Almost Prime Ideal And Almost Prime Radical

Alaa ABOUHALAKA

Department of Mathematics

Çukurova University, Adana, Turkey

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Introduction.

- Almost prime ideals appear by studying of unique factorization in Noetherian domains introduced by P.S. Bhatwdekar and P.K. Sharma through a paper intitled **unique factorization and birth of almost prime** in 2005. They defined it as the following.

Definition

A proper ideal P of a commutative ring with identity is an almost prime ideal if $ab \in P \setminus P^2$ implies $a \in P$ or $b \in P$.

- Many studies follow the previous definition in commutative rings.

Introduction.

Here we set the standard definitions of prime, weakly prime, and idempotent ideals of a noncommutative ring R.

Definition

A right ideal P of R is called prime if $AB \subseteq P$ implies $A \subseteq P$ or $B \subseteq P$ for all right ideals A, B of R.

Definition

A right ideal P of R is called weakly prime if $0 \neq AB \subseteq P$ implies $A \subseteq P$ or $B \subseteq P$, for all right ideals A, B of R.

Definition

A right ideal P of R is called idempotent if $P^2 = P$.

Introduction.

Now we introduce the concept of almost prime ideal of a ring not necessary commutative as the following.

Definition

An (A right) ideal P of a ring R is called almost prime if $AB \subseteq P$, and $AB \nsubseteq P^2$ implies $A \subseteq P$ or $B \subseteq P$ for all (right) ideals A, B of R.

Note that every idempotent, weakly prime, and prime right ideal is an almost prime right ideal. Hence the concept of almost prime ideal is a generalization of prime ideal. However, any almost prime right ideal does not need to be a prime right ideal.

Introduction

Definition

Let R be a ring and I, J be right ideals of R. The right ideal $I : J$ (known as colon right ideal) of R is defined as
$I : J = \{x \in R \mid Jx \subseteq I\}$.
Similarly we define the set $(I : J)^* = \{x \in R \mid xJ \subseteq I\}$. Recall that in the case of I, J being ideals, so are $I : J$ and $(I : J)^*$.

Here, we do the following.

- Show the analogy between almost prime (right) ideals, prime (right) ideals, and weakly prime (right) ideals.
- Investigate images and the inverse images of almost prime (right) ideals under ring homomorphisms.
- Characterize the non-local rings in which every (right) ideal is an almost prime.
- Introduce the concept of almost prime radical of an ideal.

Our Results.

Theorem

Let R be ring with identity, and P be an ideal of R then the following statements are equivalent:

(1) *P is an almost prime ideal.*

(2) *If $\langle a\rangle\langle b\rangle \subseteq P$, $\langle a\rangle\langle b\rangle \nsubseteq P^2$, then either $a \in P$ or $b \in P$, where $a, b \in R$.*

(3) *If $aRb \subseteq P$, $aRb \nsubseteq P^2$, then either $a \in P$ or $b \in P$, where $a, b \in R$.*

(4) *$P : \langle a\rangle = P \cup (P^2 : \langle a\rangle)$ and $(P : \langle a\rangle)^* = P \cup (P^2 : \langle a\rangle)^*$ for all $a \in R\backslash P$.*

(5) *Either $P : \langle a\rangle = P$ or $P : \langle a\rangle = P^2 : \langle a\rangle$, and either $(P : \langle a\rangle)^* = P$ or $(P : \langle a\rangle)^* = (P^2 : \langle a\rangle)^*$ for all $a \in R\backslash P$.*

Definition

A ring in which every (right) ideal is an almost prime (right) ideal is called a fully almost prime (right) ring.

In the following lines, we are going to illustrate some (original) examples of a fully almost prime ring, and a fully almost prime right ring.

Example

Let $R = \{0, a, b, c\}$ be the noncommutative ring with binary operations

+	0	a	b	c
0	0	a	b	c
a	a	0	c	b
b	b	c	0	a
c	c	b	a	0

.	0	a	b	c
0	0	0	0	0
a	0	a	a	0
b	0	b	b	0
c	0	c	c	0

Then, the right ideals of R are $I = \{0, b\}$, $J = \{0, c\}$, and $P = \{0, a\}$.

Our Results.

Example

■ Note that $IJ = 0 \subseteq P$, $I \not\subseteq P$, and $J \not\subseteq P$. Hence the right ideal P is not a prime right ideal. However, P is an almost prime right ideal.

■ $I = \{0, b\}$ and $J = \{0, c\}$ are also almost prime right ideals. Hence, the ring R is a fully almost prime right ring.

Our Results.

Example

Let $R=\left\{ \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} : a, b \in \mathbb{Z}_4, \text{ and } b \in \{0, 2\} \right\}$. Then the proper nonzero ideals of R are:

$$P_1 = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix} \right\}$$

$$P_2 = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} \right\}$$

$$M = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 2 & 2 \\ 0 & 0 \end{pmatrix} \right\}$$

The ring R is a fully almost prime ring.

Our Results.

Theorem

Let R be a ring with identity, and P be an ideal of R. Then the following statements are equivalent.
(1) *P is an almost prime right ideal.*
(2) *P is an almost prime ideal.*

Theorem

Let R be ring with identity, and P be a right ideal of R such that $(P^2 : P) \subseteq P$. Then the following statements are equivalent.
(1) *P is a prime right ideal.*
(2) *P is an almost prime right ideal.*

Our Results.

Theorem

Let R be a ring and P be an ideal of R. Then, the following statements are equivalent.
(1) *P is an almost prime right ideal of R.*
(2) *P/P^2 is weakly prime right ideal of R/P^2.*

Theorem

Let R be a ring, and I be an ideal of R. Let P be a right ideal of R such that $I \subseteq P$. If P is an almost prime right ideal of R, then P/I is an almost prime right ideal of R/I.

Our Results.

Theorem

Let $f : R \to S$ be a ring epimorphism, and P be an almost prime right ideal of R such that $\ker f \subseteq P$. Then $f(P)$ is an almost prime right ideal of S.

Corollary

Let $f : R \to S$ be a ring epimorphism, and B be a right ideal of S such that $f^{-1}(B)$ is an almost prime right ideal of R. Then, B is an almost prime right ideal of S.

Our Results.

Theorem

Let $f : R \to S$ be a ring epimorphism, and P be a right ideal of R such that $\mathrm{ker} f \subseteq P^2$. If $f(P)$ is an almost prime right ideal of S, then P is an almost prime right ideal of R.

Corollary

Let $f : R \to S$ be a ring epimorphism, and B be an almost prime right ideal of S such that $\mathrm{ker} f \subseteq (f^{-1}(B))^2$. Then, $f^{-1}(B)$ is an almost prime right ideal of R.

Our Results.

The following theorems describe fully almost prime (right) rings which are not local.

Theorem

Let R be a fully almost prime ring which is right $\mathbb{S}$-unital. If M_1 and M_2 are two distinct maximal ideals of R (thus R is not a local ring), then followings hold.
(1) $M_1M_2 = M_2M_1$.
(2) $M_1 \cap M_2 = M_1M_2$, *and it is idempotent.*

Theorem

Let the ring R with identity be a fully almost prime right ring which is not local. Then, for any two maximal right ideals M_1 and M_2, either M_1M_2 or M_2M_1 is idempotent.

Example

An application of our previous theorem is the subring R of $M_2(\mathbb{Z}_2)$, where

$$R = \{0, e_{11} + e_{12}, e_{21} + e_{22}, e_{11} + e_{12} + e_{21} + e_{22}\}.$$

The right ideals $I = \{0, e_{21} + e_{22}\}$, $J = \{0, e_{11} + e_{12} + e_{21} + e_{22}\}$, and $K = \{0, e_{11} + e_{12}\}$ are almost prime right ideals. Thus R is fully almost prime right ring. Note that all of I, J, and K are maximal right ideals. Clearly $IJ = 0$, $JI = J$; $IK = I$, $KI = K$; and $JK = J$, $KJ = 0$. Moreover, 0, I and K are idempotent, but J is not idempotent.

Our Results.

Definition

A nonempty set $S \subseteq R$ is called an almost m-system if, $I \cap S \neq \emptyset$, $J \cap S \neq \emptyset$, and $IJ \not\subseteq (R - S)^2$. Then $IJ \cap S \neq \emptyset$. For any ideals I, J of R.

Theorem

An ideal P of a ring R is almost prime ideal if and only if $R - P$ is almost m-system.

Theorem

Let $S \subseteq R$ be an almost m-system, and let P be an ideal maximal with respect to the property that P is disjoint from S, and $P^2 = (R - S)^2$ Then P is almost prime ideal of R.

Our Results.

Definition

Let I be an ideal of a ring R, and let P be an almost prime ideal of R such that $I \subseteq P$, and $I^2 = P^2$. Then we define $\sqrt{I}$ as the following.

- $\sqrt{I}$ = { $r \in R$: every almost m-system S containing r, with $I^2 = (R - S)^2$ meets I }.
- If there is no any almost prime ideal P such that $I \subseteq P$, with $I^2 = P^2$ then we say $\sqrt{I} = R$.

We call it the almost prime radical of the ideal I.

Our Results.

It is not clear from the definition above whether the $\sqrt{I}$ is an ideal of R or not, the following thorem shows the answer.

Theorem

$\sqrt{I}$ in the above definition is equal to the intersection of all almost prime ideals P of R containing I, with $I^2 = P^2$. Otherwise, $\sqrt{I} = R$.

Example

Let $R = M_{2\times 2}(\mathbb{Z}_{16})$. The nonzero proper ideals of R are

$$P_1 = \left\{ \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} : a_i \in \mathbb{Z}_{16},\ a_i = 2k, k \in \mathbb{Z} \right\}$$

$$P_2 = \left\{ \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} : a_i \in \mathbb{Z}_{16},\ a_i = 4k, k \in \mathbb{Z} \right\}$$

$$P_3 = \left\{ \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} : a_i \in \mathbb{Z}_{16},\ a_i = 8k, k \in \mathbb{Z} \right\}$$

Example

It is straightforward to show that $P_1^2 = P_2$, and $P_2^2 = P_3^2 = 0$. In addition, P_1, P_2 are almost prime ideals, thus $\sqrt{P_1} = P_1$, and $\sqrt{P_2} = P_2$; however, P_3 is not, since $P_1 P_2 \subseteq P_3$ and $P_1 P_2 \not\subseteq P_3^2$, and neither $P_1 \subseteq P_3$ nor $P_2 \subseteq P_3$. Additionally, P_2 is the only almost prime ideal such that $P_3 \subseteq P_2$, and $P_2^2 = P_3^2$. Thus $\sqrt{P_3} = P_2$.

Definition

An ideal P of a ring R is called an almost radical ideal if $\sqrt{P} = P$.

Note that every almost prime ideal is an almost radical ideal.

Corollary

Let R be a ring. If $\sqrt{rad(R)} \neq R$, then $\sqrt{rad(R)} = rad(R)$, where $rad(R)$ is the prime radical of R.

Proof.

Let $\sqrt{rad(R)} = \underset{i\in L}{\cap} P_i$ where P_i is an almost prime ideal of R such that $rad(R) \subseteq P_i$, and $[rad(R)]^2 = P_i^2$ for all $i \in L$. Then $P_i^2 \subseteq rad(R)$, thus $P_i \subseteq rad(R)$ for all $i \in L$, because $rad(R)$ is semiprime ideal. Hence $\sqrt{rad(R)} = \underset{i\in L}{\cap} P_i \subseteq rad(R)$, and since $rad(R) \subseteq \sqrt{rad(R)}$ we get $\sqrt{rad(R)} = rad(R)$. □

Our Results.

Corollary

Every ideal of a local ring (R, M) with $M^2 = 0$ is an almost radical ideal.

Example

Let $R = \left\{ \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} : a, b \in \mathbb{Z}_4, \text{ and } b \in \{0, 2\} \right\}$. Then the proper nonzero ideals of R are:

$$P_1 = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix} \right\}$$

$$P_2 = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} \right\}$$

$$M = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 2 & 2 \\ 0 & 0 \end{pmatrix} \right\}$$

The ring R is fully almost prime ring, and $P_1^2 = P_2^2 = M^2 = 0$. Thus, $\sqrt{P_1} = P_1 \cap M = P_1$. Also $\sqrt{P_2} = P_2 \cap M = P_2$, $\sqrt{M} = M$.

Our Results.

Theorem

Let $f : R \to S$ be a ring isomorphism, and let I be any ideal of R. If $\sqrt{I} \neq R$, then $f(\sqrt{I}) = \sqrt{f(I)}$.

Theorem

Let I be an ideal of a ring R. If $\sqrt{I} \neq R$, then $[\sqrt{I}]^2 = I^2$.

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