Algebra and Logic Seminar
Institute of Mathematics and Informatics
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Bicommutative algebras from commutative point of view

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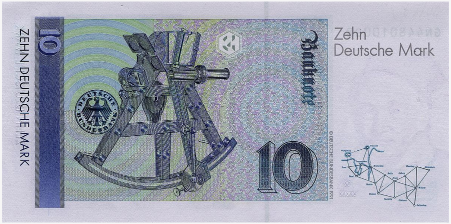
Instead of motto

Carl Friedrich Gauß:
Die Mathematik ist die Königin der Wissenschaften und die Zahlentheorie ist die Königin der Mathematik.
(Mathematics is the queen of the sciences and number theory is the queen of mathematics.)



I am not a commutative algebraist. But for me personally:
Die Ringtheorie ist die Königin der Algebra und die kommutative Algebra ist die Königin der Ringtheorie.
(Ring theory is the queen of algebra and commutative algebra is the queen of ring theory.)

At the Second International Conference "Mathematics Days in Sofia 2017", July 10-14, 2017 I presented a talk on varieties of bicommutative algebras. I started my talk with the claim that bicommutative algebras are not very central objects in the theory of algebras with polynomial identities.

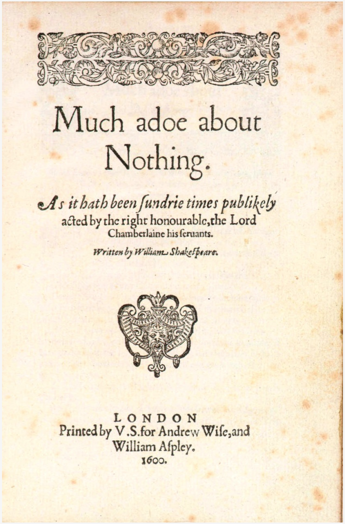
If it is commonly accepted that the very center of nonassociative ring theory is around Lie and Jordan algebras, bicommutative algebras are again in the periphery of the theory.

Nevertheless they have a rich theory which demonstrates the powerful methods of combinatorial nonassociative ring theory. Additionally, there are beautiful combinatorics, nice history and hidden commutative algebra behind the results.

The main purpose of the talk is to make more transparent the classical commutative algebra hidden there. I believe that these ideas may be used to solve also other problems.

My comment about these efforts was that maybe this is:

Much ado about (almost) nothing.



Let K be a field of any characteristic. A K-algebra R is called *right-commutative* if it satisfies the polynomial identity

$$(x_1x_2)x_3 = (x_1x_3)x_2,$$

i.e., $(r_1r_2)r_3 = (r_1r_3)r_2$ for all $r_1, r_2, r_3 \in R$. Similarly one defines *left-commutative* algebras as algebras which satisfy the identity

$$x_1(x_2x_3) = x_2(x_1x_3).$$

Algebras which are both right- and left-commutative are called *bicommutative*.

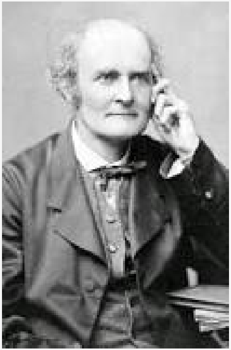
We denote by $\mathfrak{B}$ the variety of all bicommutative algebras, i.e., the class of all algebras satisfying the identities of right- and left-commutativity and by $F_d(\mathfrak{B})$ and $F(\mathfrak{B}) = F_\infty(\mathfrak{B})$ – the free bicommutative algebra generated by $X_d = \{x_1, \ldots, x_d\}$ and $X = \{x_1, x_2, \ldots\}$, respectively. In particular, if $K\{X\}$ is the free nonassociative algebra (the algebra of polynomials in countably many noncommuting and nonassociative variables), then

$$F(\mathfrak{B}) \cong K\{X\}/B,$$

where B is the ideal generated by all

$$(u_1u_2)u_3 - (u_1u_3)u_2 \text{ and } u_1(u_2u_3) - u_2(u_1u_3), \quad u_1, u_2, u_3 \in K\{X\}.$$

One-sided commutative algebras appeared first in the paper by Cayley in 1857.



- A. Cayley, *On the theory of analytical forms called trees*, Phil. Mag. **13** (1857), 172-176. Collected Math. Papers, University Press, Cambridge, Vol. 3, 1890, 242-246.

A SYMBOL such as $A\partial_x + B\partial_y + \ldots$, where A, B, &c. contain the variables x, y, &c. in respect to which the differentiations are to be performed, partakes of the natures of an operand and an operator, and may be therefore called an Operandator. Let P, Q, R .. be any operandators, and let U be a symbol of the same kind, or to fix the ideas, a mere operand; PU denotes the result of the operation P performed on U, and QPU denotes the result of the operation Q performed on PU; and generally in such combinations of symbols, each operation is considered as affecting the operand denoted by means of all the symbols on the right of the operation in question. Now considering the expression QPU, it is easy to see that we may write

$$QPU = (Q \times P)U + (QP)U,$$

where on the right-hand side $(Q \times P)$ and (QP) signify as follows: viz. $Q \times P$ denotes the mere algebraical product of Q and P, while QP (consistently with the general notation as before explained) denotes the result of the operation Q performed upon P as operand; and the two parts $(Q \times P)U$ and $(QP)U$ denote respectively the results of the operations $(Q \times P)$ and (QP) performed each of them upon U as operand. It is proper to remark that $(Q \times P)$ and $(P \times Q)$ have precisely the same meaning, and the symbol may be written in either form indifferently.

In the modern language this is the *right-symmetric Witt algebra* W_1^{rsym} in one variable.

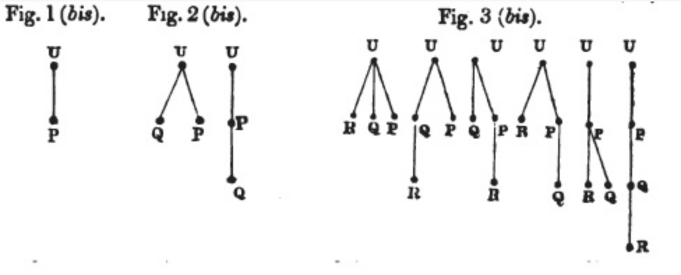
$$W_1^{\mathsf{rsym}} = \left\{ f\frac{d}{dx} \mid f \in K[x] \right\}$$

equipped with the multiplication

$$\left(f_1\frac{d}{dx}\right) * \left(f_2\frac{d}{dx}\right) = \left(f_2\frac{df_1}{dx}\right)\frac{d}{dx}$$

which is left-commutative.

The algebra W_1^{rsym} is left-commutative and right-symmetric.
(Right-symmetric algebras satisfy the polynomial identity $(x_1,x_2,x_3)=(x_1,x_3,x_2)$, where $(x_1,x_2,x_3)=(x_1x_2)x_3-x_1(x_2x_3)$ is the associator.)
Cayley also considered the realization of the right-symmetric Witt algebras W_d^{rsym} in terms of rooted trees.



An important subclass of the class of all one-sided commutative algebras is the class of *Gelfand-Dorfman-Novikov algebras*. These algebras were introduced in the 1970s and 1980s by Israel Gelfand and Irina Dorfman in their study of the Hamiltonian operator in finite-dimensional mechanics and by Alexander Balinsky and Sergei Novikov in relation with the equations of hydrodynamics.

Originally *Novikov algebras* are left-commutative and right-symmetric:

- A.A. Balinskii, S.P. Novikov, *Poisson brackets of hydrodynamic type, Frobenius algebras and Lie algebras* (Russian), Dokl. Akad. Nauk SSSR **283** (1985), No. 5, 1036-1039. Translation: Sov. Math., Dokl. **32** (1985), 228-231.

The opposite algebras of Novikov algebras are *Gelfand-Dorfman algebras*:

- I.M. Gel'fand, I.Ya. Dorfman, *Hamiltonian operators and algebraic structures structures related to them* (Russian), Funktsional. Anal. i Prilozhen. **13** (1979), No. 4, 13-30. Translation: Funct. Anal. Appl. **13** (1980), 248-262.

Askar Dzhumadil'daev, Nurlan Ismailov and Kaisar Tulenbaev described the free bicommutative algebra $F(\mathfrak{B})$ of countable rank and its main numerical invariants. In particular, they found a basis of $F(\mathfrak{B})$ as a K-vector space, computed the Hilbert series of the d-generated free bicommutative algebra $F_d(\mathfrak{B})$, the cocharacter (when char$K = 0$) and codimension sequences of the variety $\mathfrak{B}$.

- A.S. Dzhumadil'daev, N.A. Ismailov, K.M. Tulenbaev, *Free bicommutative algebras*, Serdica Math. J. **37** (2011), No. 1, 25-44.

Dzhumadil'daev, Ismailov and Tulenbaev showed:
Lemma. $F^2(\mathfrak{B})$ *is spanned by the products*

$$x_{i_1}(\cdots(x_{i_{m-1}}((\cdots(x_{i_m}x_{j_1})\cdots)x_{j_n}))\cdots), \quad m,n\geq 1,$$

and

$$x_{i_1}(\cdots(x_{i_{m-1}}((\cdots(x_{i_m}x_{j_1})\cdots)x_{j_n}))\cdots)$$
$$=x_{i_{\sigma(1)}}(\cdots(x_{i_{\sigma(m-1)}}((\cdots(x_{i_{\sigma(m)}}x_{j_{\tau(1)}})\cdots)x_{j_{\tau(n)}}))\cdots),\sigma\in S_m,\tau\in S_n,$$

where S_k is the symmetric group of degree k. Hence we may assume that

$$i_1\geq\cdots\geq i_{m-1}\geq i_m \text{ and } j_1\leq\cdots\leq j_n.$$

They proved more:

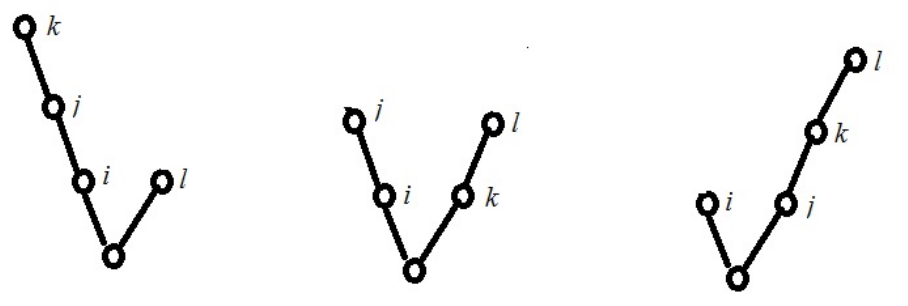
Theorem. *The products*

$$x_{i_1}(\cdots(x_{i_{m-1}}((\cdots(x_{i_m}x_{j_1})\cdots)x_{j_n}))\cdots), \quad m,n \geq 1,$$

$$i_1 \geq \cdots \geq i_{m-1} \geq i_m \text{ and } j_1 \leq \cdots \leq j_n.$$

form a basis of $F^2(\mathfrak{B})$.

The spanning elements of $F^2(\mathfrak{B})$ can be presented graphically as rooted binary trees.



$x_k(x_j(x_i x_l))$ $\quad$ $x_j((x_i x_k)x_l)$ $\quad$ $((x_i x_j)x_k)x_l$

$k \geq j \geq i$ $\quad$ $j \geq i,\ k \leq l$ $\quad$ $j \leq k \leq l$

Monomials of degree 4

If

$$x_{i_1}(\cdots(x_{i_{m-1}}((\cdots(x_{i_m}x_{j_1})\cdots)x_{j_n}))\cdots)\in F_d^2(\mathfrak{B}),$$

$$i_1\geq\cdots\geq i_{m-1}\geq i_m \text{ and } j_1\leq\cdots\leq j_n,$$

$$(i_m,i_{m-1},\ldots,i_1)=(\underbrace{1,\ldots,1}_{a_1\text{ times}},\ldots,\underbrace{d,\ldots,d}_{a_d\text{ times}}),a_1+\cdots+a_d\geq 1,$$

$$(j_1,\ldots,i_n)=(\underbrace{1,\ldots,1}_{b_1\text{ times}},\ldots,\underbrace{d,\ldots,d}_{b_d\text{ times}}),b_1+\cdots+b_d\geq 1,$$

then Dzhumadil'daev, Ismailov and Tulenbaev encoded the monomial $x_{i_1}(\cdots(x_{i_{m-1}}((\cdots(x_{i_m}x_{j_1})\cdots)x_{j_n}))\cdots)$ as

$$(a,b)=(a_1,\ldots,a_d;b_1,\ldots,b_d)\in\mathbb{N}_0^d\times\mathbb{N}_0^d.$$

Theorem. (Dzhumadil'daev, Ismailov, Tulenbaev) *Let G_d be the algebra with basis*

$$X_d \cup (\mathbb{N}_0^d \setminus (0,\ldots,0)) \times (\mathbb{N}_0^d \setminus (0,\ldots,0))$$

and multiplication rules

$$x_i * x_j = (\underbrace{0,\ldots,0}_{i-1 \text{ times}},1,0,\ldots,0;\underbrace{0,\ldots,0}_{j-1 \text{ times}},1,0,\ldots,0),$$

$$(a,b)*(c,d) = (a+c,b+d), a,b,c,d \in \mathbb{N}_0^d \setminus (0,\ldots,0),$$

$$x_i * (a_1,\ldots,a_d;b_1,\ldots,b_d) = (a_1,\ldots,a_i+1,\ldots,a_d;b_1,\ldots,b_d),$$

$$(a_1,\ldots,a_d;b_1,\ldots,b_d) * x_i = (a_1,\ldots,a_d;b_1,\ldots,b_i+1,\ldots,b_d).$$

Then $G_d \cong F_d(\mathfrak{B})$.

The *Hilbert* (or *Poincaré*) *series* of a $\mathbb{Z}$-graded vector space $V = \bigoplus_{n\geq 0} V^{(n)}$ is

$$H(V,t) = \sum_{n\geq 0} \dim(V^{(n)})t^n.$$

In a similar way one defines the Hilbert series of a $\mathbb{Z}^d$-graded vector space.

Corollary. *The Hilbert series of $F_d(\mathfrak{B})$ as a $\mathbb{Z}$-graded and a $\mathbb{Z}^d$-graded algebra are, respectively*

$$H(F_d(\mathfrak{B}),t) = dt + \left(\frac{1}{(1-t)^d} - 1\right)^2,$$

$$H(F_d(\mathfrak{B}),t_1,\ldots,t_d) = t_1 + \cdots + t_d + \left(\prod_{i=1}^{d}\frac{1}{(1-t_i)} - 1\right)^2.$$

Corollary. *The square* $F_d^2(\mathfrak{B})$ *of* $F_d(\mathfrak{B})$ *is an associative commutative algebra isomorphic to* $((\mathbb{N}_0^d \setminus (0,\dots,0)) \times (\mathbb{N}_0^d \setminus (0,\dots,0)), +)$.

In virtue of this corollary it is naturally to try to involve the powerful classical methods of commutative algebra in the study of bicommutative algebras. This idea was further developed by the speaker and Bekzat Zhakhayev.

- V. Drensky, B.K. Zhakhayev, *Noetherianity and Specht problem for varieties of bicommutative algebras*, J. Algebra **499** (2018), 570-582.

First step: Translation in the language of commutative algebra

We consider the polynomial algebra $K[Y_d, Z_d] = K[y_1, \dots, y_d, z_1, \dots, z_d]$. Let

$$Y_d^a = y_1^{a_1} \cdots y_d^{a_d}, Z_d^b = z_1^{b_1} \cdots z_d^{b_d}, |a| = a_1 + \cdots + a_d, |b| = b_1 + \cdots + b_d.$$

We identify the algebra G_d with the algebra with basis

$$X_d \cup \{Y_d^a Z_d^b \mid |a|, |b| \geq 1\}$$

and multiplication rules

$$x_i x_j = y_i z_j, x_i(Y_d^a Z_d^b) = y_i Y_d^a Z_d^b, (Y_d^a Z_d^b) x_j = Y_d^a Z_d^b z_j,$$

$$(Y_d^a Z_d^b)(Y_d^c Z_d^d) = Y_d^{a+c} Z_d^{b+d}.$$

Proof of the theorem for the basis of $F_d(\mathfrak{B})$.

- The algebra G_d is bicommutative (easy verification).
- It is a homomorphic image of $F_d(\mathfrak{B})$. Both algebras are graded and hence the homogeneous component $F_d^{(n)}(\mathfrak{B})$ of degree n maps onto the homogeneous component $G_d^{(n)}$. Hence $\dim(F_d^{(n)}(\mathfrak{B})) \geq \dim(G_d^{(n)})$.
- There is a 1-to-1 correspondence between the monomials

$$x_{i_1}(\cdots(x_{i_{m-1}}((\cdots(x_{i_m}x_{j_1})\cdots)x_{j_n}))\cdots) \in F_d^{(n)}(\mathfrak{B})$$

  and the corresponding monomials $Y_d^a Z_d^b$. Since the $Y_d^a Z_d^b$s are linearly independent in G_d, the same holds for their preimages in $F_d(\mathfrak{B})$, i.e. the above monomials in $F_d^{(n)}(\mathfrak{B})$ are also linearly independent and $\dim(F_d^{(n)}(\mathfrak{B})) = \dim(G_d^{(n)})$. Hence $F_d(\mathfrak{B})$ and G_d are isomorphic.

Since the square $F^2(\mathfrak{B})$ of the algebra $F(\mathfrak{B})$ is a commutative and associative algebra, one should expect that the algebra $F(\mathfrak{B})$ itself has many properties typical for commutative and associative algebras.

(Counter)example. (V.D. & B. Zhakhayev) *The free bicommutative algebra $F_1 = F_1(\mathfrak{B})$ is not noetherian. Its left ideal generated by the monomials*

$$y_1 z_1^{\delta}, \quad \delta = 1, 2, \dots,$$

is not finitely generated.

Theorem. (V.D. & B. Zhakhayev) *Finitely generated bicommutative algebras are weakly noetherian, i.e., satisfy the ascending chain condition for two-sided ideals.*

Idea of the proof

It is sufficient to work in the square $F_d^2(\mathfrak{B})$ of the d-generated free algebra $F_d(\mathfrak{B})$. The left and right multiplications by the generators $X_d = \{x_1, \dots, x_d\}$ of $F_d(\mathfrak{B})$ make $F_d^2(\mathfrak{B})$ a $K[Y_d, Z_d]$-module generated by the finite number of monomials $y_i z_j$, $i, j = 1, \dots, d$. Since the ideals $I \subset F_d^2(\mathfrak{B})$ of $F_d(\mathfrak{B})$ are $K[Y_d, Z_d]$-modules of $F_d^2(\mathfrak{B})$, they are also finitely generated.

The Specht property in any characteristic

Theorem. (V.D. & B. Zhakhayev) *Over a field of arbitrary characteristic every variety of bicommutative algebras is finitely based.*

Methods of the proof

We use the classical method of Higman-Cohen which, up to the 1970s, was one of the few methods to handle into affirmative the Specht problem:

- G. Higman, *Ordering by divisibility in abstract algebras*, Proc. Lond. Math. Soc., III. Ser. **2** (1952), 326-336.
- D.E. Cohen, *On the laws of a metabelian variety*, J. Algebra **5** (1967), 267-273.

Partially well ordered sets

The set A is *partially ordered* (or a *poset*, if it is equipped with a binary relation $\preceq$ such that

- $a \preceq a$ for all $a \in A$ (reflexivity);
- if $a \preceq b$ and $b \preceq a$, then $a = b$ (antisymmetry);
- if $a \preceq b$ and $b \preceq c$, then $a \preceq c$ (transitivity).

The poset $(A, \preceq)$ is *partially well ordered* if for any subset B of A there is a finite subset B_0 of B such that for every $b \in B$ there exists a $b_0 \in B_0$ such that $b_0 \preceq b$. Equivalently:

- A does not contain infinite descending chains;
- A does not have infinitely many pairwise incomparable elements.

The Higman lemma. *Let* $(A, \preceq)$ *be a partially ordered set and let* $(\overline{A}, \preceq)$ *be the set of all finite sequences of elements of* A *with the following partial ordering:*

$$(a_1, \dots, a_m) \preceq (b_1, \dots, b_n), \quad a_i, b_j \in A,$$

if there exists a subsequence $(b_{j_1}, \dots, b_{j_m})$ *such that* $a_i \preceq b_{j_i}$*,* $i = 1, \dots, m.$
If $(A, \preceq)$ *is partially well ordered, then* $(\overline{A}, \preceq)$ *is also partially well ordered.*

Corollary. *We define a partial ordering on the set* $[Y, Z]$ *of all monomials* $Y^\alpha Z^\beta$ *by*

$$Y^\alpha Z^\beta = y_1^{\alpha_1} \cdots y_m^{\alpha_m} z_1^{\beta_1} \cdots z_m^{\beta_m} \preceq y_1^{\gamma_1} \cdots y_n^{\gamma_n} z_1^{\delta_1} \cdots z_n^{\delta_n} = Y^\gamma Z^\delta$$

if there exists a sequence $i_1 < \cdots < i_m$ *such that* $y_{i_1}^{\alpha_1} \cdots y_{i_m}^{\alpha_m} z_{i_1}^{\beta_1} \cdots z_{i_m}^{\beta_m}$ *divides* $Y^\gamma Z^\delta$. *Then the set* $[Y, Z]$ *is partially well ordered.*

Meaning of the corollary and the end of the proof of the theorem

Any sequence $i_1 < i_2 < \cdots$ of positive integers defines an endomorphism of the multiplicative semigroup $[Y, Z]$. The corollary gives the noetherianity of $[Y, Z]$ with respect to the ideals which are invariant under such endomorphisms. Then the proof of the theorem for the Specht property of the varieties of bicommutative algebras repeats the final steps of the proof of the Hilbert basis theorem.

Now we assume that the base field K is of characteristic 0. The symmetric group S_n acts on the vector space $P_n(\mathfrak{B})$ of the multilinear polynomials of degree n in $F(\mathfrak{B})$:

$$\sigma : f(x_1, \ldots, x_n) \to f(x_{\sigma(1)}, \ldots, x_{\sigma(n)}), \sigma \in S_n, f(X_n) \in P_n(\mathfrak{B}),$$

and $P_n(\mathfrak{B})$ has the structure of a left KS_n-module. One of the most important numerical invariants of a variety $\mathfrak{V}$ of algebras in characteristic 0 are the *codimension sequence*

$$c_n(\mathfrak{V}) = \dim(P_n(\mathfrak{V})), n = 1, 2, \ldots,$$

and the *cocharacter sequence*

$$\chi_n(\mathfrak{V}) = \chi_{S_n}(P_n(\mathfrak{V})), n = 1, 2, \ldots.$$

The irreducible characters χ_λ of the symmetric group S_n are indexed by partitions λ of n:

$$\lambda = (\lambda_1, \ldots, \lambda_n) \vdash n, \lambda_1 \geq \cdots \geq \lambda_n \geq 0, \lambda_1 + \cdots + \lambda_n = n,$$

and

$$\chi_n(\mathfrak{V}) = \sum_{\lambda \vdash n} m_\lambda \chi_\lambda, n = 1, 2, \ldots .$$

Theorem. (Dzhumadil'daev, Ismailov, Tulenbaev) *If* $\mathrm{char}(K) = 0$, *then*

$$c_n(\mathfrak{B}) = \begin{cases} 1, & \text{if } n = 1, \\ 2^n - 2, & \text{if } n > 1; \end{cases}$$

$$\exp(\mathfrak{B}) = \lim_{n\to\infty} \sqrt[n]{c_n(\mathfrak{B})} = 2;$$

$$\chi_n(\mathfrak{B}) = \begin{cases} \chi_{(1)}, & \text{if } n = 1, \\ (n-1)\chi_{(n)} + \sum_{i=1}^{[(n-1)/2]}(n-2i+1)\chi_{(n-i,i)}, & \text{if } n > 1. \end{cases}$$

Corollary. *The variety* $\mathfrak{B}$ *is generated by* $F_2(\mathfrak{B})$.

Idea of the proof

We combine ideas of the original proof with the approach in

- V. Drensky, *Varieties of bicommutative algebras*, Serdica Math. J. **45** (2019), No. 2, 167-188.

Identifying the algebras $F_d(\mathfrak{B})$ and G_d and working in G_d, the vector space $P_n(\mathfrak{B})$, $n \geq 2$, has a basis

$$y_{i_1} \cdots y_{i_k} z_{j_1} \cdots z_{j_{n-k}}, \quad i_1 < \cdots < i_k, j_1 < \cdots < j_{n-k},$$

$$\{i_1, \ldots, i_k, j_1, \ldots, j_{n-k}\} = \{1, \ldots, n\}, k = 1, \ldots, n-1.$$

Hence for $n \geq 2$

$$c_n(\mathfrak{B}) = \dim(P_n(\mathfrak{B})) = \sum_{k=1}^{n-1} \binom{n}{k} = 2^n - 2.$$

This also gives

$$\exp(\mathfrak{B}) = \lim_{n\to\infty} \sqrt[n]{c_n(\mathfrak{B})} = 2.$$

The general linear group $GL_d(K)$ acts canonically on the vector space KX_d with basis X_d and for a variety $\mathfrak{V}$ this action is extended diagonally on $F_d(\mathfrak{V})$. As a $GL_d(K)$-module

$$F_d(\mathfrak{V}) \equiv \bigoplus_{n\geq 1} \bigoplus_{\lambda\vdash n} m_\lambda W_d(\lambda),$$

where $W_d(\lambda)$ is the irreducible polynomial $GL_d(K)$-module indexed by the partition λ. It is known (a result of Allan Berele and the speaker from the early 1980s) that m_λ is the same as the multiplicity of χ_λ in the cocharacter sequence of $\mathfrak{V}$. This can be used to compute the cocharacter sequence.

Working in G_d again, as a $GL_d(K)$-module

$$G_d^2 \cong \omega(K[Y_d]) \otimes \omega(K[Z_d]),$$

where $\omega(K[Y_d])$ is the augmentation ideal of $K[Y_d]$ (the ideal of polynomials without constant term) and similarly for $\omega(K[Z_d])$. It is well known that

$$\omega(K[Y_d]) \cong \bigoplus_{m\geq 1} W_d(m)$$

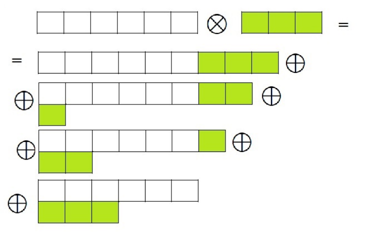
which implies that

$$G_d^2 \cong \bigoplus_{m,n\geq 1} W_d(m) \otimes W_d(n).$$

By the Young rule (a partial case of the Littlewood-Richardson rule for the tensor product of irreducible polynomial $GL_d(K)$-modules), if $m \geq n$, then

$$W_d(m) \otimes W_d(n) \cong \bigoplus_{k=0}^{n} W_d(m+n-k,k).$$

In the language of Young diagrams:



The formula for $\chi_n(\mathfrak{B})$ follows by easy counting of number of $W_d(m+n-k,k)$.

Other results for the variety $\mathfrak{B}$

Theorem. (V.D.) *The variety $\mathfrak{B}$ is generated by the two-dimensional algebra with basis $\{r, r^2\}$ and multiplication rules*

$$r \cdot r^2 = r^2, r^2 \cdot r = -r^2, r^2 \cdot r^2 = -r^2.$$

Theorem. (V.D.) *The variety $\mathfrak{B}$ is a minimal variety of exponent 2. If $\mathfrak{V} \subsetneqq \mathfrak{B}$, then the codimension sequence $c_n(\mathfrak{V})$, $n = 1, 2, \ldots$, is of polynomial growth and $\exp(\mathfrak{V}) = 1$.*

From now on we assume that the base field K is of arbitrary characteristic.

Gröbner bases

Let the semigroup $[X_d]$ of the monomials in d associative commutative variables be *well ordered* with ordering $\prec$, which is *compatible with the multiplication*, i.e. the ordering $\prec$ is linear, every subset of $[X_d]$ has a minimal element and

$$u \prec v \Rightarrow uw \prec vw \text{ for any } w \in [X_d],$$

If $0 \neq f(X_d) \in K[X_d]$, then we denote by $\overline{f}$ the leading monomial of $f(X_d)$ (with respect to the ordering $\prec$). Similarly, if I is an ideal of the polynomial algebra $K[X_d]$, then $\overline{I}$ is the set of the leading monomials of the polynomials in I.
The subset G of the ideal I is a *Gröbner basis* of I if for every $0 \neq f(X_d) \in I$ there exists $f_i(X_d) \in G$ such that $\overline{f_i}$ divides $\overline{f}$.

Gröbner bases were introduced by Bruno Buchberger in his Ph.D. Thesis and are named after his advisor Wolfgang Gröbner.

- B. Buchberger, *Ein Algorithmus zum Auffinden der Basiselemente des Restklassenringes nach einem nulldimensionalen Polynomideal*, Innsbruck: Univ. Innsbruck, Mathematisches Institut (Diss.), 1965.

Bruno Buchberger

Wolfgang Gröbner

It follows from the Hilbert Basisatz applied to the monomial ideal of $K[X_d]$ generated by $\overline{I}$ (or by the Dickson lemma), that every ideal of $K[X_d]$ has a finite Gröbner basis.

2. LEMMA A. *Any set S of functions of the type*

$$F = x_1^{e_1} x_2^{e_2} \dots x_n^{e_n} \qquad (e\text{'s integers} \geqq 0) \tag{1}$$

contains a finite number of functions $F_1, \dots, F_k$ such that each function F of the set S can be expressed as a product $F_i f$, where f is of the form (1), *but is not necessarily in the set S.*

- L.E. Dickson, *Finiteness of the odd perfect and primitive abundant numbers with n distinct prime factors*, Am. J. Math. **35** (1913), 413-422.

Dickson mentioned that his lemma follows from the Hilbert Basissatz applied to monomial ideals.

Algorithm of Buchberger

Let the ideal I of $K[X_d]$ be generated by the set $G = \{f_1, \ldots, f_p\}$. If

$$f_1(X_d) = \overline{f_1} + \cdots, f_2(X_d) = \overline{f_2} + \cdots \in G,$$

then we apply one of the two operations:

- If $\overline{f_2} = \overline{f_1} u(X_d)$ for some monomial $u(X_d) \in [X_d]$, then we replace in G the polynomial $f_2(X_d)$ with $f_2(X_d) - f_1(X_d)u(X_d)$.
- If $\overline{f_1} u(X_d) = \overline{f_2} v(X_d)$ for some monomials $1 \neq u(X_d), v(X_d) \in [X_d]$, then we add to G the polynomial

$$f_1(X_d)u(X_d) - f_2(X_d)v(X_d).$$

The process stops after a finite number of steps and the obtained set of polynomials G is the Gröbner basis of the ideal I.

Gröbner bases are widely used for many computations in commutative algebra and algebraic geometry.

- The simplest example: The factor algebra $K[X_d]/I$ has a vector space basis consisting of all monomials which are not divisible by any leading monomial of the polynomials in the Gröbner basis of I.

There are standard packages which compute Gröbner bases of ideals of $K[X_d]$.

Ideas, similar to the ideas of Buchberger appeared also in the work of other mathematicians, some of them even before him. But the significance of the contributions of Buchberger are undeniable. In particular, his algorithm is in the base of many of the computations with polynomials.

- H. Hironaka, *Resolution of singularities of an algebraic variety over a field of characteristic zero, I, II*, Ann. Math. (Second Ser.) **79** (1964), 109-203, 205-326.
- N.M. Günter, *Über einige Zusammenhnge zwischen den homogenen Gleichungen* (Russian), Sbornik Inst. Inzh. Putej Soobshct. (Samml. des Inst. der Verkehrwege) **84** (1913), 1-20. JFM 44.0156.01.
- N. Gunther, *Sur les modules des formes algébriques*, Travaux de L'Institut Math. de Tbilissi **9** (1941), 97-206.

Heisuke Hironaka

Nikolai Günter

Nikolai Günter is known mainly for his results in differential and integral equation, analysis, mathematical physics and hydrodynamics. Maybe due to this his papers in commutative algebra were overlooked by the experts. A detailed description of his contributions to algebra can be found in

- B. Renschuch, H. Roloff, G.G. Rasputin, *Beiträge zur konstruktiven Theorie der Polynomideale. XXIII: Vergessene Arbeiten des Leningrader Mathematikers N. M. Gjunter zur Theorie der Polynomideale*, Wiss. Z. Pädagog. Hochsch. Karl Liebknecht, Potsdam **31** (1987), 111-126.
  Translation (M. Abramson): *Contributions to constructive polynomial ideal theory XXIII: Forgotten works of Leningrad mathematician N.M. Gjunter on polynomial ideal theory*, ACM SIGSAM Bulletin, **37** (2003), No 2, 35-48.

Investigations in the spirit of those of Buchberger appeared independently also in noncommutative algebra. In 1962 Anatoliĭ Shirshov developed his algorithmic approach to Lie algebras. His approach works in a much larger cases and nowadays one speaks about Gröbner-Shirshov bases.

- A.I. Shirshov, *Some algorithm problems for Lie algebras* (Russian), Sibirsk. Mat. Zh., **3** (1962),292-296.

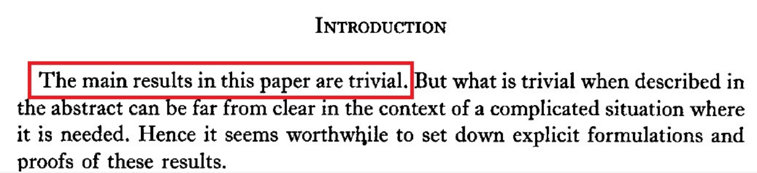
The case for associative algebras was developed by George Bergman in 1978.

- G.M. Bergman, *The diamond lemma for ring theory*, Adv. Math. **29** (1978), 178-218.

Anatoliĭ Shirshov

George Bergman

The paper by Bergman has 41 pages and starts very interesting:


INTRODUCTION

$\boxed{\text{The main results in this paper are trivial.}}$ But what is trivial when described in the abstract can be far from clear in the context of a complicated situation where it is needed. Hence it seems worthwhile to set down explicit formulations and proofs of these results.

Recently (the paper was posted in arXiv in July 2021) Yuxiu Bai, Yuqun Chen and Zerui Zhang established:

Theorem. *The ideals of finitely generated free bicommutative algebras have finite Gröbner-Shirshov bases.*

- Y. Bai, Y. Chen, Z. Zhang, *Gelfand-Kirillov dimension of bicommutative algebras*, Linear and Multilinear Algebra, published online 07 Nov. 2021.

Their results are a demonstration of the power of the methods of Shirshov for the study of ideals of nonassociative algebras.

The theorem of Bai, Chen and Zhang can be obtained as a corollary of the corresponding theorem in the commutative case. If we identify $F_d^2(\mathfrak{B})$ with the ideal $\omega(K[Y_d])\omega(K[Z_d])$ of $K[Y_d, Z_d]$ generated by all monomials $Y_d^a Z_d^b$, $|a|, |b| > 0$, and introduce an arbitrary linear ordering on the monomials of $F_d(\mathfrak{B})$, then the set $\overline{I}$ of the leading monomials of the ideal I of $F_d(\mathfrak{B})$ consists of a finite number of elements in X_d and of the ideal $\overline{(F_d(\mathfrak{B}) \cap I)} \cap [Y_d, Z_d]$ of the associative commutative semigroup $[Y_d, Z_d]$. By the Hilbert Basissatz or by the Dickson lemma this ideal has a finite number of generators. This immediately implies the theorem because a subset of $\{Y_d^a Z_d^b \mid |a|, |b| > 0\}$ is an ideal of $[Y_d, Z_d]$ if and only if it is stable under multiplications from both sides by elements in X_d.

Gelfand-Kirillov dimension

Let R be an arbitrary finitely generated algebra and let V be the subspace of R generated by an arbitrary finite system of generators. The *growth function* of R with respect to V is defined as follows:

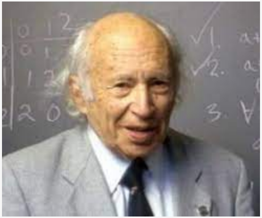
$$g_V(n) = \dim(V^0 + V^1 + \cdots + V^n), n = 1, 2, \ldots,$$

where $V^0 = K$ for unitary algebras and $V^0 = 0$ for algebras without 1. The *Gelfand-Kirillov dimension* of R is defined as

$$\mathsf{GKdim}(R) = \limsup_{n\to\infty} \log_n(g_V(n))$$

and does not depend on the choice of the finite generated system of R.

The Gelfand-Kirillov dimension was introduced by Israel Gelfand and Alexander Kirillov in 1966.



Israel Gelfand

Alexander Kirillov

Announced:

- I.M. Gel'fand, A.A. Kirillov, *Fields associated with enveloping algebras of Lie algebras* (Russian), Doklady AN SSSR **167** (1966), No. 3, 503-507. Translation: Sov. Math., Dokl. **7** (1966), 407-409.

Published with complete proofs:

- I.M. Gelfand, A.A. Kirillov, *Sur les corps liés aux algèbres enveloppantes des algèbres de Lie*, Publ. Math., Inst. Hautes Étud. Sci. **31** (1966), 506-523.

It is well known that the Gelfand-Kirillov dimension of a finitely generated associative commutative algebra is a nonnegative integer.
In their paper Bai, Chen and Zhang give a simple combinatorial algorithm for computing the Gelfand-Kirillov dimension of the factor algebra $K[X_d]/I$. The input consists of the leading monomials of the Gröbner basis of the ideal I and the output is $\text{GKdim}(K[X_d]/I)$.
For a finite nonempty set S of monomials in $[X_d]$ we define

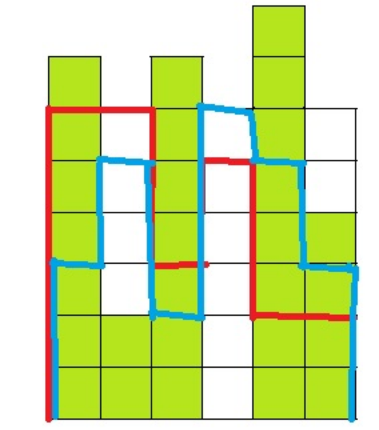
$$C_S = \min\{n \in \mathbb{N} \mid s \text{ divides } x_1^n \cdots x_d^n, s \in S\}$$

and for $u = x_1^{n_1} \cdots x_d^{n_d} \in [X_d]$

$$\text{Sdeg}(u) = \#\{i \mid n_i > C_S, i = 1, \ldots, d\}.$$

If I is the ideal of $[X_d]$ generated by S and $\text{Irr}(S) = [X_d] \setminus I$, then

$$\text{Sdeg}(\text{Irr}(S)) = \max\{\text{Sdeg}(u) \mid u \in \text{Irr}(S)\}.$$



$s_1 = x_1^6 x_2^6 x_3^3 x_4^5 x_5^2 x_6^2$, $s_2 = x_1^3 x_2^5 x_3^2 x_4^6 x_5^5 x_6^3$,

$u = x_1^7 x_2^2 x_3^7 x_5^8 x_6^4$, $\mathsf{Sdeg}(u) = 3$

Theorem. (Bai, Chen and Zhang) *Let* $\mathrm{Gröbner}(I)$ *be a Gröbner basis of the nontrivial ideal* I *of* $K[X_d]$ *and let* S *be the set of the leading monomials of* $\mathrm{Gröbner}(I)$*. Then*

$$\mathrm{GKdim}(K[X_d]/I) = \mathrm{Sdeg}(\mathrm{Irr}(S)).$$

Remark. As the authors state it is sufficient to consider only these elements $u \in \mathsf{Irr}(S)$ with the property

$$\deg_{x_i} \leq C_S + 1, i = 1, \ldots, d,$$

i.e. it is sufficient to check only $\leq (C_S + 2)^d$ monomials.

Idea of the proof. If

$$u = x_1^{n_1} \cdots x_d^{n_d} \in \mathsf{Irr}(S), \mathsf{Sdeg}(u) = \mathrm{Sdeg}(\mathrm{Irr}(S)) = m,$$

then only m of the indices $n_1, \ldots, n_d$ are larger than C_S. Let for example $n_1, \ldots, n_m > C_S$. Then all monomials $x_1^{k_1} \cdots x_m^{k_m}$, $k_i \geq 0$, belong to $\mathsf{Irr}(S)$. Hence the algebra $K[X_d]/I$ grows not slower than the polynomial algebra $K[X_m]$ and

$$\mathsf{GKdim}(K[X_d]/I) \geq \mathsf{GKdim}(K[X_m]) = m.$$

On the other hand in the monomials $x_1^{n_1} \cdots x_d^{n_d} \in \mathsf{Irr}(S)$ not more than m degrees n_i are greater than C_S. Hence $K[X_d]/I$ behaves as a finitely generated $K[X_m]$-module and

$$\mathsf{GKdim}(K[X_d]/I) \leq \mathsf{GKdim}(K[X_m]) = m.$$

As a consequence of the existence of a Gröbner-Shirshov bases of the ideals of $F_d(\mathfrak{B})$ and their algorithm for computing of the Gelfand-Kirillov dimension of the ideals of $K[X_d]$ Bai, Chen and Zhang prove:

Theorem. *Finitely generated bicommutative algebras have an integral Gelfand-Kirillov dimension.*

Proof from commutative point of view.
Let $R = F_d(\mathfrak{B})/I$ be a finitely generated bicommutative algebra, let V be the vector space spanned by the images of X_d in R and let

$$g_V(n) = V + V^2 + \cdots + V^n, \quad n = 1, 2, \ldots,$$

be the growth function of R with respect to V.
We consider a linear ordering first by degree of the monomials and then in an arbitrary (e.g. lexicographic) ordering. Let $\overline{I}$ be the set of the leading monomials of the ideal I and $S = F_d(\mathfrak{B})/(\overline{I})$ be the factor algebra modulo the monomial ideal generated by $\overline{I}$. It is well known that the algebras R and S have the same Gelfand-Kirillov dimension because they have the same growth functions.

The algebra S is graded and its homogeneous component $S^{(n)}$ of degree n is of dimension equal to the number of monomials of degree n in $F_d(\mathfrak{B})$ which are not in $\bar{I}$, i.e. $h(n) = g_V(n) - g_V(n-1)$, $n = 2, 3, \ldots$, (and $h(1) = g_V(1)$). The Hilbert series of S is

$$H(S,t) = \sum_{n\geq 1} \dim(S^{(n)})t^n = \sum_{n\geq 1} h(n)t^n$$

$$= g_V(1)t + \sum_{n\geq 2}(g_V(n) - g_V(n-1))t^n.$$

The algebra $S' = \sum_{n\geq 2} S^{(n)}$ is a $K[Y_d, Z_d]$-module generated by the images of the products y_iz_j, $i, j = 1, \ldots, d$, in S. By the Hilbert-Serre theorem the Hilbert series of S'

$$H(S', t) = \sum_{n\geq 2} h(n)t^n$$

is a rational function with denominator which is a product of binomials of the form $1 - t^a$, i.e.

$$H(S, t) = g_V(1)t + p(t) \prod_{m=1}^{b} \frac{1}{(1 - t^a)^{c^a}}, p(t) \in \mathbb{Z}[t].$$

It is well known that if the Hilbert series of an algebra is a rational function with polynomial growth of the coefficients and has a pole of the m-th order for $t = 1$, then the Gelfand-Kirillov dimension of the algebra is equal to m.

- V.A. Ufnarovskij, Combinatorial and asymptotic methods in algebra (Russian), Itogi Nauki Tekh., Ser. Sovrem. Probl. Mat., Fundam. Napravleniya 57 (1990), 5-177. English translation: Algebra VI. Encycl. Math. Sci. 57 (1995), 1-196 (1995).

Hence the bicommutative algebra S has an integral Gelfand-Kirillov dimension and the same holds for the fintely generated algebra R.

I was a referee of the paper by Bai, Chen and Zhang and recommended them to choose one of the following options:

- *To revise the paper in the spirit of my talk today.*
  The authors may revise the paper as I suggest. The advantage of this solution is that the paper will be easier to follow by algebraists who are not experts in bicommutative algebras. Another advantage is that the paper will be much shorter than now.

- *To leave the text as it was and to add an appendix, and I am ready to write the appendix.*
  The authors may leave the paper as it is taking into account the other comments only. The advantage now is that the authors will demonstrate the power of the methods typical for the theory of Gröbner-Shirshov bases. In this case it would be nice if they add an appendix "Suggestions of the referee for alternative approach". I also can write the appendix but in this case the authors will know who was the referee. The disadvantage of having an appendix is that the paper will be too long.

- *Finally, the authors may leave the text as it is but after the publishing of their paper I may write a short article with title close to the title of my talk.*

  Finally, if the authors decide to leave the things without changing and without an appendix, in some moment after the paper is published, I may write a short paper "Gröbner-Shirshov bases and Gelfand-Kirillov dimension of bicommutative algebras – from the point of view of commutative algebra". I think that after the publishing of the paper this will be not against the restriction that the referee cannot use the reviewed paper for other purposes. Of course, in this case the authors will also know who was the referee.

The answer of the authors was:

- The referee's proofs are very interesting, and according to the referee's report, we decide not to rewrite the sections dealing with bicommutative algebras and not to add an appendix. We believe that it is better to revise the article in this way, so that the readers will know the referee's ideas from the referee's own article.

Instead of conclusion

During most of the talk we commented results initially obtained with methods typical for nonassociative algebra and how they can be obtained naturally with classical methods of commutative algebra.
MY OPINION IS THAT THESE RESULTS ARE VERY INTERESTING AND THEIR INITIAL PROOFS ARE HIGHLY NONTRIVIAL.
BUT ONCE THE RESULTS WERE OBTAINED IT WAS NATURAL TO LOOK AT THEM ALSO FROM ANOTHER POINT OF VIEW.

THANK YOU VERY MUCH FOR YOUR ATTENTION!