory of T-ideals (= ideals of polynomial identities) which is similar to the theory of ideals in polynomial algebras. See for an account:

- A.R. Kemer, Ideals of Identities of Associative Algebras, Translations of Math. Monographs 87, AMS, Providence, RI, 1991.

We shall mention two consequences only.

- The polynomial identities of every finitely generated PI-algebra are the same as the polynomial identities of some finite-dimensional algebra. For PI-algebras which are not finitely generated there is an analogue stated in terms of finite-dimensional superalgebras.
- The affirmative answer to the Specht problem: The polynomial identities of every PI-algebra follow from a finite number of identities.
- A.R. Kemer, Finite basis property of identities of associative algebras (Russian), Algebra Logika 26 (1987), No. 5, 597-641. Translation: Algebra Logic 26 (1987), No. 5, 362-397.

Infinitely based varieties of associative algebras, char$(K) > 0$:

The first examples of infinitely based varieties of associative algebras in positive characteristic were presented by Belov in 1999 and almost in the same time by Grishin and Shchigolev.

- A.Ya. Belov, On non-Spechtian varieties. (Russian), Fundam. Prikl. Mat. 5 (1999), No. 1, 47-66.
- A.V. Grishin, Examples of T-spaces and T-ideals over a field of characteristic 2 without the finite basis property (Russian), Fundam. Prikl. Mat. 5 (1999), No. 1, 101-118.
- V.V. Shchigolev, Examples of infinitely based T-ideals (Russian), Fundam. Prikl. Mat. 5 (1999), No. 1, 307-312.

See also

- A.Ya. Belov, Counterexamples to the Specht problem (Russian), Mat. Sb. 191 (2000), No. 3, 13-24. Translation: Sb. Math. 191 (2000), No. 3, 329-340 (2000).
- A.V. Grishin, The variety of associative rings, which satisfy the identity $x^{32} = 0$, is not Specht, in D. Krob, Daniel et al. (eds), Formal Power Series and Algebraic Combinatorics. Proceedings of the 12th international conference, FPSAC'00, Moscow, Russia, June 26-30, 2000. Berlin: Springer, 2000, 686-691.

The simplest examples in characteristic 2

- C.K. Gupta, A.N. Krasilnikov, A non-finitely based system of polynomial identities which contains the identity $x^6 = 0$, Q. J. Math. 53 (2002), No. 2, 173-183.

$$x^6 = 0, w_n = u_1 x_1^2 \cdots x_n^2 u_2 u_1 u_2 = 0, u_i = [[y_i, z_i], t_i], n \geq 0.$$

- C.K. Gupta, A.N. Krasilnikov, A simple example of a non-finitely based system of polynomial identities, Commun. Algebra 30 (2002), No. 10, 4851-4866.

$$[x, y^2] x_1^2 \cdots x_n^2 [x, y^2]^3 = 0, n \geq 0.$$

Conjecture

Over an infinite field K of characteristic 2 the associative algebra $M_2(K)$ of 2×2 matrices does not have a finite basis of its polynomial identities.

Identities in finite objects – positive results

If R is a finite group (Oates and Powell), a finite associative (Kruse and Lvov), Lie (Bahturin and Olshanskij) or Jordan (Medvedev) ring or algebra, then it has a finite basis for its identities.

- S. Oates, M.B. Powell, Identical relations in finite groups, J. Algebra 1 (1964), 11-39.
- R.L. Kruse, Identities satisfied by a finite ring, J. Algebra 26, 298-318 (1973).
- I.V. L'vov, Varieties of associative rings. I (Russian), Algebra Logika 12 (1973), 269-297. Translation: Algebra Logic 12(1973), 150-167.
- Yu.A. Bakhturin, A.Yu. Olshanskij, Identical relations in finite Lie rings (Russian), Mat. Sb., N. Ser. 96(138) (1975), 543-559. Translation: Math. USSR, Sb. 25(1975), 507-523.
- Yu.A. Medvedev, Identities of finite Jordan Φ-algebras.(Russian), Algebra Logika 18 (1979), 723-748. Translation: Algebra Logic 18 (1979), 460-478.

Idea of the proofs

Structure theory of groups and rings. (For Lie rings the proof involves also quasi-identities.) Every variety in consideration generated by a finite object contains only a finite number of critical objects. It has a finite number of subvarieties only, all generated by some of the critical objects in the variety.

Problem

- G. Birkhoff, Lattice Theory, Rev. ed. American Mathematical Society Colloquium Publications. 25. New York: AMS, 1948.

Does every finite algebraic system with a finite system of operations possess a finite set of identities from which all others are derivable?

Answer into affirmative for systems with two elements

It is perhaps surprising that this problem is not entirely trivial even for algebras that contain only two elements. (Comment by Lyndon.)

- R.C. Lyndon, Identities in two-valued calculi, Trans. Am. Math. Soc. 71 (1951), 457-465 (1951).

Finite systems with infinitely based identities

- R.C. Lyndon, Identities in finite algebras, Proc. Am. Math. Soc. 5 (1954), 8-9.

The algebraic system M consists of 7 elements $M = \{0, e, b_1, b_2, c, d_1, d_2\}$. It has a zero and one binary operation. The only nonzero products are

$$ce = c, cb_j = d_j, d_je = d_j, d_jb_k = d_j, \quad j, k = 1, 2.$$

Identities (with left normed notation, e.g. $xyz = (xy)z$):

$$0x = 0, x0 = 0, x(yz) = 0,$$

$$x_1x_2\cdots x_nx_1 = 0, \quad n = 1, 2, \ldots,$$

$$x_1x_2\cdots x_nx_2 = x_1x_2\cdots x_n, \quad n = 1, 2, \ldots.$$

Problem

What is the minimal number of elements of a finite algebraic system M without finite basis of identities?

Answer: $|M| = 3$ or 4

- V.V. Vishin, Identical transformations in four-place logic (Russian), Dokl. Akad. Nauk SSSR 150 (1963), 719-721. Translation: Sov. Math., Dokl. 4 (1963), 724-726.

The following 4-element binary magma is not finitely based:

$$M = \{0, 1, 2, 3\}$$

and the only nonzero products are

$$1 * 2 = 1, 3 * 3 = 3, 3 * 2 = 1.$$

It satisfies the identities

$$x0 = 0x = x_1(x_2(x_3x_4)) = 0$$

and the left normed identities

$$x_1x_2x_2 = x_1x_2,$$

$$x_1x_2x_3x_4 \cdots x_nx_2 = x_1x_3x_2x_4 \cdots x_nx_3, n \geq 3,$$

$$x_1 \cdots x_kx_{k+1} \cdots x_nx_2 = x_1 \cdots x_{k+1}x_k \cdots x_nx_2, k = 4, \ldots, n-1, n \geq 5,$$

$$x_1x_2 \cdots x_nx_1 = 0, n \geq 2.$$

These left normed identities form a basis of the left normed part of the identities of M.

Final answer: $|M| = 3$

- V.L. Murskii, The existence in three-valued logic of a closed class with finite basis, not having a finite complete system of identities (Russian), Dokl. Akad. Nauk SSSR 163 (1965), 815-818. Translation: Sov. Math., Dokl. 6 (1965), 1020-1024.

The following 3-element binary magma is not finitely based:

$\times$	0	1	2
0	0	0	0
1	0	0	1
2	0	2	2

Idea of the proof

For $n \geq 3$ the magma M satisfies the identity

$$x_1(x_2(x_3 \cdots (x_{n-1}(x_n x_1)) \cdots)) = (x_1 x_2)(x_n(x_{n-1} \cdots (x_4(x_3 x_2)) \cdots))$$

which does not follow from any system of identities in less than n variables.

Relations with logic – problem

Is there an algorithm which, given an arbitrary finite algebra R as input, determines whether the variety $\mathrm{var}(R)$ is finitely based?

- A. Tarski, Equational logic and equational theories of algebras, Contrib. Math. Logic, Proc. Logic Colloq., Hannover 1966, 275-288, 1968.

  Tarski attributed the problem to Perkins:

- P. Perkins, Decision Problems for Equational Theories of Semigroups and General Algebras, Ph.D. Thesis, University of California, Berkeley, 1966.

Negative answer

- R. McKenzie, Tarskis finite basis problem is undecidable, Int. J. Algebra Comput. 6 (1996), No. 1, 49-104.

The paper contains a construction which produces, for every Turing machine $\mathcal{T}$, an algebra $F(\mathcal{T})$ (finite and of finite type) such that the Turing machine halts if and only if the algebra has a finite basis for its identities.
For more results related with logic see also

- P. Perkins, Unsolvable problems for equational theories, Notre Dame J. Formal Logic 8 (1967), 175-185.

To the best of my knowledge the problem is still open for finite semigroups.

Finite semigroups with infinite bases of identities

- P. Perkins, Bases for equational theories of semigroups, J. Algebra 11 (1969), 298-314.

The following 6-element semigroup of 2×2 matrices is not finitely based:

$$\left\{\begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix}, \begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}, \begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}, \begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}, \begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}, \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\right\}.$$

Semigroups with 6 elements

Up to isomorphism or anti-isomorphism, there exist 15973 distinct semigroups with 6 elements (and 1373 of them are monoids). Four of them (2 of them are monoids) are not finitely based. Each of the remaining 15969 semigroups with 6 elements has finitely based identities. (Lee and Li for monoids; Lee and Zhang for the other semigroups):

- E.W.H. Lee, J.R. Li, Minimal non-finitely based monoids, Diss. Math. 475 (2011), 65 p.
- E.W.H. Lee, W.T. Zhang, Finite basis problem for semigroups of order six, LMS J. Comput. Math. 18 (2015), 1-129.

Semigroups with ≤ 5 elements

In his paper in the Proc. Logic Colloq., Hannover 1966 Tarksi raised explicitly the finite basis problem for semigroups of order five or less which attracted the interest of several mathematicians (Bol'bot, Edmunds, Karnofsky, Tishchenko, Trahtman) A solution to this problem was eventually completed by Trahtman:

Theorem. Every semigroup with ≤ 5 elements has a finite basis of its identities.

- A. N. Trahtman, Finiteness of identity bases of five-element semigroups (Russian), in: Semigroups and Their Homomorphisms, E.S. Lyapin (ed.), Ross. Gos. Ped. Univ., Leningrad, 1991, 76-97.

If a variety of algebraic systems satisfies an infinite system of independent identities (i.e. removing one of the identities we obtain a larger variety) then it has a continuum of subvarieties. (For the proof, apply the Cantor diagonal argument.)

There are examples of six-element semigroups which generate a variety with a continuum of subvarieties. The semigroup may have an infinite basis of identities (Trahtman) or may be finitely based (Edmunds, Lee, Lee).

- A.N. Trahtman, A six-element semigroup that generates a variety with a continuum of subvarieties (Russian), Ural. Gos. Univ. Mat. Zap. 14(3) (1988), 138143.
- C.C. Edmunds, E.W.H. Lee, K.W.K. Lee, Small semigroups generating varieties with continuum many subvarieties, Order 27 (2010), No. 1, 83-100.

All semigroups with ≤ 5 elements generate varieties with finite or countably many subvarieties with one exception. We do not know whether this holds for the semigroup with multiplication given in the table

$\times$	a	b	c	d	e
a	a	a	a	a	a
b	a	a	a	b	c
c	c	c	c	c	c
d	a	b	c	d	e
e	e	e	e	e	e

(see the paper by Edmunds, Lee, Lee in the previous slide.)

For more information concerning results on identities in semigroups before 1985 see Shevrin and Volkov and for more recent results Jackson and Lee:

- L.N. Shevrin, M.V. Volkov, Identities of semigroups (Russian), Izv. Vyssh. Uchebn. Zaved., Mat. (1985), No. 11(282), 3-47. Translation: Sov. Math. 29 (1985), No. 11, 1-64.
- M. Jackson, E.W.H. Lee, Monoid varieties with extreme properties, Trans. Am. Math. Soc. 370 (2018), No. 7, 4785-4812.

Two recent exotic examples

The following 5-element semigroups generate Spechtian varieties of semigroups but non-Specht varieties of involution semigroups.

- E.W.H. Lee, Non-Specht variety generated by an involution semigroup of order five, Tr. Mosk. Mat. O.-va 81 (2020), No. 1, 105-115. Trans. Mosc. Math. Soc. 2020, 87-95.

A_2	0	a	e	ae	ea
0	0	0	0	0	0
a	0	0	ae	0	0
e	0	ea	e	e	ea
ae	0	a	ae	ae	a
ea	0	0	e	0	ea

$$A_2 = \langle a, e \mid a^2 = 0, aea = a, e^2 = eae = e \rangle,$$

$$a^* = a, e^* = e.$$

- M. Gao, W.T. Zhang,Y.F. Luo, A non-finitely based involution semigroup of order five, Algebra Univers. 81 (2020), No. 3, Paper No. 31, 14 p.

A_0^1	0	fe	f	e	1
0	0	0	0	0	0
fe	0	0	0	fe	fe
f	0	fe	f	fe	f
e	0	0	0	e	e
1	0	fe	f	e	1

$$A_0^1 = \langle e, f \mid e^2 = e, f^2 = f, ef = 0 \rangle,$$

$$e^* = f, f^* = e.$$

Finite rings and finite-dimensional algebras

Polin showed that there are finite nonassociative algebras without finite bases of their polynomial identities. His algebras are left nilpotent and satisfy the identity

$$x(yz) = 0.$$

- S.V. Polin, Identities of finite algebras (Russian), Sib. Math. J. 17(1976), 992-999. Translation: Sib. Mat. Zh. 17 (1976), 1356-1366.

The infinitely based 6-dimensional algebra of Lvov

Lvov suggested the following construction which in the sequel was used and improved to construct finite-dimensional nonassociative algebras without finite bases of their polynomial identities.
For any field K the algebra of Lvov is of the form

$$R = V \oplus M_2(K),$$

where $M_2(K)$ acts from the right on the two-dimensional vector space V and the multiplication in R is defined by

$$(v_1 + a_1)(v_2 + a_2) = v_1 a_2, \quad m_1, m_2 \in V, a_1, a_2 \in M_2(K).$$

Again, R satisfies the polynomial identity $x(yz) = 0$.

- I.V. L'vov, Finite-dimensional algebras with infinite bases of identities (Russian), Sib. Mat. Zh. 19 (1978), 91-99. Translation: Sib. Math. J. 19 (1978), 63-69.

Five-dimensional algebra in characteristic 0

- Yu.N. Maltsev, V.A. Parfenov, A nonassociative algebra having no finite basis for its laws (Russian), Sib. Mat. Zh. 18 (1977), 1420-1421. Translation: Sib. Math. J. 18 (1977), 1007-1008.

Over a field K of characteristic 0 Maltsev and Parfenov constructed a 5-dimensional algebra with infinite basis of its polynomial identities. The nonzero multiplications between the basis elements of the algebra in the paper are

$$e_1e_3 = e_2e_3 = e_1, e_1e_4 = e_2e_4 = e_2, e_1e_5 = e_1, e_2e_5 = e_2.$$

It is easy to see that translated in the construction of Lvov (and changing the basis of R as a vector space) this is the algebra $R = V \oplus T_2(K)$ where $T_2(K)$ is the algebra of 2×2 upper triangular matrices.

The algebra of Maltsev and Parfenov has a basis of polynomial identities consisting of

$$x(yz) = 0, x[y,z]x_1 \cdots x_n[u,v] = 0, n = 0, 1, 2, \ldots,$$

the products are left normed and $a[b,c]$ is a shortcut for $abc - acb$.

The basis of the identities of this algebra over a finite field and over an arbitrary infinite field of positive characteristic are given in

- I.M. Isaev, Essentially infinitely based varieties of algebras (Russian), Sib. Mat. Zh. 30 (1989), No. 6(178), 75-77. Translation: Sib. Math. J. 30 (1989), No. 6, 892-894.
- I.M. Isaev, A.V. Kislitsin, Identities in vector spaces and examples of finite-dimensional linear algebras having no finite basis of identities (Russian), Algebra Logika 52 (2013), No. 4, 435-460. Translation: Algebra Logic 52 (2013), No. 4, 290-307.

Four-dimensional algebra in any characteristic

The following 4-dimensional algebra has an infinite basis of its identities over any field

$$R = V + \text{span}\{e_{11} + e_{12}, e_{22}\}, \dim(V) = 2.$$

(Isaev, Kislitsin, Vestn. Novosib. Gos. Univ. for finite fields and Sib. Èlektron. Mat. Izv. for infinite fields);

- I.M. Isaev, A.V. Kislitsin, On identities of vector spaces embedded in finite associative algebras. (Russian) Zbl 1349.17001 Vestn. Novosib. Gos. Univ., Ser. Mat. Mekh. Inform. 15, No. 3, 69-77 (2015).
- I.M. Isaev, A.V. Kislitsin, The identities of vector spaces embedded in a linear algebra (Russian. English summary), Sib. Èlektron. Mat. Izv. 12 (2015), 328-343.

Weak polynomial identities

Let R be a (nonassociative) algebra generated by its vector subspace V. The polynomial $f(x_1,\dots,x_n)$ in the free nonassociative algebra $K\{X\}$ is a weak polynomial identity for the pair (R,V) if

$$f(v_1,\dots,v_n)=0 \text{ in } R,\quad v_1,\dots,v_n\in V.$$

Similarly one considers weak polynomial identities in the free associative algebra $K\langle X\rangle$ when the algebra R is associative.
We shall restrict our considerations for associative algebras R only.

Weak polynomial identities were introduced by Razmyslov who applied them to construct central polynomials for $M_d(K)$ and to find the basis of the polynomial identities in $M_2(K)$ and $sl_2(K)$, char$(K) = 0$.

- Yu.P. Razmyslov, On a problem of Kaplansky (Russian), Izv. Akad. Nauk SSSR, Ser. Mat. 37 (1973), 483-501. Translation: Math. USSR, Izv. 7 (1973), 479-496.
- Yu.P. Razmyslov, Finite basing of the identities of a matrix algebra of second order over a field of characteristic zero (Russian), Algebra i Logika 12 (1973), 83-113. Translation: Algebra and Logic 12 (1973), 47-63.

Consequences of weak polynomial identities

The weak polynomial identities of the pair (R, V) form an ideal of $K\langle X\rangle$ which is closed under linear endomorphisms (i.e. under replacements of the variables X with their linear combinations). We shall consider such ideals only and the corresponding bases of weak polynomial identities. If $V \subset R$ is a Lie algebra with respect to the operation $[u, v] = uv - vu$ the ideals are closed under Lie substitutions and similarly when V is a Jordan algebra with operation $u \circ v = uv + vu$. Then one considers also consequences obtained by Lie or Jordan substitutions.

Other applications of weak polynomial identities

- M.R. Vaughan-Lee, Abelian-by-nilpotent varieties of Lie algebras, J. Lond. Math. Soc., II. Ser. 11 (1975), 263-266.

Although not explicitly stated, the key moment of the proof that in characteristic 2 the variety of Lie algebras $\mathfrak{A}\mathfrak{N}_2$ defined by the identity

$$[[x_1, x_2, x_3], [x_4, x_5, x_6]] = 0$$

is not Spechtian is that the system of weak polynomial identities

$$[x, y, z] = 0, [x_1, x_2][x_2, x_3] \cdots [x_{n-1}, x_n][x_n, x_1] = 0, n = 2, 3, \dots,$$

is not finitely based.

- I.B. Volichenko, Varieties of Lie algebras with identity $[[X_1, X_2, X_3], [X_4, X_5, X_6]] = 0$ over a filed of characteristic 0 (Russian), Sibirsk. Mat. Zh. 25 (1984), No. 3, 40-54. Translation: Sib. Math. J. 25 (1984), 370-382.

Again, the key moment of the proof that in characteristic 0 the variety of Lie algebras $\mathfrak{A}\mathfrak{N}_2$ is Spechtian is the Specht property for the weak polynomial identity

$$[x, y, z] = 0.$$

Problem

Is the weak polynomial identity $[x, y, z] = 0$ Spechtian in characteristic $p > 2$?
The negative answer would give an example of infinitely based subvariety of the variety of Lie algebras $\mathfrak{A}\mathfrak{N}_2$ in characteristic $p > 2$.

The proofs for the infinitely based varieties generated by algebras of dimension 6,5 and 4 (by Lvov, Maltsev and Parfenov, Isaev and Kislitsin) are based essentially on the infinite basis property of weak polynomial identities.

For further results and problems on the topic
visit the forthcoming talk by Kislitsin.

Varieties of group representations

Let $F(X)$ be the free group freely generated by $X = \{x_1, x_2, \ldots\}$ and let $KF(X)$ be the group algebra. If G is a group and M is a G-module, then $f(x_1, \ldots, x_n) \in KF(X)$ is an identity for the pair (G, M) if

$$f(g_1, \ldots, g_n)M = 0 \text{ for all } g_1, \ldots, g_n \in G.$$

The problems in the theory of varieties of group representations are in the spirit of varieties of pairs. For more information see:

- B.I. Plotkin, Varieties of group representations (Russian), Usp. Mat. Nauk 32 (1977), No. 5(197), 3-68. Translation: Russ. Math. Surv. 32 (1977), No. 5, 1-72.
- B.I. Plotkin, S.M. Vovsi, Varieties of Group Representations. General Theory, Relations and Applications (Russian), Riga: “Zinatne”, 1983.
- S.M. Vovsi, Topics in Varieties of Group Representations, London Mathematical Society Lecture Note Series. 163. Cambridge etc.: Cambridge University Press, 1991.

Examples of non-Spechtian variety of group representations

- C.K. Gupta, A.N. Krasilnikov, Some non-finitely based varieties of groups and group representations, Int. J. Algebra Comput. 5 (1995), No. 3, 343-365.

The varieties of group representations over a field of characteristic 2 defined by the identities

$$[x_1, x_2][x_3, x_4, x_5](x_6 - 1) = 0, \quad (x_1 - 1)((x_2, x_3, x_4) - 1)(x_5 - 1) = 0$$

do not satisfy the Specht property.
(Here $[u, v]$ and (u, v) are the Lie and group commutators, respectively.)

Varieties of representations of Lie algebras

Every representation $\varphi : \mathfrak{g} \to \mathfrak{gl}(L) = \mathrm{End}(V)^{(-)}$ of the Lie algebra $\mathfrak{g}$ induces a representation $\overline{\varphi} : U(\mathfrak{g}) \to \mathrm{End}(V)$ of its universal enveloping algebra $U(\mathfrak{g})$. The free associative algebra $K\langle X\rangle$ is the universal enveloping algebra of the free Lie algebra $L(X)$. The polynomial $f(x_1, \ldots, x_n) \in K\langle X\rangle$ is an identity for the pair $(\mathfrak{g}, \varphi)$ if

$$\overline{\varphi}(f(g_1, \ldots, g_n))V = 0 \text{ for all } g_1, \ldots, g_n \in \mathfrak{g}.$$

Infinitely based varieties of representations of Lie algebras in characteristic 2

- A.N. Krasilnikov, A.L. Shmelkin, Finiteness of the basis of identities of finite-dimensional representations of solvable Lie algebras Russian), Sib. Mat. Zh. 29 (1988), No. 3(169), 78-86. Translation: Sib. Math. J. 29 (1988), No. 3, 395-402.

Let K be an infinite field of characteristic 2 and let

$$\mathfrak{g} = \left\{ \begin{pmatrix} 0 & b & c & 0 \\ 0 & a & 0 & c \\ 0 & 0 & a & b \\ 0 & 0 & 0 & 0 \end{pmatrix} \mid a, b, c \in K \right\}$$

be the Lie subalgebra of $\mathfrak{gl}_4(K)$ with its natural 4-dimensional representation $\iota : \mathfrak{g} \to \mathfrak{gl}_4(K)$. Then the pair $(\mathfrak{g}, \iota)$ is not finitely based and the proof uses the identities

$$[y, z]x_1 \cdots x_n[u, v] = 0, \quad n = 0, 1, 2, \dots .$$

Problems

Prove analogues of the theorem of Gupta and Krasilnikov for group representations (for any field of characteristic $p > 2$) and of the theorem of Krasilnikov and Shmelkin (for infinite fields of characteristic $p > 2$).

Limit varieties

The variety of algebraic systems is limit (or just-non-finitely based) if it is not finitely based and all of its proper subvarieties are finitely based.
By the Zorn lemma every infinitely based variety has a limit subvariety.

We shall survey some results on explicit examples of limit varieties. The examples do not depend on the Zorn lemma.

Groups

The first examples of infinitely based varieties of groups guarantee that there is a limit variety of 2-groups only. Newman showed that for any prime $p > 2$ there is an infinitely based variety of p-groups.
Hence there are infinitely many limit varieties.
The example is a subvariety of the product of varieties $(\mathfrak{A}_p\mathfrak{A}_p \cap \mathfrak{N}_p)\mathfrak{T}_p$ where $\mathfrak{A}_p$ is the variety of abelian groups of exponent p, $\mathfrak{N}_p$ is the variety of nilpotent groups of class p (defined by the identity $(x_1, \ldots, x_{p+1}) = 1$ and $\mathfrak{T}_p$ is the variety generated by the nonabelian group of order p^3 and exponent p. The independent system of identities consists of

$$((x, y, z), (x, y, z)^{u_k}, \ldots, (x, y, z)^{u_k^{p-1}}) = 1, k = 1, 2, \ldots,$$

where $u_k = (x_1, x_2) \cdots (x_{2k-1}, x_{2k})$.

- M.F. Newman, Just non-finitely-based varieties of groups, Bull. Aust. Math. Soc. 4 (1971), 343-348.

This was improved by Kozhevnikov who established that there are uncountably many limit varieties of groups.

- P. A. Kozhevnikov, Varieties of Groups of Prime Exponent and Identities with High Powers (Russian), Ph.D. Thesis, Moscow State University, 2000.
- P.A. Kozhevnikov, On nonfinitely based varieties of groups of large prime exponent, Commun. Algebra 40 (2012), No. 7, 2628-2644.

No examples are known for limit varieties of groups.
Gupta and Krasilnikov (1999) constructed a non-finitely based subvariety of $\mathfrak{N}_2\mathfrak{N}_2$ which seems to be very close to limit. Its intersection with $\mathfrak{A}\mathfrak{N}_2$ satisfies the Specht property.
They (2001) constructed a limit variety of bigroups. (Bigroups are generalizations of integral representations of groups.)
See also the survey (2003).

- C.K. Gupta, A.N. Krasilnikov, A solution of a problem of Plotkin and Vovsi and an application to varieties of groups, J. Aust. Math. Soc., Ser. A 67 (1999), No. 3, 329-355.
- C.K. Gupta, A.N. Krasilnikov, A just non-finitely based variety of bigroups, Commun. Algebra 29 (2001), No. 9, 4011-4046.
- C.K. Gupta, A. Krasilnikov, The finite basis question for varieties of groups – some recent results, Ill. J. Math. 47 (2003), No. 1-2, 273-283.

Semigroups

The first example of a limit variety of semigroups was constructed by Volkov.

Let $\mathfrak{N}$ be the variety of semigroups defined by the identities

$$xyzw = xzyw, x^2y^2 = y^2x^2, x^2zy^2 = y^2zx^2$$

and let $\mathfrak{V}$ be the subvariety of $\mathfrak{N}$ defined by the identities

$$yx^2y = y^2x^p, \quad p \geq 2 \text{ prime.}$$

Then $\mathfrak{V}$ is the only limit subvariety of $\mathfrak{N}$.

- M.V. Volkov, An example of a limit variety of semigroups, Semigroup Forum 24 (1982), 319-326.

More examples including infinite series of limit varieties generated by finite groups can be found in:

- G. Pollák, A new example of limit variety, Semigroup Forum 38 (1989), 283-303.
- M.V. Sapir, On Cross semigroup varieties and related questions, Semigroup Forum 42 (1991), 345-364.
- E.W.H. Lee, M.V. Volkov, Limit varieties generated by completely 0-simple semigroups, Int. J. Algebra Comput. 21 (2011), No. 1-2, 257-294.

Klejman showed that the 6-element Brandt semigroup B_2^1 generates a limit variety of inverse semigroups and this is the only limit variety of inverse semigroups which is not a limit variety of groups.
(The semigroup B_2^1 consists of the 2×2 matrices 0, 1 and the matrix units e_{ij}, $i, j = 1, 2$.)

- E.I. Klejman, Bases of identities of varieties of inverse semigroups (Russian), Sib. Mat. Zh. 20 (1979), 760-777. Translation: Sib. Math. J. 20 (1979), 530-543.

Monoids

The hunting for limit varieties of monoids continues also nowadays. See for example the papers and the references there:

- E.W.H. Lee, Maximal Specht varieties of monoids, Mosc. Math. J. 12 (2012), No. 4, 787-802.
- S.V. Gusev, A new example of a limit variety of monoids, Semigroup Forum 101 (2020), No. 1, 102-120.
- S.V. Gusev, Limit varieties of aperiodic monoids with commuting idempotents, J. Algebra and Its Appl. https://doi.org/10.1142/S0219498821501607.
- S.V. Gusev, O.B. Sapir, Classification of limit varieties of $\mathcal{J}$-trivial monoids arXiv:2009.06904v1 [math.GR].
- O.B. Sapir, Limit varieties generated by finite non-$\mathcal{J}$-trivial aperiodic monoids, arXiv:2012.13598v1 [math.GR].

Lie algebras

Only one example of a limit variety of Lie algebras in characteristic 2 is explicitly known. It is a subvariety of the center-by-abelian variety $[\mathfrak{A}^2, \mathfrak{E}]$ defined by the identity

$$[[x_1, x_2], [x_3, x_4], x_5] = 0.$$

- I.B. Volichenko, On varieties of centre-by-metabelian Lie algebras, Institute of Math. Belarussian Academy of Sciences, preprint No. 16, 1980.

Finite-dimensional algebras in characteristic 0

The following variety of algebras $\mathfrak{V}$ is limit.
(The products are left normed: $(xyz = (xy)z$, $x[y,z] = xyz - xzy$ and $s_3(x_1,x_2,x_3)$ is the standard polynomial of degree 3)

$$xy + yx = 0, (xy)(zt) = 0, x_1x_2y_1y_2x_3 - x_1x_2y_2y_1x_3 = 0,$$

$$s_3(x_1,x_2,x_3)x_1^{n-5}[x_1,x_2] = 0, n = 5,6,\ldots,$$

$$\sum_{\sigma\in S_4} \mathsf{sign}(\sigma)x_{\sigma(1)}x_{\sigma(2)}x_{\sigma(3)}x_1^{n-4}x_{\sigma(4)} = 0, n = 4,5,\ldots.$$

- V. Drensky, Representations of the symmetric group and varieties of linear algebras (Russian), Mat. Sb. 115 (1981), 98-115. Translation: Math. USSR Sb. 43 (1981), 85-101.

Theorem

The variety $\mathfrak{V}$ from the previous slide is generated by the following anticommutative six-dimensional algebra with basis $\{a_1, a_2, a_3, b, c, g\}$ as a vector space and nonzero products of the basis elements

$$a_1a_2 = b, ba_3 = c, bg = b.$$

As the proof in Mat. Sb. from 1981 the proof of the theorem uses representation theory of the general linear group.

Theorem

The variety $\mathfrak{V}$ defined by the identities

$$xy - yx = 0, (xy)(zt) = 0, x_1x_2y_1y_2x_3 - x_1x_2y_2y_1x_3 = 0,$$

$$\sum_{\sigma\in S_2} \text{sign}(\sigma)x_{\sigma(1)}[x_1, x_2]x_{\sigma(2)}x_1^{n-6}[x_1, x_2] = 0, n = 6, 7, \ldots,$$

$$\sum_{\sigma\in S_3} \text{sign}(\sigma)x_{\sigma(1)}[x_1, x_2]x_1^{n-5}x_{\sigma(2)}x_{\sigma(3)} = 0, n = 5, 6, \ldots.$$

is limit and is generated by the following commutative six-dimensional algebra with basis $\{a_1, a_2, a_3, b, c, g\}$ as a vector space and nonzero products of the basis elements

$$a_1a_2 = b, ba_3 = c, bg = b.$$

The proof is similar to the proof in the anticommutative case.

Varieties of pairs in characteristic 0

The pair (R, V) where R is a the subalgebra of $M_4(K)$

$$\begin{pmatrix} * & * & 0 & 0 \\ 0 & * & 0 & 0 \\ 0 & 0 & * & * \\ 0 & 0 & 0 & * \end{pmatrix}$$

and $V = K(e_{11} + e_{44}) + K(e_{12} + e_{34})$ is infinitely based and has a basis of weak polynomial identities

$$x_1[y_1, y_2]x_2 = 0, s_3(x_1, x_2, x_3) = 0,$$

$$[x_1, x_2]y_1 \cdots y_n[x_3, x_4] = 0, n = 0, 1, 2, \dots .$$

- I.M. Isaev, A.V. Kislitsin, Identities in vector spaces and examples of finite-dimensional linear algebras having no finite basis of identities (Russian), Algebra Logika 52 (2013), No. 4, 435-460. Translation: Algebra Logic 52 (2013), No. 4, 290-307.

Theorem

The pair (R, V) from the previous slide generates a limited variety of pairs.

Idea of the proof

Using representation theory of S_n and $GL_d(K)$ (in the spirit of the paper from 1981):

- Modulo the weak identity $x_1[y_1, y_2]x_2 = 0$ it is sufficient to work in $K\langle x_1, x_2, x_3\rangle$;
- Modulo the weak identities of (R, V) any weak identity is equivalent to an identity in two variables;
- Find weak identities in 3 variables which are equivalent to the identity $[x_1, x_2]y_1 \cdots y_n[x_3, x_4] = 0$.
- Show that any weak identity in two variables which does not hold in (R, V) implies $[x_1, x_2]y_1 \cdots y_n[x_3, x_4] = 0$ for n large eniough.

In the spirit of the constructions of Lvov, Maltsev and Parfenov, Isaev and Kislitsin, consider the 6-dimensional algebra R with basis

$$\{v_1, v_2, v_3, v_4, e_{11} + e_{44}, e_{12} + e_{34}\}$$

and nonzero multiplication inspired by the rule

$$v_i e_{ij} = v_j, \quad i, j = 1, 2, 3, 4.$$

By the theorem of Isaev and Kislitsin (in the left normed notation) the identities of R follow from

$$x(yz) = 0, zx_1[y_1, y_2]x_2 = 0, ys_3(x_1, x_2, x_3) = 0,$$

$$z[x_1, x_2]y_1 \cdots y_n[x_3, x_4] = 0, n = 0, 1, 2, \dots$$

The variety generated by the algebra R is not finitely based but it is not limit. The identity $s_3(x_1, x_2, x_3)y = 0$ does not hold in R but modulo the identities $x(yz) = 0$ and $zx_1[y_1, y_2]x_2 = 0$ do not imply any identity in two variables. In particular, the identities

$$x[x, y]x^n[x, y] = 0, n = 0, 1, 2, \ldots$$

do not follow from it. Hence adding the identity $s_3(x_1, x_2, x_3)y = 0$ to the identities of R we still have an infinitely based variety.

Problem which seems to be realistic

Find a limit subvariety of var(R) and finite-dimensional algebra which generates it.
It seems that the solution should use representation theory of $GL_d(K)$.

The previous two examples are of anticommutative and commutative algebras. The solution of the problem above will give a limit variety satisfying the identity $x(yz) = 0$.

Problems

- Find limit varieties generated by algebras in positive characteristic and over finite fields.
- The same problem for associative algebras.

Further readings (on counterexamples to the Specht problem only)

- Ju.A. Bahturin, Lectures on Lie Algebras. Lectures given at Humbolt University Berlin and Lomonosov University Moscow, Studien zur Algebra und ihre Anwendungen Band 4. Berlin: Akademie-Verlag, 1978.
- Yu.A. Bakhturin, Identities in Lie Algebras (Russian), Moskva: "Nauka". Glavnaya Redaktsiya Fiziko-Matematicheskoj Literatury, 1985. Translation: Identical Relations in Lie Algebras, Utrecht: VNU Science Press, 1987.
- Yu.A. Bahturin, A.Yu. Olshanskii, Identities (Russian), Itogi Nauki Tekh., Ser. Sovrem. Probl. Mat., Fundam. Napravleniya 18, 117-240, 1988. Translation: in A.I. Kostrikin, I.R. Shafarevich (Eds.), "Algebra II", Encyclopedia of Math. Sciences 18, Springer-Verlag, 107-221, 1991.

- V. Drensky, Free Algebras and PI-algebras. Graduate Course in Algebra, Singapore, Springer, 2000.
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