Algebra and Logic Seminar
Institute of Mathematics and Informatics
Bulgarian Academy of Sciences, Sofia

Derivations of upper triangular matrix rings
vs
Derivations of upper triangular matrix semirings

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December 02, 2022

At Algebra and Logic Seminar in 2021 I compared similar results for endomorphisms of matrix rings and semirings and promised that in the next year 2022 I will talk about my results for derivations of triangular matrix rings and semirings.

These results are published in the following articles:

1. D. Vladeva, Derivations of upper triangular matrix semirings, Linear and Multilinear Algebra, 2022, 70(4): 625-641.

2. D. Vladeva, Derivations of triangular matrix rings, Linear Multilinear Algebra. 2022; DOI:10.1080/03081087.2022.2063786.

Introduction

The study of representations of an arbitrary derivation of a ring as a sum of well-known derivations, though started long ago, got a boost only after the Amitsur's seminal article[a]. He proved that an arbitrary derivation of the ring $M_n(R)$ of $n \times n$ matrices over an associative ring R with identity is a sum of an inner derivation and a hereditary derivation.

In 1978, a representation of derivations of generalized quasi-matrix algebra was obtained by Burkov[b].

In 1983, Nowicki[c] showed a similar result for special subrings of matrix rings.

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[a] S. A. Amitsur, Extension of derivations to central simple algebras, *Commun. Algebra* **10**(8) (1982) 797-803.

[b] V. D. Burkov, Derivations of generalized quasi.matrix rings, *Mat. zametki* **24**(1) (1978) 111-122.

[c] A. Nowicki, Derivations of special subrings of matrix rtings and regular graphs. *Tsukuba J. Math.* 1983; 7(2): 281-297.

In 1993, Coehlo & Milies[a] proved the result similar to those in Amitsur's article for the ring of upper triangular matrices.
In 1995, Jondrup[b] generalized the theorem of Coelho and Milies.
Further, in 2006, Chun and Park[c] determined the derivations of the niltriangular matrix ring as a sum of diagonal and strongly nilpotent derivation.
Derivations of matrix ring containing a subring of triangular matrices was described in 2011 by Kolesnikov and Mal'tsev[d].

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[a] S. Coelho, P. Milies, Derivations of Upper Triangular Matrix Rings. *Linear Algebra Appl.* 1993; 187: 263-267.

[b] S. Jondrup, Automorphisms and Derivations of Upper Triangular Matrix Rings, *Linear Algebra Appl*. 1995; 221: 205-218.

[c] J. H. Chun, J. W. Park, Derivations on subrings of matrix rings. *Bull. Korean Math. Soc.* 2006; 43(3): 635–644.

[d] S. G. Kolesnikov, N. V. Mal'tsev, Derivations of a Matrix Ring Containing a Subring of Triangular Matrices. *Izv. VUZ*. 2011; 55 (11): 18-26.

Derivations of matrix rings consisting of sums of niltriangular matrix and matrix over an ideal was studies in 2017 by Kuzucuoğlu and Sayin[a]

Similar results for an arbitrary semiring does not hold in general. For additively idempotent semirings in the paper cited above I have (in 2020) analogous investigations.

Let R be an arbitrary associative (not necessarily commutative) ring or additively idempotent semiring ($a + a = a$ for any $a \in R$).

A derivation of R is an additive map $d : R \to R$ that satisfies Leibniz's law.

A derivation d of the ring (semiring) $UTM_n(R)$ of upper triangular matrices over the ring (semiring) R will be an R-derivation if it is an R-linear map, i.e. $d(\lambda A) = \lambda d(A)$ where $\lambda \in R$ and $A \in UTM_n(R)$. This definition is used in Jacobson[b] for algebras over a commutative ring. In this talk we will work only with R-derivations.

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[a] F. Kuzucuoğlu, U. Sayin, Derivations of some classes of matrix rings. *Journ. Algebra and Appl.* 2017; 16(1): 1-12.

[b] N. Jacobson, Basic Algebra II, W.H. Freeman & Company; 1989.

Basic derivations of the semiring of triangular matrices

Let $A = (a_{ij}) \in UTM_n(R)$, where R is an additively idempotent semiring, that is $A = \sum_{\substack{i,j=1 \\ i \leq j}}^{n} a_{ij} e_{ij}$, where $a_{ij} \in R$, $i, j = 1, \dots n$ and e_{ij} are matrix units.

Let $\ell_k = e_{11} + \cdots + e_{kk}$ for $1 \leq k \leq n$.

Let us note that in arbitrary ring (semiring) R by $x \circ y = xy + yx$ for any $x, y \in R$ we denote the Jordan product of x and y.

We obtain that for $A \in UTM_n(R)$ it follows

$A \circ \ell_k = A \ell_k + \ell_k A = \ell_k A$. Moreover, ℓ_k is a left semicentral idempotent, i.e. $\ell_k A \ell_k = A \ell_k$, in sense of Birkenmeier[a].

So, we have proved

The map $\delta_k : UTM_n(R) \to UTM_n(R)$ such that $\delta_k(A) = A \circ \ell_k$ is a derivation.

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[a] G. F. Birkenmeier et all, Extensions of rings and modules. Birkhäuser; Springer; 2013.

Next we consider the matrix $r_m = e_{n-m+1\, n-m+1} + \cdots + e_{nn}$, where $1 \leq m \leq n$ and for an arbitrary $A \in UTM_n(R)$ find that $A \circ r_m = A\, r_m + r_m\, A = A\, r_m$. Furthermore, r_m is a right semicentral idempotent, that is $r_m A r_m = r_m A$.

The map $d_m : UTM_n(R) \to UTM_n(R)$ such that $d_m(A) = A \circ r_m$ is a derivation.

If $m = n$ the map $d_n = \mathrm{i}$ is an identity map, which is a derivation in any additively idempotent semiring. The derivations δ_k, $1 \leq k \leq n$, and d_m, $1 \leq m \leq n$, are called basic derivations. We obtain that $\delta_k + \delta_\ell = \delta_\ell + \delta_k = \delta_\ell$ and $\delta_k\, \delta_\ell = \delta_\ell\, \delta_k = \delta_k$, where $k \leq \ell$. Thus, if $\mathcal{D}_\ell$ is the set of derivations δ_k, then $(\mathcal{D}_\ell, +, .)$ is a semiring with a zero, which is the smallest element δ_1 and identity, which is the greatest element the identity map δ_n. Similarly $d_m + d_\ell = d_\ell + d_m = d_m$ and $d_m d_\ell = d_\ell d_m = d_\ell$, where $\ell \leq m$. So, if $\mathcal{D}_r$ is the set of derivations d_m, then $(\mathcal{D}_r, +, .)$ is a semiring with a zero, which is the smallest element d_1 and an identity, which is the greatest element the identity map d_n.

Products of derivations

Let $\mathcal{D}$ be the semiring generated by the set $\mathcal{D}_\ell \cup \mathcal{D}r$. As the elements of semirings $(\mathcal{D}_r, +, .)$ and $(\mathcal{D}_\ell, +, .)$ are derivations of the semiring $UTM_n(R)$ we can add them and their sums are derivations. Thus $\delta_k + d_m \in \mathcal{D}$.

The product $\delta_k d_m$, where $1 \leq k, m \leq n$ is well defined by the rule $\delta_k d_m(A) = d_m(\delta_k(A))$ for any $A \in UTM_n(R)$,
but in general it is not a derivation.

Let $\delta_k, d_m \in \mathcal{D}$, where $1 \leq k, m \leq n$. The map $\delta_k d_m = d_m \delta_k$ is a derivation if and only if $\delta_k + d_m$ is the identity map.

A principal connected submatrix of a matrix $A = (a_{ij}) \in UTM_n(R)$ is a square submatrix of A in which the main diagonal consists of the elements $a_{ii}, \ldots, a_{jj}$, where $i, \ldots, j$ are consecutive numbers and $1 \le i \le j \le n$. Let $A = (a_{ij}) \in UTM_n(R)$ and $A(i_k, n_k)$ is the principal connected submatrix of A with main diagonal $a_{i_k\, i_k} \ \cdots \ a_{i_k+n_k-1\, i_k+n_k-1}$. An arbitrary finite set of principal connected submatrices $A(i_k, n_k)$, where $k = 1, \ldots, s$, without common elements is called a family.

Now we can present an arbitrary derivation $\delta \in \mathcal{D}$ by the derivatives $\delta(A)$.

Theorem 1. *Let for any matrix $A = (a_{ij}) \in UTM_n(R)$ the matrix $\delta(A) \in UTM_n(R)$ has the same entries a_{ij}, $1 \le i \le n$, $1 \le j \le n$ except the elements of a family of principal connected submatrices $A(i_k, n_k)$, where $k = 1, \ldots, s$, which are zeroes. Then the map $\delta : UTM_n(R) \to UTM_n(R)$ is a derivation and $\delta \in \mathcal{D}$. Conversely, for any $\delta \in \mathcal{D}$ and matrix $A = (a_{ij}) \in UTM_n(R)$, the matrix $\delta(A)$ is of the same type.*

Basis of $\mathcal{D}$

We will construct a basis $\mathcal{B}$ of the semigroup $(\mathcal{D}, +)$, such that any element of $\mathcal{D}$ can be represented as a sum of elements of $\mathcal{B}$.
For an arbitrary derivation $\delta_i\, d_j$ the number $i + j$ is called a weight of a derivation. From Theorem 1 it follows that the weight of $\delta_i\, d_j$ belongs to the interval $[n, 2n]$. The derivations
$\delta_1 = \delta_1\, \mathrm{i} = \delta_1\, d_n, \delta_2\, d_{n-1}, \ldots, \delta_i\, d_{n-i+1}, \ldots, d_1 = \mathrm{i}\, d_1 = \delta_n\, d_1$ are of weight $n + 1$. Each other derivation of weight $n + 1$ can be represented as a sum of these derivations.
Each derivation of weight $n + p$, where $1 < p \leq n - 1$ has the form $\delta_k\, d_{n-k+p}$ (as a consequence of Theorem 1)) and can be represented as a sum of two derivations of weight $n + 1$:
$\delta_k\, d_{n-k+p} = \delta_k\, d_{n-k+1} + \delta_{k-p+1}\, d_{n-k+p}$.
The derivations $\delta_1\, d_{n-1}, \ldots, \delta_i\, d_{n-i}, \ldots, \delta_{n-1}\, d_1$ are of weight n.

For arbitrary n we construct a basis in $\mathcal{D}$ consisting of the derivations:

1. $\delta_1, \delta_2\, d_{n-1}, \ldots, \delta_{n-1}\, d_2, d_1$.

2. $\delta_1\, d_{n-1},\ \delta_2\, d_{n-2},\ \ldots,\ \delta_{n-2}\, d_2,\ \delta_{n-1}\, d_1$.

For any $\alpha \in R$ and a derivation $\delta \in \mathcal{D}$ the map $\alpha\delta : UTM_n(R) \to UTM_n(R)$, such that $(\alpha\delta)(A) = \alpha\, \delta(A)$, where $A \in UTM_n(R)$, is a derivation.
Thus the semigroup $\mathcal{D}$ is an R-semimodule. If we denote $\delta_{ii} = \delta_i\, d_{n-i+1},\ i = 1, \ldots, n,\ \delta_{j\,j+1} = \delta_j\, d_{n-j},\ j = 1, \ldots, n-1$, then the basis of the R-semimodule $\mathcal{D}$ is

$$\mathcal{B} = \{\delta_{ii},\ \delta_{j\,j+1},\ i = 1, \ldots, n,\ j = 1, \ldots, n-1\}.$$

The main theorem (the semiring case)

Theorem 2. *An arbitrary derivation $D : UTM_n(R) \to UTM_n(R)$, where R is an additively idempotent semiring, is a linear combination of elements of the basis $\mathcal{B}$ of the R-semimodule $\mathcal{D}$ with coefficients from R.*

In the proof $D(A) = \sum_{i=1}^{n} \alpha_{ii}\delta_{ii}(A) + \sum_{j=1}^{n-1} \alpha_{jj+1}\delta_{jj+1}(A)$ where $A \in UTM_n(R)$ and $\alpha_{ii}, \alpha_{j\,j+1} \in R$.

Corollary *The images of the matrices of $UTM_n(R)$, where R is an additively idempotent semiring, under arbitrary derivation form an ideal of $UTM_n(R)$.*

The set of upper triangular matrices with zeroes on the main diagonal is an ideal $N_n(R)$ of $UTM_n(R)$. We denote the subsemiring of diagonal matrices by $Diag_n(R)$.
If the semiring R contains no nilpotent elements then $UTM_n(R) = Diag_n(R) \oplus N_n(R)$.

The derivations $D_D = \sum_{i=1}^{n} \lambda_i \delta_{ii}$, $\lambda_i \in R$ are called diagonal derivations since $D_D(A)$ has almost one nonzero element on the main diagonal.

The derivations $D_N = \sum_{j=1}^{n-1} \mu_j \delta_{j\,j+1}$, $\mu_j \in R$ are called nilpotent derivations since $(D_N(A))^k = 0$ for some positive integer k.

Theorem 3. *Let the additively idempotent semiring R contain no nonzero nilpotent elements, $A \in UTM_n(R)$ and $D : UTM_n(R) \to UTM_n(R)$ be an arbitrary derivation. Then there exists a diagonal derivation D_D, nilpotent derivation D_N, diagonal matrix A_D and nilpotent matrix A_N, such that*

$$D(A) = D_D(A_D) + D_N(A_N).$$

Derivations generated by left semicentral idempotents of a ring

Now we recall some definitions.
Let R be an arbitrary associative (not necessarily commutative) ring. For a fixed $x \in R$ the map $d_x(a) = [x, a] = xa - ax$ for any $a \in R$ is a derivation called inner derivation of R determined by x.

Let R_1 be a ring of matrices over R. The map $\widetilde{d} : R_1 \to R_1$ such that $\widetilde{d}(A) = (d(a_{ij}))$ for any matrix $A = (a_{ij}) \in R_1$, where d is a derivation of R, is a derivation called a hereditary derivation generated by d.

As in the semiring case an idempotent $\ell \in R$ ($r \in R$) is called left (right) semicentral if $\ell x \ell = x \ell$ ($rxr = rx$) for all $x \in R$.

The set of left (right) semicentral idempotents of the ring R is a multiplicative semigroup with identity.

The multiplicative semigroup of left (resp. right) semicentral idempotents elements of the ring R is denoted by $(\mathcal{L}(R), .)$, (resp. $(\mathcal{R}(R), .)$).
Let $\ell_1, \ell_2 \in \mathcal{L}(R)$ and $\ell_1 \star \ell_2 = \ell_1 + \ell_2 - \ell_1\ell_2$. Similarly we denote $r_1 \star r_2 = r_1 + r_2 - r_1 r_2$, where $r_1, r_2 \in \mathcal{R}(R)$.

The semigroup $\mathcal{L}(R)$, $(\mathcal{R}(R))$ is closed under the operation $\star$.

If $\ell \in \mathcal{L}(R)$, then $r = 1 - \ell \in \mathcal{R}(R)$. So, for each left semicentral element ℓ there is a right semicentral element r such that $\ell + r = 1$. Moreover, if $\ell + r = 1$ for an arbitrary $x \in R$ we have $rx\ell = (1-\ell)x\ell = x\ell - \ell x\ell = 0$.

Let ℓ be a left semicentral and r a right semicentral idempotent of R such that $\ell + r = 1$. The map $d_\ell : R \to R$ such that $d_\ell(x) = \ell x r$ for any $x \in R$ is a derivation of R.

Why we denote this derivation by d_ℓ?
Because $d_\ell(x) = \ell x r = \ell x(1-\ell) = \ell x - x\ell = [\ell, x]$.

The composition of two derivations in general is not a derivation, but there are exceptions.

In 1957, Posner[a] has shown that for prime ring R of characteristic different from 2 the composition of two non-zero derivations is not a derivation. This result has been generalized in several ways – see Chebotar and Lee[b], Chuang[c], Krempa and Matczuk[d].

But a composition of inner derivations can be a nonzero derivation, for many examples see Lanski[e].

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[a] E. C. Posner, Derivations in prime rings, *Proc. Amer. Math. Soc.* 1957 8; 1093-1100.

[b] M. A. Chebotar, P-H Lee P-H, A Note on Compositions of Derivations of Prime Rings, Commun. Algebra. 2003 31:6; 2965-2969.

[c] C. L. Chuang, On compositions of derivations of prime rings, *Proc. Amer. Math. Soc*. 1990; 108(3): 647-652.

[d] J. Krempa, J. Matczuk, On the composition of derivations. *Rend. Circ. Mat. Palermo*, 1984 33:441-455.

[e] C. Lanski, Differential identities of prime rings, Kharchenko's theorem and applications, *Contemporary Math.* 124 (1992) 111-128.

The next result is a little step in this direction.

Theorem 4. *Let* $\ell_1, \ell_2 \in \mathcal{L}(R)$*. The composition* $d_{\ell_1} d_{\ell_2}$ *is a derivation if and only if the Jordan product is equal to the sum:* $\ell_1\ell_2 + \ell_2\ell_1 = \ell_1 + \ell_2$*.*

Basic derivations of the ring of upper triangular matrices

Let R be an associative ring with identity 1 and $UTM_n(R)$ the ring of upper triangular $n \times n$ matrices over R.
As in the semiring case we consider
$\ell_k = e_{11} + \cdots + e_{kk}$, where $1 \leq k \leq n$,
and prove that these matrices are left semicentral idempotents.
Consequently, from the result above we obtain that

the map $d_{\ell_k} : UTM_n(R) \to UTM_n(R)$, where R is a ring and $d_{\ell_k}(A) = [\ell_k, A]$, $1 \leq k \leq n-1$, for any matrix $A \in UTM_n(R)$, is a derivation of the ring $UTM_n(R)$, generated by ℓ_k.

As a consequence for any matrix $A = (a_{ij}) \in UTM_n(R)$ we obtain

$$d_{\ell_k}(A) = \begin{pmatrix} 0 & \cdots & 0 & a_{1\,k+1} & \cdots & a_{1n} \\ \vdots & \cdots & \vdots & \vdots & \cdots & \vdots \\ 0 & \cdots & 0 & a_{k\,k+1} & \cdots & a_{kn} \\ 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \cdots & \vdots & \vdots & \cdots & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & 0 \end{pmatrix},$$

where $k = 1, \ldots, n-1$.
Since $\ell_k = e_{11} + \cdots + e_{kk}$, where $1 \leq k \leq n$, we prefer to consider new derivations

$$\delta_i(A) = [e_{ii}, A]$$

for any $A \in UTM_n(R)$, $i = 1, \ldots, n$. Now $d_{\ell_k}(A) = \sum_{i=1}^{k} \delta_i(A)$.
Another reason to consider these derivations is the fact that the matrix $\delta_i(A)$ in the general case contains more zeroes than the matrix $d_{\ell_k}(A)$.

Thus we calculate

$$\delta_1(A) = d_{\ell_1}(A) = \begin{pmatrix} 0 & a_{12} & \cdots & a_{1n} \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}$$

and

$$\delta_i(A) = \begin{pmatrix} 0 & \cdots & 0 & -a_{1i} & 0 & \cdots & 0 \\ \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & \cdots & 0 & -a_{i-1\,i} & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 0 & a_{i\,i+1} & \cdots & a_{in} \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \end{pmatrix},$$

where $i = 2, \ldots, n$ and $A = (a_{ij}) \in UTM_n(R)$.

Some properties of these basic derivations of $UTM_n(R)$ are:
(i) *The derivation* δ_1 *is an idempotent.*
(ii) *For all* $\mathrm{i} = 2, \ldots, \mathrm{n}$ *it follows* $\delta_{\mathrm{i}}^3 = \delta_{\mathrm{i}}$*, i.e. the derivations* δ_{i} *are tripotent.*
(iii) *For any* $\mathrm{i}, \mathrm{j} = 1, \ldots, \mathrm{n}$ *where* $\mathrm{i} < \mathrm{j}$ *and* $\mathrm{A} = (\mathrm{a_{ij}}) \in \mathrm{UTM_n(R)}$ *it follows* $\delta_{\mathrm{i}}(\delta_{\mathrm{j}}(\mathrm{A})) = \delta_{\mathrm{j}}(\delta_{\mathrm{i}}(\mathrm{A})) = -\mathrm{a_{ij}}$.
(iv) $\delta_1 + \delta_2 + \cdots + \delta_{\mathrm{n}} = 0$.

Let us denote by $\mathcal{D}$ the additive group of derivations generated by $\delta_2, \ldots, \delta_n$. Since $\delta_2, \ldots, \delta_n$ are R-derivations, it follows (as in the semiring case) that $\mathcal{D}$ is an R-module.

The derivations $\delta_2, \ldots, \delta_n$ *forms a basis of the* R*-module* $\mathcal{D}$*.*

All derivations d_{ℓ_k}, $k = 1, \ldots, n$ are elements of $\mathcal{D}$.

Are there any other left semicentral idempotents $\ell \in UTM_n(R)$ such that $d_\ell \in \mathcal{D}$?
The answer is positive. Denote by ℓ^R an arbitrary left semicentral idempotent of R. Similarly to $\ell_k = \sum_{i=1}^{k} e_{ii}$ we consider

$$\ell_k^R = \sum_{i=1}^{k-1} e_{ii} + \ell^R e_{kk}.$$

For an arbitrary matrix $A = (a_{ij}) \in UTM_n(S)$ we find that $\ell_k^R A \ell_k^R = A\ell_k^R$.
Note that unlike d_{ℓ_k}, $k = 1, \ldots, n$, the new derivations $d_{\ell_k^R}$, where ℓ^R is any left semicentral idempotent of R, are not R-derivations.
Another difference between derivations d_{ℓ_k} and the new derivations $d_{\ell_k^R}$ is in the following reasoning.
The composition $d_{\ell_k} d_{\ell_m}$ is not a derivation since $\ell_k \ell_m = \ell_m \ell_k = \ell_k$ and $\ell_k + \ell_m = \ell_m$, where $k < m$ and then the equality of Jordan product and sum of idempotents, considered in Theorem 4, fail to hold.

Let $\ell', \ell'' \in \mathcal{L}(R)$. Consider the left semicentral idempotents

$$\ell'_k = \sum_{i=1}^{k-1} e_{ii} + \ell' e_{kk} \text{ and } \ell''_k = \sum_{i=1}^{k-1} e_{ii} + \ell'' e_{kk}.$$

Now by Theorem 4 if $\ell'\ell'' + \ell''\ell' = \ell' + \ell''$, it follows that the composition $d_{\ell'_k} d_{\ell''_k}$ is a derivation.

Are there another derivations of $UTM(R)$?

To answer this question we return again to a left (respectively right) semicentral idempotents of the ring R. Since $d_\ell(x) = \ell x r$ for any $x \in R$ is a derivation of the ring R, it follows that the corresponding $\widetilde{d}_\ell$ is a hereditary derivation of $UTM(R)$. Since $\widetilde{d}_\ell(A) = \ell A r$ for matrix $A \in UTM(R)$ and ℓ is in general not a central idempotent we see that $\widetilde{d}_\ell$ is not an R-derivation.

Representation of arbitrary derivations of triangular matrices

In Theorem 2 (the case when R is an additively idempotent semiring) we proved that an arbitrary derivation $D: UTM_n(R) \to UTM_n(R)$ is a linear combination of elements of the basis of the R-semimodule $\mathcal{D}$ with coefficients from R.

Note that this result is stronger than the next result.

Theorem 5. *Let $D: UTM_n(R) \to UTM_n(R)$ be an arbitrary R-derivation of the ring $UTM_n(R)$ and $A = (a_{ij}) \in UTM_n(R)$. Then there exist matrices $M^D, M^D_{ij}, N^D_{ij} \in UTM_n(R)$, $i, j = 1, \dots, n$, such that*

$$D(A) = \sum_{i=1}^{n} a_{ii}\delta_i\left(M^D\right) + \sum_{j=2}^{n}\left(\sum_{\substack{i=1\\ i<j}}^{n-1} a_{ij}(\delta_i\left(M^D_{ij}\right) + \delta_j\left(N^D_{ij}\right))\right),$$

where δ_i, $i = 1, \dots, n$, are the basic derivations.

Note that the entries a_{ij} of the matrix A are the coefficients of the linear combinations in the right side of the equality.
Since the proof of this theorem is constructive we can show the entries of the matrices M^D, M^D_{ij} and N^D_{ij}.

If we denote $D(e_{pq}) = \sum_{\substack{i,j=1 \\ i \le j}}^{n} \alpha_{ij}^{(p,q)} e_{ij}$, where $1 \le p \le n$, $1 \le q \le n$,

we construct the matrix

$$M^D = \begin{pmatrix} 0 & \alpha_{12}^{(1,1)} & \alpha_{13}^{(1,1)} & \cdots & \alpha_{1n}^{(1,1)} \\ 0 & 0 & \alpha_{23}^{(2,2)} & \cdots & \alpha_{2n}^{(2,2)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \alpha_{n-1\,n}^{(n-1,n-1)} \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

Then we prove that $D(e_{ii}) = \delta_i(M^D)$ for $i = 1, \ldots, n$.

In the final step of the proof we construct new matrices M_{ij}^D and N_{ij}^D, where $i < j$ and represent the derivative $D(e_{ij})$ by the derivatives of these matrices.
Thus

$$M_{ij}^D = \alpha_{ij}^{(i,j)} e_{ij} + \sum_{l=j+1}^{n} \alpha_{jl}^{(j,j)} e_{il} \ \text{ and}$$

$$N_{ij}^D = -\sum_{k=1}^{i-1} \alpha_{ki}^{(i,i)} e_{kj}.$$

Then

$$D(e_{ij}) = \delta_i \left(M_{ij}^D \right) + \delta_j \left(N_{ij}^D \right)$$

Some consequences of the theorem. Immediately follows

If D is an arbitrary R-derivation of the ring $UTM_n(R)$, then $D(E) = 0$, where E is the identity matrix.

As we know from Theorem 2 of Amitsur's article, cited above, an arbitrary derivation is a sum of a hereditary derivation and an inner derivation.

What happens when the considered derivation is an R-derivation.

Let $\delta : R \to R$ be a derivation and $D = \widetilde{\delta}$ a hereditary derivation of $UTM_n(R)$. Since D is an R-derivation we have

$$D(\lambda A) = \lambda D(A) = \lambda\widetilde{\delta}(A) = \lambda(\delta(a_{ij})),$$

where $\lambda \in R$ and $A \in UTM_n(R)$. On the other hand

$$D(\lambda A) = (\delta(\lambda a_{ij})) = (\delta(\lambda)a_{ij} + \lambda\delta(a_{ij})) = \delta(\lambda)A + \lambda(\delta(a_{ij})).$$

Then $\delta(\lambda)A = 0$. Since A is an arbitrary matrix we have $\delta(\lambda) = 0$ and since λ is an arbitrary element of R we obtain that δ is a zero derivation.

Hence there are no nonzero hereditary derivations which are R-derivations. In other words

any R-derivation $D : UTM_n(R) \to UTM_m(R)$ is an inner derivation.

Thus there is a fixed matrix $X = (x_{ij}) \in UTM_n(R)$ such that $D(A) = D_X(A)) = [X, A]$. So

$$D_X(A) = \begin{pmatrix} [x_{11}, a_{11}] & * & \cdots & * \\ 0 & [x_{22}, a_{22}] & \cdots & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & [x_{nn}, a_{nn}] \end{pmatrix}.$$

By Theorem 5. it follows that $[x_{ii}, a_{ii}] = 0$, $i = 1, \ldots, n$. Thus the elements on the main diagonal of the matrix X belongs to the centre $C(R)$.

To describe the entries of the matrix $D_X(A)$ over the main diagonal we consider

$$\Delta_X^i(a_{jk}) = x_{ij}a_{jk} - a_{ij}x_{jk},$$

where $1 \leq i \leq j \leq k \leq n$, but $i < n$ and call this difference an index commutator of elements of the matrices X and A.

All entries of the matrix $D_X(A)$ are represented by sums of index commutators:

$$D_X(A) =$$

$$= \begin{pmatrix} 0 & \Delta_X^1(a_{12}) + \Delta_X^1(a_{22}) & \cdots & \sum_{i=1}^n \Delta_X^1(a_{in}) \\ 0 & 0 & \cdots & \sum_{i=2}^n \Delta_X^2(a_{in}) \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \Delta_X^{n-1}(a_{n-1\,n}) + \Delta_X^{n-1}(a_{nn}) \\ 0 & 0 & \cdots & 0 \end{pmatrix}.$$

Now for the given matrix $X = (x_{ij})$ we consider the map $\Delta_X^i : R \to R$ such that $\Delta_X^i(a_{jk})$ is the index commutator, $i = 1, \ldots, n-1$ and $A = (a_{ij}) \in UTM_n(R)$.
Let $B = (b_{ij}) \in UTM_n(R)$. Then we find $\Delta_X^i(a_{jk} + b_{jk}) = \Delta_X^i(a_{jk}) + \Delta_X^i(b_{jk})$, i.e. Δ_X^i is a linear map. We obtain $\Delta_X^i(a_{jk})b_{k\ell} + a_{ij}\Delta_X^j(b_{k\ell}) = \Delta_X^i(a_{jk}b_{k\ell})$.

Hence the index commutator is similar to derivation.
The fact that the index commutators are not derivations emphasizes the crucial role of derivations δ_i for studying arbitrary R-derivations.

THANK YOU !