Bootstrap percolation: merging operations for polytopes[1]

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Algebra and Logic seminar, Sofia

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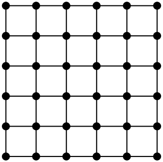

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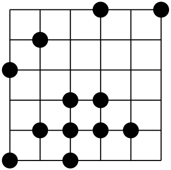

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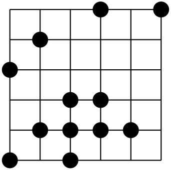

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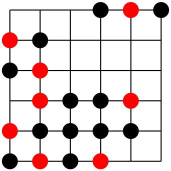

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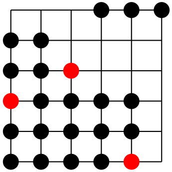

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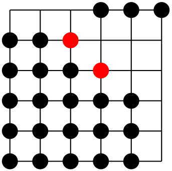

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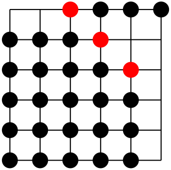

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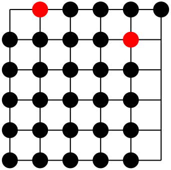

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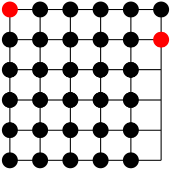

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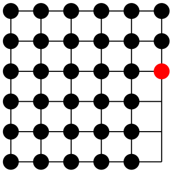

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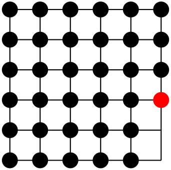

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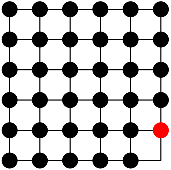

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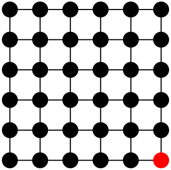

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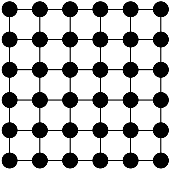

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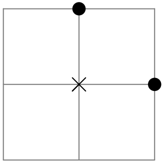

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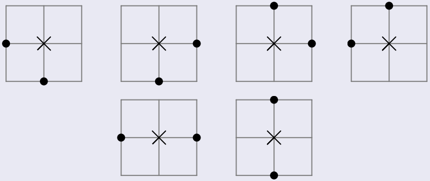

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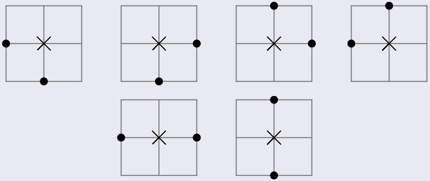

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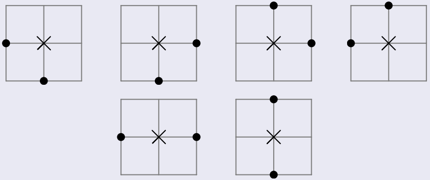

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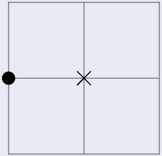
Examples

East



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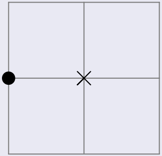

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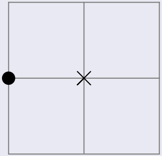

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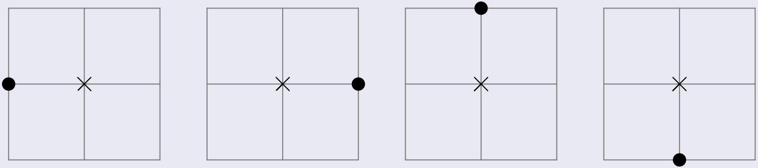

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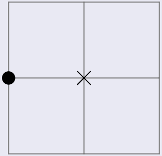

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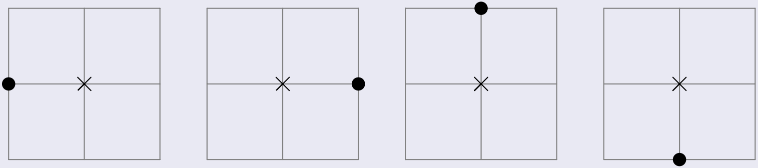

$p_c = 0$ $\tau_0 \approx 1/\sqrt{p}$

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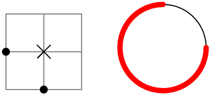

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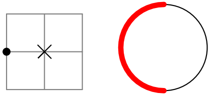

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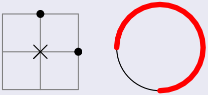
Theorem (BSU15)

An update family $\mathcal{U}$ is supercritical iff there is an open semi-circle of unstable directions.

North-East/Oriented percolation



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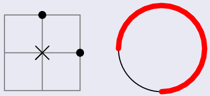

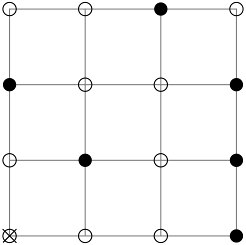

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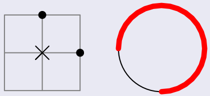

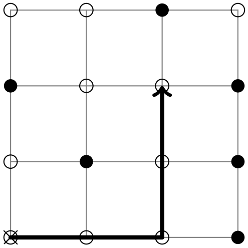

$\tau_0 = 5$

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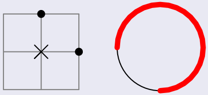

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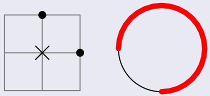

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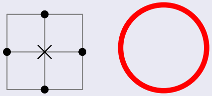
4-neighbour bootstrap percolation



$$p_c = 1$$

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For all $\mathcal{U}$ and $p > p_{\mathrm{c}}$, τ_0 has an exponential moment.

Theorem (H'22)

For all $\mathcal{U}$ supported on a half-space the conjecture holds.

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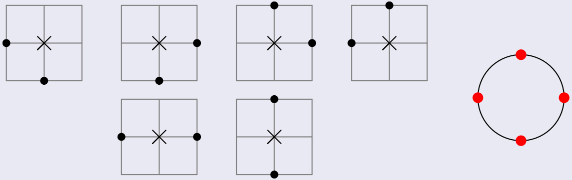
2-neighbour bootstrap percolation



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If $\mathcal{U}$ is critical, then $p_c = 0$ and $\tau_0 = \exp(p^{-\Theta(1)})$.

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Definition (Difficulty)

The *difficulty* $\alpha(u) \in \{1, 2, \dots\}$ of an isolated stable direction $u \in S^1$ is the smallest cardinal of a set of $Z \subset \mathbb{Z}^2$ such that $Z \cup \mathbb{H}_u$ can infect an infinite set. We set $\alpha(u) = \infty$ for non-isolated stable directions and $\alpha(u) = 0$ for unstable ones.

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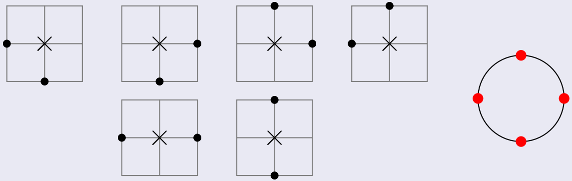
2-neighbour bootstrap percolation



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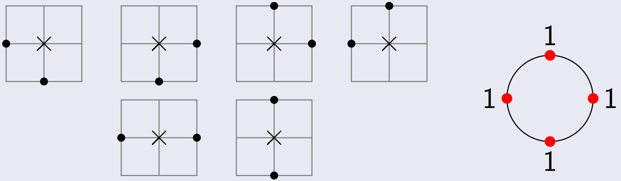

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Theorem (H–Mezei'20)

The difficulty α is computable, but NP-hard to determine.

Theorem (BSU15 lower bound)

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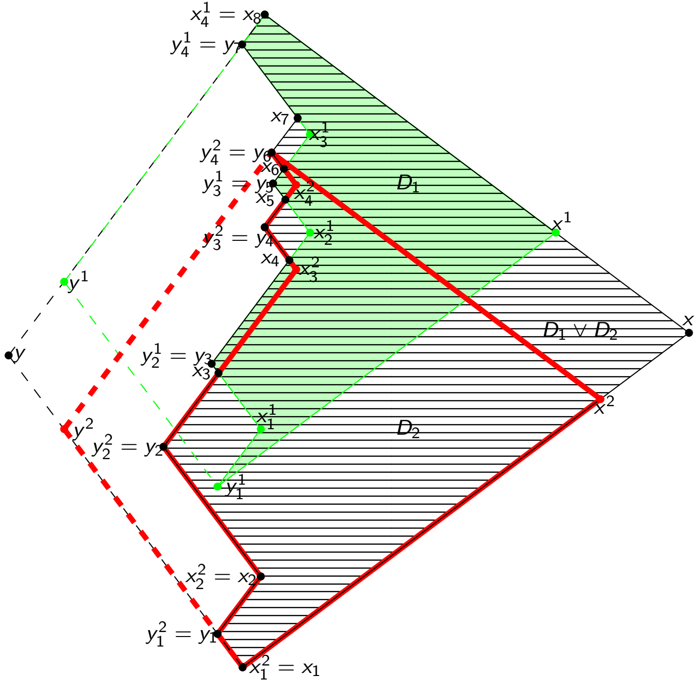
- Subadditivity.
- Aizenman–Lebowitz lemma.
- Extremal bound.
- Union bound on critical droplet probability.



$x_4^1 = x_8$
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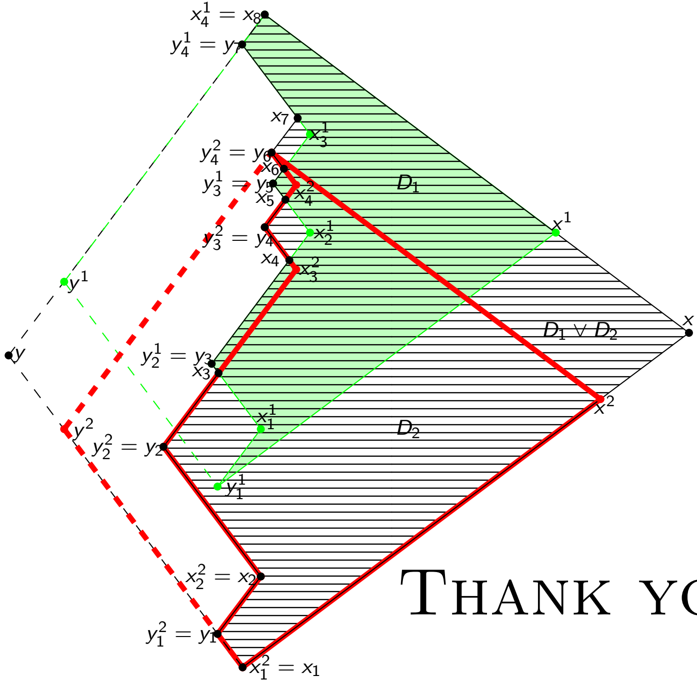
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Thank you!

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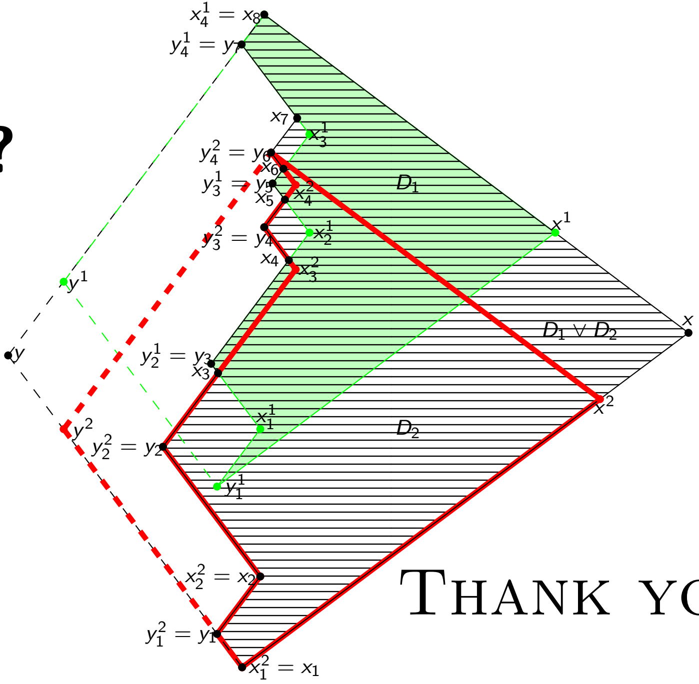

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$y_3^2 = y_4$
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Thank you!

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