Modal definability of some classes of modal products

Two commuting equivalence relations

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Motivation

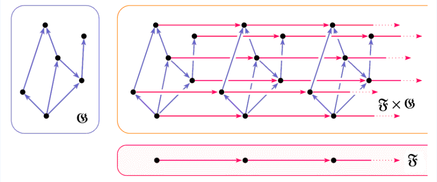
Motivation, 1

- The modal logic S5 is very interesting from different viewpoints (epistemic logic, AI, philosopy and theology, etc.).
- S5 axiomatizes the one-variable fragment of FOL[1], the validity problem w.r.t. it is decidable and S5 is Kripke complete (for example w.r.t. the class of all partitions $\mathcal{K}_{\text{partition}}$).
- It is also finitely axiomatizable and has the polynomial finite model property **FMP**.
- The modal definability problem w.r.t. the class of all partitions is **PSPACE-complete** as well as the correspondence problem and the first-order definability problem is in **PSPACE** w.r.t. the same class.
- A surprise is that the validity problem w.r.t. the class of structures having two equivalence relations $\mathcal{K}_{2S5}$ is undecidable and consequently the modal definability problem.
- But if we add the condition that the two equivalence relations are in local agreement both problems become decidable!
- Since S5 enjoys so many interesting properties, it is natural that the products of logics of the type $L \times S5$ for L a normal modal logic, will be subjected to study. That is how arrive to the current study of $S5 \times S5$ ($S5^2$).

Definition (Product of structures)

We will use two types of products of structures in this work: direct product (classical model theory definition), denoted $\times$ and the modal product $\underset{mod}{\times}$.
If $\mathfrak{F} = \langle W, R\rangle$ and $\mathfrak{G} = \langle U, S\rangle$ be two Kripke structures, then $\mathfrak{F} \underset{mod}{\times} \mathfrak{G} = \langle F \times G, \mathbb{H}, \mathbb{V}\rangle$ is called **the modal product of** $\mathfrak{F}$ **and** $\mathfrak{G}$ is a Kripke frame defined as follows:

- The universe is $F \times G$;
- $\langle\langle a_1, b_1\rangle, \langle a_2, b_2\rangle\rangle \in \mathbb{H} \iff [\langle a_1, a_2\rangle \in R \,\&\, b_1 = b_2]$;
- $\langle\langle a_1, b_1\rangle, \langle a_2, b_2\rangle\rangle \in \mathbb{V} \iff [a_1 = a_2 \,\&\, \langle b_1, b_2\rangle \in S]$;



Definition (Product of modal logics)

Let L_1 and L_2 be two Kripke complete unimodal logics.
Then their **product** is defined as following:

$$L_1 \times L_2 = \mathrm{Log}(\{\mathfrak{F}_1 \underset{mod}{\times} \mathfrak{F}_2 \mid \mathfrak{F}_1 \in \mathrm{Fr}(L_1) \,\&\, \mathfrak{F}_2 \in \mathrm{Fr}(L_2)\}).$$

- The language of $S5^2$ is the propositional language based on a fixed countably infinite set of propositional variables and equipped with the two modal operators $\blacksquare$ (vertical) and $\blacksquare$ (horizontal).
- Satisfiability of modal formulae in $S5^2$ is **NEXPTIME-complete** and validity of modal formulae is **co-NEXPTIME-complete**;
- A key feature of $S5^2$ is that it corresponds to the equality and substitution free fragment of two-variable first-order logic FOL^2, via the standard translation of modal formulae to first-order formulae. There is a wide variety of proofs available of the decidability of FOL^2;
- $S5^2$ has the exponential **FMP**;
- $S5^2$ is Kripke complete w.r.t. to the class of all structures with two commuting equivalence relations and other;
- It is interesting that $S5 \times S5 \times S5$ has an undecidable satisfiability problem. In general $S5^n$ for $n \in \mathbb{N}_{>2}$ has an undecidable satisfiability problem;

- Let $\mathcal{K}_{\text{commute}}$ be the class of structures for the language $\mathfrak{L}(R_1, R_2, \doteq)$ in which the two equivalence relations commute and let $\mathcal{K}_{\text{partition}}$ be the class of structures for the language $\mathfrak{L}(R, \doteq)$ of all partitions.
- We take an interest in $\mathcal{K}_{\text{commute}}$ and some of its subclasses. S5^2 is Kripke complete w.r.t. it and the models over the structures from the class $\mathcal{K}_{\text{commute}}$ we will call **the nonstandard models of this logic**.
- Let us define two other subclasses of this class of structures which have more “standard” modal semantics: $\mathcal{K}_{\text{rectangle}} \leftrightharpoons \{\mathfrak{A}_1 \underset{mod}{\times} \mathfrak{A}_2 \mid \mathfrak{A}_1, \mathfrak{A}_2 \in \mathcal{K}_{\text{partition}}\}$ and $\mathcal{K}_{\text{square}} \leftrightharpoons \{\mathfrak{A} \underset{mod}{\times} \mathfrak{A} \mid \mathfrak{A} \in \mathcal{K}_{\text{partition}}\}$.
- S5^2 also is Kripke complete w.r.t. $\mathcal{K}_{\text{rectangle}}$ and $\mathcal{K}_{\text{square}}$, i.e., $\text{Log}(\mathcal{K}_{\text{commute}}) = \text{Log}(\mathcal{K}_{\text{rectangle}}) = \text{Log}(\mathcal{K}_{\text{square}}) = \text{S5} \times \text{S5}$.

- Our main goal is to study the modal definability problem w.r.t. $\mathcal{K}_{\text{rectangle}}$ and $\mathcal{K}_{\text{square}}$. We need also study the modal definability problem w.r.t. $\mathcal{K}_{\text{commute}}$ to get a better understanding of the former.
- These classes almost fit the criteria of the definition of a **stable class** conjured by Balbiani and Tinchev, so we will call them **pre-stable** (by modifying the methods for stable classes).
- By being pre-stable this means that we can reduce the problem of deciding the validity of sentences in each of the class in question to the modal definability problem w.r.t. the same class. Alas, this only gives us a lower bound of the complexity of the modal definability problems w.r.t. each of these classes.

Definition (Modal definability problem w.r.t. a class of structures $\mathcal{K}$)

Let $\mathcal{K}$ be a class of structures for a relational FOL language $\mathfrak{L}$. A FOL sentence φ **defines** a modal formula $\mathtt{A}$ **w.r.t.** $\mathcal{K}$ or alternatively $\mathtt{A}$ **defines** φ **w.r.t.** $\mathcal{K}$ if for every structure $\mathfrak{F}$: $$(\forall \mathfrak{F} \in \mathcal{K})[\mathfrak{F} \models \mathtt{A} \iff \mathfrak{F} \models \varphi].$$

A FOL sentence φ is called **modally definable w.r.t.** $\mathcal{K}$ if there exists a modal formula $\mathtt{A}$ which defines her w.r.t. $\mathcal{K}$. **The modal definability problem w.r.t.** $\mathcal{K}$ is whether there is an algorithm which given a FOL sentence can determine whether it is modally definable w.r.t. $\mathcal{K}$?

- So the decidability of the first-order theories of these classes become an essential part of our study of the modal definability problem w.r.t. them.
- Our hopes of measuring the complexity of the modal definability problems w.r.t. $\mathcal{K}_{\text{rectangle}}$ and $\mathcal{K}_{\text{square}}$ using the complexity of the modal definability problem w.r.t. $\mathcal{K}_{\text{commute}}$ are in vain, because the latter is undecidable.
- Also, we will see that the first-order theories of $\mathcal{K}_{\text{rectangle}}$ and $\mathcal{K}_{\text{square}}$ are decidable, giving us only a lower bound of the modal definability problems w.r.t. the classes.

In proving all these statements we will use various techniques from classical and finite model theory and other. Some of them will be methods from like *Ehrenfeucht-Fraïssé games* and *results on generalized products* started by Mostowski and continued by Feferman and Vaught to demonstrate decidabilities of the first-order validity problems and the possession of the **FMP**.
We will use *Relative elementary definability* introduced by Ershov is derived from Tarski's *Method of interpretations* which is one of the methods for proving undecidability of first-order theories as well as FOL relativization and bounded morphisms lemma, etc.

We will review the decidability of first-order theories of the following of $\mathcal{K}_{\text{commute}}$, $\mathcal{K}_{\text{rectangle}}$, $\mathcal{K}_{\text{square}}$ and some constrainted variants of $\mathcal{K}_{\text{commute}}$:

- Let for each positive natural number n $\mathcal{K}^{R_1 \leq n}_{\text{commute}}$ be the class of all structures from $\mathcal{K}_{\text{commute}}$ such that for each matrix in the structure the rows have $\leq n$ number of cells;
- Let for each positive natural number n $\mathcal{K}^{R_1 \leq n, R_2 < \omega}_{\text{commute}}$ be the class of all structures from $\mathcal{K}^{R_1 \leq n}_{\text{commute}}$ such that for each matrix in the structure the columns have a finite number of cells;
- Let for each positive natural numbers n, m $\mathcal{K}^{R_1 \leq n, R_2 \leq m}_{\text{commute}}$ be the tighter subclass of $\mathcal{K}^{R_1 \leq n, R_2 < \omega}_{\text{commute}}$;

Note that because $\mathcal{K}^{R_2 \leq n}_{\text{commute}}$ is similar to $\mathcal{K}^{R_1 \leq n}_{\text{commute}}$ and $\mathcal{K}^{R_2 \leq n, R_1 < w}_{\text{commute}}$ is similar to $\mathcal{K}^{R_1 \leq n, R_2 < w}_{\text{commute}}$, we will only discuss the decidability problem of $\mathcal{K}^{R_1 \leq n}_{\text{commute}}$ and $\mathcal{K}^{R_1 \leq n, R_2 < w}_{\text{commute}}$. The same reasoning can be applied for obtaining the results for the other two classes.

How can we describe the structures?

How we represent a structure from $\mathcal{K}_{\text{commute}}$

Let $\langle A, R_1^{\mathfrak{A}}, R_2^{\mathfrak{A}} \rangle \in \mathcal{K}_{\text{commute}}$. Then we can prove that $R_1^{\mathfrak{A}} \circ R_2^{\mathfrak{A}} \in \mathcal{E}quiv(A)$, and, thus, A is a set of blocks (equivalence classes) w.r.t. the equivalence relation $R_1^{\mathfrak{A}} \circ R_2^{\mathfrak{A}}$.
The following proposition is true:

Proposition

Let $c \in A$ and $a, b \in [c]_{R_1^{\mathfrak{A}} \circ R_2^{\mathfrak{A}}}$.
Then $[a]_{R_1^{\mathfrak{A}}} \subseteq [c]_{R_1^{\mathfrak{A}} \circ R_2^{\mathfrak{A}}}$ and $[b]_{R_2^{\mathfrak{A}}} \subseteq [c]_{R_1^{\mathfrak{A}} \circ R_2^{\mathfrak{A}}}$ and $[a]_{R_1^{\mathfrak{A}}} \cap [b]_{R_2^{\mathfrak{A}}} \neq \emptyset$.

How can we describe the structures?, 2

How we represent a structure from $\mathcal{K}_{\text{commute}}$, (Cont.)

Let $c \in A$ and let $p \leftrightharpoons [c]_{R_1^{\mathfrak{A}} \circ R_2^{\mathfrak{A}}}$.

Enumerate all the blocks of $R_1^{\mathfrak{A}}$ w.r.t. p: $\{a_\alpha\}_{\alpha<\lambda}$ and enumerate all the blocks of $R_2^{\mathfrak{A}}$ w.r.t. p: $\{b_\beta\}_{\beta<\mu}$, where $\mathbf{card}(p/R_1^{\mathfrak{A}}) = \lambda$ and $\mathbf{card}(p/R_2^{\mathfrak{A}}) = \mu$.

Denote $c_{\alpha,\beta} \leftrightharpoons a_\alpha \cap b_\beta$.

We have that:

- $c_{\alpha,\beta} \neq \emptyset$;
- $c_{\alpha,\beta} \cap c_{\alpha',\beta'} = \emptyset$ for $\langle \alpha, \beta \rangle \neq \langle \alpha', \beta' \rangle$;
- $\bigcup\limits_{\substack{\alpha<\lambda \\ \beta<\mu}} c_{\alpha,\beta} = p$.

⋱	⋮	⋮	⋮	⋮	⋮	⋮	⋰
…	..	..	**67**	..	..	**1**	…
…	..	..	..	..	..	..	…
…	**2**	$\aleph_0$	**21**	..	..	**2**	…
…	..	..	..	..	**500**	..	…
…	$\aleph_1$	..	**2**	..	..	2^{179}	…
⋰	⋮	⋮	⋮	⋮	⋮	⋮	⋱

I.e., the family $M = \{c_{\alpha,\beta}\}_{\substack{\alpha<\lambda \\ \beta<\mu}}$ is a partition of p. So we can think of p as a "**matrix of the type** $\lambda \times \mu$" of non-empty, mutually disjoint sets. We will call an element of the family $\{a_\alpha\}_{\alpha<\lambda}$ a "**row**" and we will call an element of the family $\{b_\beta\}_{\beta<\mu}$ a "**columns**". A set $c_{\alpha,\beta}$ we will call a "**cell**". All the structures $\mathfrak{A} = \langle A, R_1^{\mathfrak{A}}, R_2^{\mathfrak{A}} \rangle \in \mathcal{K}_{\text{commute}}$ are **a collection of matrices** $\{M(\gamma)\}_{\gamma<\xi}$ of the type $\lambda_\gamma \times \mu_\gamma$, $\gamma < \xi$, for $\#_{R_1^{\mathfrak{A}} \circ R_2^{\mathfrak{A}}} = \xi$.

How we represent a structure from $\mathcal{K}^{R_1\leq n}_{\text{commute}}$, $\mathcal{K}^{R_1\leq n, R_2<\omega}_{\text{commute}}$ and $\mathcal{K}^{R_1\leq n, R_2\leq m}_{\text{commute}}$

The structures are going to be similar to those of the wiser class $\mathcal{K}_{\text{commute}}$ with some restrictions of the number of cells a matrix can have in a row/column. For example if we take a structure from $\mathfrak{A} \in \mathcal{K}^{R_1\leq n}_{\text{commute}}$, then a matrix from $\mathfrak{A}$ will be of the type $m \times \mu$ for $m \leq n$.

How we represent a structure from $\mathcal{K}_{\text{rectangle}}$ and $\mathcal{K}_{\text{square}}$

Both the structures in the classes $\mathcal{K}_{\text{rectangle}}$ and in $\mathcal{K}_{\text{square}}$ are models of the formula $\forall x \forall y(x = y \leftrightarrow R_1(x, y) \wedge R_2(x, y))$; therefore, automatically all cells have cardinality one. Hence, all matrices in a structure from $\mathcal{K}_{\text{rectangle}}$ are rectangles and are essentially modal products of two structures from the class $\mathcal{K}_{\text{partition}}$, i.e.
$\mathcal{K}_{\text{rectangle}} = \{\mathfrak{A} \underset{mod}{\times} \mathfrak{B} \mid \mathfrak{A}, \mathfrak{B} \in \mathcal{K}_{\text{partition}}\}$.
The same goes for the structures from $\mathcal{K}_{\text{square}}$ with the exception that the modal product is of the same structure from $\mathcal{K}_{\text{partition}}$, i.e. $\mathcal{K}_{\text{square}} = \{\mathfrak{A} \underset{mod}{\times} \mathfrak{A} \mid \mathfrak{A} \in \mathcal{K}_{\text{partition}}\}$.

We will review the decidability of first-order theories of the following of $\mathcal{K}_{\text{commute}}$, $\mathcal{K}_{\text{rectangle}}$, $\mathcal{K}_{\text{square}}$ and some constrained variants of $\mathcal{K}_{\text{commute}}$:

- Let for each positive natural number n $\mathcal{K}^{R_1 \leq n}_{\text{commute}}$ be the class of all structures from $\mathcal{K}_{\text{commute}}$ such that for each matrix in the structure the rows have $\leq n$ number of cells;
- Let for each positive natural number n $\mathcal{K}^{R_1 \leq n, R_2 < \omega}_{\text{commute}}$ be the class of all structures from $\mathcal{K}^{R_1 \leq n}_{\text{commute}}$ such that for each matrix in the structure the columns have a finite number of cells;
- Let for each positive natural numbers n, m $\mathcal{K}^{R_1 \leq n, R_2 \leq m}_{\text{commute}}$ be the tighter subclass of $\mathcal{K}^{R_1 \leq n, R_2 < \omega}_{\text{commute}}$;

Note that because $\mathcal{K}^{R_2 \leq n}_{\text{commute}}$ is similar to $\mathcal{K}^{R_1 \leq n}_{\text{commute}}$ and $\mathcal{K}^{R_2 \leq n, R_1 < w}_{\text{commute}}$ is similar to $\mathcal{K}^{R_1 \leq n, R_2 < w}_{\text{commute}}$, we will only discuss the decidability problem of $\mathcal{K}^{R_1 \leq n}_{\text{commute}}$ and $\mathcal{K}^{R_1 \leq n, R_2 < w}_{\text{commute}}$. The same reasoning can be applied for obtaining the results for the other two classes.

A quick overview of some properties

Do the theories differ?

- $\mathrm{Th}(\mathcal{K}_{\text{commute}}) \subsetneq \mathrm{Th}(\mathcal{K}_{\text{rectangle}})$. For example the formula $\varphi_{=}(x, y) \leftrightharpoons (R_1(x, y) \wedge R_2(x, y))$ is true in all structures from $\mathcal{K}_{\text{rectangle}}$, but not all of $\mathcal{K}_{\text{commute}}$.
- $\mathrm{Th}(\mathcal{K}_{\text{rectangle}}) \subsetneq \mathrm{Th}(\mathcal{K}_{\text{square}})$. Define:

$$\varphi_{R_1 \circ R_2}(x, y) \leftrightharpoons \exists z(R_1(x, z) \wedge R_2(z, y)).$$

$$\varphi_{\text{oneBlock}R_1 \circ R_2} \leftrightharpoons \exists x \forall y(\varphi_{R_1 \circ R_2}(x, y)).$$

$$\varphi_{\text{twoOrLessIndividuals}} \leftrightharpoons \exists x \exists y \forall z(x \doteq z \vee y \doteq z).$$

$$\varphi_{\text{oneIndividual}} \leftrightharpoons \exists x \forall y(x \doteq y).$$

Let ψ_{dot} be the following sentence:

$$\psi_{\text{dot}} \leftrightharpoons \varphi_{\text{oneBlock}R_1 \circ R_2} \wedge \varphi_{\text{twoOrLessIndividuals}} \to \varphi_{\text{oneIndividual}}.$$

Then ψ_{dot} is true for all structures of $\mathcal{K}_{\text{square}}$. But ψ_{dot} is not true in all structures of $\mathcal{K}_{\text{rectangle}}$, because the matrices can be rectangular.

Do the theories differ?, (Cont.)

- Th($\mathcal{K}_{\text{commute}}$) $\subsetneq$ Th($\mathcal{K}^{R_1 \leq n}_{\text{commute}}$). Let for each $k \in \omega^+$ define the formula:

$$\varphi_{\mathsf{KcellsIn}R_1}(x_1, x_2, \dots, x_k) \leftrightharpoons \bigwedge_{1 \leq i < j \leq k} \neg(x_i \doteq x_j) \wedge \bigwedge_{1 \leq i < j \leq k} R_1(x_i, x_j) \wedge$$
$$\bigwedge_{1 \leq i < j \leq k} \neg R_2(x_i, x_j) \wedge \forall y (R_1(y, x_1) \wedge \bigwedge_{1 \leq i \leq k} \neg(y \doteq x_i) \rightarrow \bigvee_{1 \leq i \leq k} R_2(x_i, y)).$$

  For each positive natural number k, the formula says that $x_1, x_2, \dots, x_k$ forms a row with exactly k non-empty cells. Now define the formula:

$$\varphi_{R_1 \leq n} \leftrightharpoons \forall y \bigvee_{1 \leq k \leq n} \exists x_1 \dots \exists x_{k-1} \varphi_{\mathsf{KcellsIn}R_1}(y, x_1, \dots, x_{k-1}).$$

  This formula proves that the theories differ.
- Th($\mathcal{K}^{R_1 \leq n}_{\text{commute}}$) $\subsetneq$ Th($\mathcal{K}^{R_1 \leq n, R_2 < \omega}_{\text{commute}}$). The collection of infinitely many formulae $\varphi_{\mathsf{KcellsIn}R_2}$ for $m \in \omega$ (it is essentially $\varphi_{\mathsf{KcellsIn}R_1}$ where the symbol R_1 is substituted for R_2) help us to establish this statement.
- Th($\mathcal{K}^{R_1 \leq n, R_2 < \omega}_{\text{commute}}$) $\subsetneq$ Th($\mathcal{K}^{R_1 \leq n, R_2 \leq m}_{\text{commute}}$). The formula $\varphi_{R_2 \leq m}$ (written similarly as $\varphi_{R_1 \leq n}$) helps us to differ them.

Remark

With $\mathcal{K}^{\text{fin}}$ we will denote the class of all the structure of a class of structures $\mathcal{K}$ having a **finite universe**.

Definition (Axiomatized class of structures)

Let $\Sigma \subseteq \mathcal{S}\text{ent}(\mathfrak{L})$ and $\mathcal{K}$ be a class of structures for $\mathfrak{L}$.
Σ **axiomatizes the class of structures** $\mathcal{K}$ if for all structures $\mathfrak{A}$ for $\mathfrak{L}$
$[\mathfrak{A} \models \Sigma \iff \mathfrak{A} \in \mathcal{K}]$.

Definition (Finitely axiomatized class of structures)

Let $\varphi \in \mathcal{S}\text{ent}(\mathfrak{L})$ and $\mathcal{K}$ be a class of structures for $\mathfrak{L}$.
φ **finitely axiomatizes the class of structures** $\mathcal{K}$ if for all structures $\mathfrak{A}$ for $\mathfrak{L}$
$[\mathfrak{A} \models \varphi \iff \mathfrak{A} \in \mathcal{K}]$.

Are the theories axiomatizable?

- $\mathcal{K}_{\text{commute}}$ is finitely axiomatizable and so are $\mathcal{K}_{\text{commute}}^{R_1 \leq n}$ and $\mathcal{K}_{\text{commute}}^{R_1 \leq n, R_2 \leq m}$. $\mathcal{K}_{\text{commute}}^{\text{fin}}$, $(\mathcal{K}_{\text{commute}}^{R_1 \leq n})^{\text{fin}}$, $(\mathcal{K}_{\text{commute}}^{R_1 \leq n, R_2 \leq m})^{\text{fin}}$, $\mathcal{K}_{\text{commute}}^{R_1 \leq n, R_2 < \omega}$ and $(\mathcal{K}_{\text{commute}}^{R_1 \leq n, R_2 < \omega})^{\text{fin}}$ are only axiomatizable.
- All $\mathcal{K}_{\text{rectangle}}$, $\mathcal{K}_{\text{square}}$, $\mathcal{K}_{\text{rectangle}}^{\text{fin}}$ and $\mathcal{K}_{\text{square}}^{\text{fin}}$ are not closed w.r.t. isomorphisms. That is because if we take a structure $\mathfrak{A} \in \mathcal{K}_{\text{rectangle}}$, then it is of the type $\langle A_1 \times A_2, R_1^{\mathfrak{A}}, R_2^{\mathfrak{A}} \rangle$. Let the set B be such that $\mathbf{card}(A) = \mathbf{card}(B)$ and the elements of B are **not tuples**. Then $\mathfrak{A} \cong \langle B, R_1^{\mathfrak{A}}, R_2^{\mathfrak{A}} \rangle$, but $\langle B, R_1^{\mathfrak{A}}, R_2^{\mathfrak{A}} \rangle \notin \mathcal{K}_{\text{rectangle}}$ (the same reasoning can be applied for the other classes).
  Therefore, $\mathcal{K}_{\text{rectangle}}$, $\mathcal{K}_{\text{square}}$, $\mathcal{K}_{\text{rectangle}}^{\text{fin}}$ and $\mathcal{K}_{\text{square}}^{\text{fin}}$ are not axiomatizable.
  The question is if we close the classes w.r.t. isomorphisms, can we (finitely) axiomatize the new classes?
  Let us denote with $I(\mathcal{K}) \leftrightharpoons \{\mathfrak{A} \mid (\exists \mathfrak{B} \in \mathcal{K})[\mathfrak{A} \cong \mathfrak{B}]\}$ the **closure of the class** $\mathcal{K}$ **w.r.t. isomorphisms**.
  Even after we close them w.r.t. ismorphisms $I(\mathcal{K}_{\text{rectangle}})$ and $I(\mathcal{K}_{\text{square}})$ are not axiomatizable (can be seen by using Ehrenfeucht–Fraïssé games, Downward Löwenheim–Skolem theorem, etc.). $I(\mathcal{K}_{\text{rectangle}}^{\text{fin}})$ and $I(\mathcal{K}_{\text{square}}^{\text{fin}})$ are not finitely axiomatizable (can be seen by using Ehrenfeucht–Fraïssé games).

Are the theories decidable?
What about the modal definability problem w.r.t. these classes?

First-order validity problem w.r.t. $\mathcal{K}_{\text{commute}}$

First-order validity problem w.r.t. $\mathcal{K}_{\text{commute}}$, 1

Some of the tools needed

- Let us define a class of structures which will be of interest to us: $\mathcal{K}_{\text{irref, sym}} \leftrightharpoons \{\langle A, R^{\mathfrak{A}}\rangle \mid R^{\mathfrak{A}}$ is symmetric and irreflexive in $A\}$ for the language $\mathfrak{L}(R, \doteq)$.
- Th($\mathcal{K}_{\text{irref, sym}}$) and Th($\mathcal{K}^{\text{fin}}_{\text{irref, sym}}$) are hereditarily undecidable (Ershov, 1980).

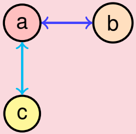
Theorem for hereditary undecidability, (Ershov, 1980)

If the class of structures $\mathcal{K}_0$ is **relatively elementary definable** in the class of structures $\mathcal{K}_1$ and the theory Th($\mathcal{K}_0$) is hereditarily undecidable, then the theory Th($\mathcal{K}_1$) is also hereditarily undecidable.

The idea

Let $k, m \in \omega^+$ be positive natural numbers such that $k \neq m$. For simplifying the following steps let us fix $k = 1$ and $m = 2$.

Let $\mathfrak{A} \in \mathcal{K}_{\text{irref, sym}}$ be the simple finite structure:



Then we can represent $\mathfrak{A}$ within a structure $\mathfrak{B}$ with the following matrix:

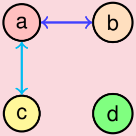
1	**2**	**2**
2	**1**	**3**
2	**3**	**1**

First-order validity problem w.r.t. $\mathcal{K}_{\text{commute}}$, 3

The idea, (Cont.)

Let $k, m \in \omega^+$ be positive natural numbers such that $k \neq m$. For simplifying the following steps let us fix $k = 1$ and $m = 2$.

Let $\mathfrak{A} \in \mathcal{K}_{\text{irref, sym}}$ be the simple finite structure:



Then we can represent $\mathfrak{A}$ within a structure $\mathfrak{B}$ with the following matrix:

1	**2**	**2**	**3**
2	**1**	**3**	**3**
2	**3**	**1**	**3**
3	**3**	**3**	**1**

The idea, (Cont.)

Let $\mathcal{K}^{\text{uni}}_{\text{commute}}$ be all the structures which have exactly one matrix.

- We will prove that the class $\mathcal{K}_{\text{irref, sym}}$ is relatively elementary definable in the class $\mathcal{K}^{\text{uni}}_{\text{commute}}$.
- For an arbitrary structure $\mathfrak{A} \in \mathcal{K}_{\text{irref, sym}}$ we will construct a structure $\mathfrak{B} \in \mathcal{K}^{\text{uni}}_{\text{commute}}$.
- Then we will "filter" a structure $\mathfrak{C}$ from $\mathfrak{B}$ and we will have explicitly from the construction that $\mathfrak{A} \cong \mathfrak{C}$.
- From Th($\mathcal{K}^{\text{uni}}_{\text{commute}}$) is hereditarily undecidable, rending Th($\mathcal{K}_{\text{commute}}$) hereditarily undecidable.

Theorem: RED of $\mathcal{K}_{\text{irref, sym}}$ in $\mathcal{K}^{\text{uni}}_{\text{commute}}$

The class $\mathcal{K}_{\text{irref, sym}}$ is relatively elementary definable in the class $\mathcal{K}^{\text{uni}}_{\text{commute}}$.

Theorem: Undecidability

Th($\mathcal{K}^{\text{uni}}_{\text{commute}}$), Th($\mathcal{K}_{\text{commute}}$), Th(($\mathcal{K}^{\text{uni}}_{\text{commute}}$)$^{\text{fin}}$), Th($\mathcal{K}^{\text{fin}}_{\text{commute}}$) are hereditarily undecidable and therefore undecidable.

Corollary (Janiczak, Rogers)

Let $\mathcal{K}_{2S5}$ the class of all structures for $\mathfrak{L}(R_1, R_2, \doteq)$ such that the relation symbols are interpreted as two equivalence relations on the universe of the structure.
Th($\mathcal{K}_{2S5}$) is undecidable: (Janiczak, 1953) and (H. Rogers, 1956).

Corollary: Undecidability

Th($\mathcal{K}_{2S5}$) and Th($\mathcal{K}^{\text{fin}}_{2S5}$) are hereditarily undecidable and therefore undecidable.

Additional properties of $\mathcal{K}_{\text{commute}}$

Th($\mathcal{K}_{\text{commute}}$) is not essentially undecidable and $\mathcal{K}_{\text{commute}}$ does not have **FMP**.

Modal definability problem w.r.t. $\mathcal{K}_{\text{commute}}$

Some of the tools needed

Relativization theorem

Let $\mathfrak{A}$ and $\mathfrak{A}_0$ are structures for $\mathfrak{L}$, $\varphi(x, x_1, \ldots, x_n) \in \mathcal{F}\text{orm}(\mathfrak{L})$ and $\bar{a}$ be a list of individuals in A.
If $\mathfrak{A}_0$ is a **relativized substructure** of $\mathfrak{A}$ w.r.t. $\varphi(x, x_1, \ldots, x_n)$ and $\bar{a}$, then for all FOL formula $\chi(y_1, \ldots, y_m)$ and all lists of individuals $\bar{c}$ in A_0: $\mathfrak{A} \models (\chi)^{\varphi}_{x}[\![\bar{a}; \bar{c}]\!] \iff \mathfrak{A}_0 \models \chi[\![\bar{c}]\!]$.

Ehrenfeucht-Fraïssé theorem

For all $k \in \omega$, for all finite FOL languages without function symbols $\mathfrak{L}$ and for all structures $\mathfrak{A}$ and $\mathfrak{B}$ for $\mathfrak{L}$ the following are equivalent:
The $\mathcal{D}uplicator$ has a winning strategy for $G_k(\mathfrak{A}, \mathfrak{B}) \iff \mathfrak{A} \equiv_k \mathfrak{B}$.

Bounded morphism lemma

Let $\mathfrak{F}$ and $\mathfrak{F}'$ be frames.
If $\mathfrak{F}'$ is a bounded morphic image of $\mathfrak{F}$ then $\mathfrak{F} \leq \mathfrak{F}'$.

Definition (Quantifier rank of a formula)

Let $\varphi \in \mathcal{F}\text{orm}(\mathfrak{L})$.
The **quantifier rank** $qr(\varphi) \in \omega$ of φ is defined in the following manner.

- $\varphi \in \mathcal{A}tomic_{\mathfrak{L}}$, then $qr(\varphi) = 0$;
- $\varphi \rightleftharpoons \neg\psi$, then $qr(\varphi) = qr(\psi)$;
- $\varphi \rightleftharpoons (\psi_1 \vee \psi_2)$, then $qr(\varphi) = max\{qr(\psi_1), qr(\psi_2)\}$;
- $\varphi \rightleftharpoons \exists x\psi$ for $x \in \mathcal{V}ar_{\mathfrak{L}}$, then $qr(\varphi) = 1 + qr(\psi)$;

A ***k*-rank formula** is a formula having quantifier rank exactly *k*.

Definition (Partial isomorphism)

Let be a $\mathfrak{L}$ finite RFOL language and let $\mathfrak{A}$ and $\mathfrak{B}$ be structures for $\mathfrak{L}$.
Let *h* be a mapping such that Dom$(h) \subseteq A$ and Range$(h) \subseteq B$. *h* is called a **partial isomorphism** from $\mathfrak{A}$ to $\mathfrak{B}$ if it is injective and for each *n*-ary relation symbol $p \in \mathcal{P}red_{\mathfrak{L}}$ and for every $a_1, \ldots, a_n \in$ Dom(h):

$$\langle a_1, \ldots, a_n \rangle \in p^{\mathfrak{A}} \iff \langle h(a_1), \ldots, h(a_n) \rangle \in p^{\mathfrak{B}}.$$

We will denote the **set of all partial isomorphisms** from $\mathfrak{A}$ to $\mathfrak{B}$ with $\mathcal{P}art(\mathfrak{A}, \mathfrak{B})$.

Definition (k-equivalent structures)

Let $\mathfrak{A}$ and $\mathfrak{B}$ be structures for $\mathfrak{L}$.
The structures $\mathfrak{A}$ and $\mathfrak{B}$ are called ***k*-equivalent**, denoted $\mathfrak{A} \equiv_k \mathfrak{B}$, if they satisfy the same *i*-rank first-order sentences for $0 \leq i \leq k$.

Definition (Elementarily equivalent structures)

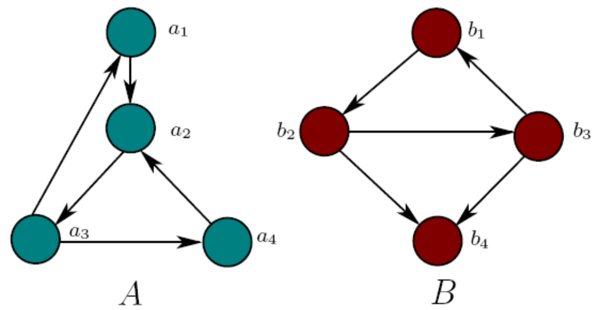
Let $\mathfrak{A}$ and $\mathfrak{B}$ be structures for $\mathfrak{L}$.
$\mathfrak{A}$ and $\mathfrak{B}$ are called **elementarily equivalent**, denoted $\mathfrak{A} \equiv \mathfrak{B}$, if they satisfy the same first-order sentences , i.e., $\text{Th}(\mathfrak{A}) = \text{Th}(\mathfrak{B})$.

Remark

Let $\mathfrak{A}$ and $\mathfrak{B}$ be structures for $\mathfrak{L}$ and $\mathbf{card}(\mathfrak{L}) \leq \aleph_0$.
If $\mathfrak{A} \equiv_k \mathfrak{B}$ for all $k \in \omega$, then $\mathfrak{A} \equiv \mathfrak{B}$.

How the game is played?

Let $\mathfrak{A}$ and $\mathfrak{B}$ be the following structures for the language $\mathfrak{L}(R, \doteq)$:



The **Ehrenfeucht–Fraïssé game** $G_k(\mathfrak{A}, \mathfrak{B})$ is played by two players called the $\mathcal{S}poiler$ and the $\mathcal{D}uplicator$. Each player has to make k moves in the course of a play. The players take turns.

How the game is played?, (Cont.)

1. In his i-th move the $\mathcal{S}poiler$ first selects a structure, $\mathfrak{A}$ or $\mathfrak{B}$, and an element in this structure.
2. If the $\mathcal{S}poiler$ chooses $s_i \in A$ then the $\mathcal{D}uplicator$ in his i-th move must choose an element $d_i \in B$. If the $\mathcal{S}poiler$ chooses $d_i \in B$ then the $\mathcal{D}uplicator$ must choose an element $s_i \in A$.
3. Let $\{\langle s_i, d_i\rangle \mid 1 \leq i \leq k\}$ be the corresponding choices for each turn. The $\mathcal{D}uplicator$ **wins** if and only if $\bar{s} \mapsto \bar{d} \in \mathcal{P}art(\mathfrak{A}, \mathfrak{B})$. Otherwise, the $\mathcal{S}poiler$ wins.
4. Given Ehrenfeucht–Fraïssé theorem that means that $\mathcal{D}uplicator$ **wins** if and only if $\mathfrak{A} \equiv_k \mathfrak{B}$.
5. A **strategy** is a system of rules which tells the player what move to make, depending on the history of the game up to the current moment.
6. We say that a player has a **winning strategy** in $G_k(\mathfrak{A}, \mathfrak{B})$, or shortly, a player wins $G_k(\mathfrak{A}, \mathfrak{B})$, if it is guaranteed that he is always the winner of the game (following mindlessly the strategy).

Theorem: Reducibility $\mathcal{K}_{\text{commute}}$

The problem of deciding the validity of sentences in $\mathcal{K}_{\text{commute}}$ is reducible to the modal definability problem w.r.t. $\mathcal{K}_{\text{commute}}$.

Beginning of proof

Let $\varphi \in \mathcal{F}\text{orm}(\mathfrak{L}(R_1, R_2, \doteq))$ be defined $\varphi(x, x_1) \leftrightharpoons \neg\exists z(R_1(x_1, z) \wedge R_2(z, x))$.
Let $\chi \in \mathcal{S}\text{ent}(\mathfrak{L}(R_1, R_2, \doteq))$ be such that $qr(\chi) = k$ for some $k \in \omega$ and define the sentence ψ to depend on $qr(\chi)$ in the following manner:

$$\psi \leftrightharpoons \exists y_1 \ldots \exists y_{k+1}\Big(\bigwedge_{1 \leq i < j \leq k+1} (R_1(y_i, y_j) \wedge R_2(y_i, y_j) \wedge \neg(y_i \doteq y_j))\Big).$$

Let $\theta \leftrightharpoons \exists x_1(\exists x\varphi(x, x_1) \wedge \neg(\chi)_x^{\varphi(x,x_1)}) \wedge \psi$ be a sentence of $\mathfrak{L}(R_1, R_2, \doteq)$.
We need only prove that $\mathcal{K}_{\text{commute}} \models \chi \iff \theta$ is modally definable w.r.t. $\mathcal{K}_{\text{commute}}$.

Corollary

The modal definability problem w.r.t. $\mathcal{K}_{\text{commute}}$ is undecidable.

Theorem: Reducibility $\mathcal{K}^{\text{fin}}_{\text{commute}}$

The problem of deciding the validity of sentences in $\mathcal{K}^{\text{fin}}_{\text{commute}}$ is reducible to the modal definability problem w.r.t. $\mathcal{K}^{\text{fin}}_{\text{commute}}$.

Corollary

The modal definability problem w.r.t. $\mathcal{K}^{\text{fin}}_{\text{commute}}$ is undecidable.

First-order validity problem w.r.t. $\mathcal{K}^{R_1 \leq n}_{\text{commute}}$ and related subclasses

How we will accomplish the task

We will show decidability of $\mathcal{K}^{R_1 \leq n}_{\text{commute}}$ and the related subclasses by demonstrating that they have the "strong" **FMP** (to see if a FOL sentence $\varphi \in \text{Th}(\mathcal{K}^{R_1 \leq n}_{\text{commute}})$ we will need only to check that $\varphi \in \text{Th}(\mathcal{C})$, where $\mathcal{C}$ is a finite class of finite structures w.r.t. isomorphisms directly yielding the decidability of the respective theory). For the task we will use Ehrenfeucht–Fraïssé games and so some basic definitions and theorems are needed to be remembered.

Definition (Finite model property (FMP))

A class of structures $\mathcal{K}$ for a RFOL language $\mathfrak{L}$ has **FMP** if for any sentence φ of the language $\mathfrak{L}$: $\text{Th}(\mathcal{K}) \not\models \varphi \Longrightarrow (\exists \mathfrak{B} \in \mathcal{K}^{\text{fin}})[\mathfrak{B} \not\models \varphi]$.
An analogous definition is that $\text{Th}(\mathcal{K}) = \text{Th}(\mathcal{K}^{\text{fin}})$.

Theorem

Let φ be a sentence in $\mathfrak{L}(R_1, R_2, \doteq)$. Then for all positive natural numbers n:

$$\varphi \notin \mathrm{Th}(\mathcal{K}_{\text{commute}}^{R_1 \leq n}) \Longrightarrow (\exists \mathfrak{A}_0^{\text{fin}} \in (\mathcal{K}_{\text{commute}}^{R_1 \leq n})^{\text{fin}})[\mathfrak{A}_0^{\text{fin}} \not\models \varphi]$$

Proof.

- Let φ be a sentence in $\mathfrak{L}(R_1, R_2, \doteq)$ such that $qr(\varphi) = k$ for some natural number k;
- Take a witness $\mathfrak{A} \in \mathcal{K}_{\text{commute}}^{R_1 \leq n}$ such that $\mathfrak{A} \not\models \varphi$;
- Refine $\mathfrak{A}$ to a structure $\mathfrak{A}'$ with the property that in every cell in every matrix of $\mathfrak{A}'$ is of cardinality not greater that k;
- Next for every matrix $M \in \mathfrak{A}'$ of type $m \times \mu$ for $m \leq n$ we will do the following:
  - We will set an ordering of the columns. In this way we can extract a set of m–tuples $\mathcal{R}_m$ of the cardinalities of the cells of rows in the matrix.
  - Afterward for every m–tuple from $\mathcal{R}_m$ we will leave not more than k rows with that "type" and discard the rest from the matrix. We obtain a new matrix M' which is **finite**;
- After applying the previous step for every matrix in $\mathfrak{A}'$ and replace them with finite matrices, we obtain a new structure $\mathfrak{A}''$ which consists only of finite matrices;

First-order validity problem w.r.t. $\mathcal{K}^{R_1 \leq n}_{\text{commute}}$, 3

Proof., (Cont.)

- Fix an ordering of the columns of all matrices. For each matrix take the set of m-tuples $\mathcal{R}'_m$ and the number of every such m-tuple of $\mathcal{R}'_m$ in the respective matrix (Defined on the previous slide. They are a finite number of sets.).
  The last step is to leave no more than k matrices of $\mathfrak{A}''$ with the above described property. In this way we end up with a **finite** structure $\mathfrak{A}_0$.
- Checking that the $\mathcal{D}uplicator$ has a winning strategy for the game $G_k(\mathfrak{A}, \mathfrak{A}_0)$ is immediate and by Ehrenfeucht–Fraïssé theorem that means that $\mathfrak{A} \equiv_k \mathfrak{A}_0$.
- Since $qr(\varphi) = k$ and $\mathfrak{A} \not\models \varphi$ then $\mathfrak{A}_0 \not\models \varphi$.

∎

Theorem

The theory of $\mathcal{K}^{R_1 \leq n}_{\text{commute}}$ is decidable for all positive natural numbers n.

Proof.

- Fix a positive natural number n.
- We know $\mathcal{K}^{R_1 \leq n}_{\text{commute}}$ has **FMP**;
- Let φ be a sentence from $\mathfrak{L}(R_1, R_2, \doteq)$ and let $qr(\varphi) = k$. Let red(k) denote the compositions of the operation of refinement of the structures in the previous theorem proving **FMP**. Then $(\mathcal{K}^{R_1 \leq n}_{\text{commute}})^{\text{red}(k)}$ is a class in which for each structure, the matrices in it have cardinality no more than some natural number - the maximal cardinality of a matrix produced after applying red(k) to the original matrix, that is, all structures are finite. Therefore, $(\mathcal{K}^{R_1 \leq n}_{\text{commute}})^{\text{red}(k)}$ is a finite class of structures w.r.t. isomorphism. As a result, $(\mathcal{K}^{R_1 \leq n}_{\text{commute}})^{\text{red}(k)} \equiv_k \mathcal{K}^{R_1 \leq n}_{\text{commute}}$. Finally in order to see if φ is a theorem of $\mathcal{K}^{R_1 \leq n}_{\text{commute}}$ we need only check validity of φ in a finite number of finite structures.

∎

Corollary

The theory of $(\mathcal{K}^{R_1\leq n}_{\text{commute}})^{\text{fin}}$ is decidable for all positive natural numbers n.

Corollary

The theory of $(\mathcal{K}^{R_1\leq n}_{\text{commute}})^{\text{uni}}$ is decidable for all positive natural numbers n, where $\mathcal{K}^{\text{uni}}$ means that there is only one matrix in the universe of the strucres in the class.

Using the same arguments we can prove that:

Theorem

The following classes have decidable theories for all positive natural numbers n, m:

- $\mathcal{K}^{R_1\leq n, R_2<\omega}_{\text{commute}}$, $(\mathcal{K}^{R_1\leq n, R_2<\omega}_{\text{commute}})^{\text{uni}}$, $(\mathcal{K}^{R_1\leq n, R_2<\omega}_{\text{commute}})^{\text{fin}}$;
- $\mathcal{K}^{R_1\leq n, R_2\leq m}_{\text{commute}}$, $(\mathcal{K}^{R_1\leq n, R_2\leq m}_{\text{commute}})^{\text{uni}}$, $(\mathcal{K}^{R_1\leq n, R_2\leq m}_{\text{commute}})^{\text{fin}}$.

Modal definability problem w.r.t. $\mathcal{K}^{R_1 \leq n}_{\text{commute}}$ and related subclasses

Let by $\mathcal{K}$ we mean any of the classes $\mathcal{K}^{R_1 \leq n}_{\text{commute}}$, $\mathcal{K}^{R_1 \leq n, R_2 < \omega}_{\text{commute}}$ or $\mathcal{K}^{R_1 \leq n, R_2 \leq m}_{\text{commute}}$ for n, m positive natural numbers.

Theorem: Reducibility $\mathcal{K}$

The problem of deciding the validity of sentences in $\mathcal{K}$ is reducible to the modal definability problem w.r.t. $\mathcal{K}$.

Theorem: Reducibility $\mathcal{K}^{\text{fin}}$

The problem of deciding the validity of sentences in $\mathcal{K}^{\text{fin}}$ is reducible to the modal definability problem w.r.t. $\mathcal{K}^{\text{fin}}$.

The inspection of the proofs shows that the proof for $\mathcal{K}_{\text{commute}}$ and $\mathcal{K}^{\text{fin}}_{\text{commute}}$ is a proof for $\mathcal{K}$ and $\mathcal{K}^{\text{fin}}$ respectfuly.

First-order validity problem w.r.t. Th($\mathcal{K}_{\text{rectangle}}$) and Th($\mathcal{K}_{\text{square}}$)

Th($\mathcal{K}_{\text{rectangle}}$) and Th($\mathcal{K}_{\text{square}}$) are decidable, 1

How can we prove the decidabilities?, 1

- We can prove it by using old results on decidability of generalized products and powers from the 50-ties started by Mostowski and continued by Feferman and Vaught, we will prove a corner case corollary which will yield one means with which we will show the decidability of Th($\mathcal{K}_{\text{rectangle}}$) and Th($\mathcal{K}_{\text{square}}$).
- The original papers are (Mostowski, 1952) and (Feferman and Vaught, 1959).

How can we prove the decidabilities?, 2

- Another solution to the decidability problems can be described using Ehrenfeucht–Fraïssé games to prove that they have the strong **FMP**.
- We will use that $\mathcal{K}_{\text{partition}}$ has **FMP** is a folklore fact.
- This proof depends on this property of Ehrenfeucht–Fraïssé games w.r.t. direct product of structures.
- This method gives us more information about the exact upper bound of the complexity of the membership problem to Th($\mathcal{K}_{\text{rectangle}}$) and Th($\mathcal{K}_{\text{square}}$).

Th($\mathcal{K}_{\text{rectangle}}$) and Th($\mathcal{K}_{\text{square}}$) are decidable, 3

How we can represent the structures of $\mathcal{K}_{\text{rectangle}}$ and $\mathcal{K}_{\text{square}}$

- Let $\mathfrak{A}_1$ and $\mathfrak{A}_2$ be structures from $\mathcal{K}_{\text{partition}}$. But how is the direct product of $\mathfrak{A}_1 \times \mathfrak{A}_2$ remotly close to the modal product $\mathfrak{A}_1 \underset{mod}{\times} \mathfrak{A}_2 \in \mathcal{K}_{\text{rectangle}}$?
- Let us have a structure $\mathfrak{A} = \langle A, R^{\mathfrak{A}} \rangle$ for the language $\mathfrak{L}(R, \doteq)$.
- We will effectively generate two new in a way expansions of $\mathfrak{A}$ for the language $\mathfrak{L}(R_1, R_2, \doteq)$ in the following manner:
  - Let $\mathfrak{A}^{=2} = \langle A, R_1^{\mathfrak{A}^{=2}}, R_2^{\mathfrak{A}^{=2}} \rangle$ be such that the interpretation of $R_1^{\mathfrak{A}^{=2}}$ is the same as that of $R^{\mathfrak{A}}$ and the interpretation of $R_2^{\mathfrak{A}^{=2}}$ will be that of equality of individuals of A (formal equality in the structure $\mathfrak{A}$).
  - Similarly, we generate an expansion $\mathfrak{A}^{=1} = \langle A, R_1^{\mathfrak{A}^{=1}}, R_2^{\mathfrak{A}^{=1}} \rangle$.

How we can represent the structures of $\mathcal{K}_{\text{rectangle}}$ and $\mathcal{K}_{\text{square}}$, (Cont.)

- Let us take a formula φ from $\mathfrak{L}(R, \doteq)$. We can effectively generate two formulae $\varphi^{=2}$ and $\varphi^{=1}$ like this:
  - For $\varphi^{=2}$ we substitute **all occurrences** of the predicate symbol R for R_1, and we substitute **all occurrences** of the formal equality $\doteq$ for R_2.
  - For $\varphi^{=1}$ we substitute **all occurrences** of the formal equality $\doteq$ for R_1, and we substitute **all occurrences** of the predicate symbol R for R_2.
- In turn by taking a formula φ from $\mathfrak{L}(R_1, R_2, \doteq)$ we can obtain a formula from the language $\mathfrak{L}(R, \doteq)$ by substituting all occurrences of the symbol R_1 with R and substituting all occurrences of the symbol R_2 with $\doteq$. We will denote it as $\varphi[R_1/R, R_2/\doteq]$ or $tr_2(\varphi)$.
- We can do also this translation $\varphi[R_1/\doteq, R_2/R]$ or $tr_1(\varphi)$.

Example

Let $\varphi \leftrightharpoons \forall x \forall y \forall z((R(x, y) \wedge x \doteq z) \vee x \doteq y)$.
Then $\varphi^{=1} \rightleftharpoons \forall x \forall y \forall z((R_2(x, y) \wedge R_1(x, z)) \vee R_1(x, y))$ and
$tr_1(\varphi^{=1}) \rightleftharpoons \forall x \forall y \forall z((R(x, y) \wedge x \doteq z) \vee x \doteq y)$.

Properties of the translation

If we want to return to the language $\mathfrak{L}(R_1, R_2, \doteq)$ we do not know which $\doteq$ comes from a substitution of the symbol R_i with $\doteq$ or was originally $\doteq$; thus, we do not have injectivity of the translation, but at least every formula from $\mathfrak{L}(R_1, R_2, \doteq)$ has a translation.

Th($\mathcal{K}_{\text{rectangle}}$) and Th($\mathcal{K}_{\text{square}}$) are decidable, 6

We can prove:

Lemma

For any formula $\varphi(x_1, \ldots, x_n)$ from the language $\mathfrak{L}(R, \doteq)$, for any structure $\mathfrak{A}$ for $\mathfrak{L}(R, \doteq)$ and for any individuals $a_1, \ldots, a_n \in A$ we have:

$$\mathfrak{A} \models \varphi[\![a_1, \ldots, a_n]\!] \iff \mathfrak{A}^{=2} \models \varphi^{=2}[\![a_1, \ldots, a_n]\!] \iff \mathfrak{A}^{=1} \models \varphi^{=1}[\![a_1, \ldots, a_n]\!].$$

and

Lemma

For any formula $\varphi(x_1, \ldots, x_n)$ from the language $\mathfrak{L}(R_1, R_2, \doteq)$, for any structure $\mathfrak{A}$ for $\mathfrak{L}(R, \doteq)$ and for any individuals $a_1, \ldots, a_n \in A$ we have:

$$\mathfrak{A}^{=1} \models \varphi[\![a_1, \ldots, a_n]\!] \iff \mathfrak{A}^{=2} \models \varphi[\![a_1, \ldots, a_n]\!] \iff \mathfrak{A} \models tr_i(\varphi)[\![a_1, \ldots, a_n]\!].$$

Corollary

If $\mathfrak{A}$ has a decidable theory, then so do the structures $\mathfrak{A}^{=1}$ and $\mathfrak{A}^{=2}$.

Corollary

If $\mathcal{K}$ has a decidable theory, then so do the classes $\mathcal{K}^{=1}$ and $\mathcal{K}^{=2}$.

Why was it all necessary?

Why was all this introduced, and why is it useful? The reason is that it gives us a deconstruction of the models of $\mathcal{K}_{\text{rectangle}}$.
If we have $\mathfrak{A}_1, \mathfrak{A}_2 \in \mathcal{K}_{\text{partition}}$, then $\mathfrak{A}_1^{=2} \times \mathfrak{A}_2^{=1}$ will be such a structure that:

- the universe is $A_1 \times A_2$;
- $R_1^{\mathfrak{A}_1^{=2} \times \mathfrak{A}_2^{=1}}$ is such that for any $\langle a, b\rangle, \langle c, d\rangle \in A_1 \times A_2$:
  $\langle\langle a, b\rangle, \langle c, d\rangle\rangle \in R_1^{\mathfrak{A}_1^{=2} \times \mathfrak{A}_2^{=1}} \iff \langle a, c\rangle \in R_1^{\mathfrak{A}_1^{=2}} \;\&\; \langle b, d\rangle \in R_1^{\mathfrak{A}_2^{=1}} \iff \langle a, c\rangle \in R_1^{\mathfrak{A}_1^{=2}} \;\&\; b = d$;
- $R_2^{\mathfrak{A}_1^{=2} \times \mathfrak{A}_2^{=1}}$ is such that for any $\langle a, b\rangle, \langle c, d\rangle \in A_1 \times A_2$:
  $\langle\langle a, b\rangle, \langle c, d\rangle\rangle \in R_2^{\mathfrak{A}_1^{=2} \times \mathfrak{A}_2^{=1}} \iff \langle a, c\rangle \in R_2^{\mathfrak{A}_1^{=2}} \;\&\; \langle b, d\rangle \in R_2^{\mathfrak{A}_2^{=1}} \iff a = c \;\&\; \langle b, d\rangle \in R_2^{\mathfrak{A}_2^{=1}}$.

Proposition

1. Therefore, we have that $\mathfrak{A}_1 \underset{mod}{\times} \mathfrak{A}_2 = \mathfrak{A}_1^{=2} \times \mathfrak{A}_2^{=1}$, i.e., the direct product and the modal product coincides for these specific structures.
2. $\mathcal{K}_{\text{rectangle}} \overset{\text{def.}}{=} \{\mathfrak{A}_1 \underset{mod}{\times} \mathfrak{A}_2 \mid \mathfrak{A}_1, \mathfrak{A}_2 \in \mathcal{K}_{\text{partition}}\} = \mathcal{K}_{\text{partition}} \underset{mod}{\times} \mathcal{K}_{\text{partition}} =$ $\{\mathfrak{A}_1^{=2} \times \mathfrak{A}_2^{=1} \mid \mathfrak{A}_1, \mathfrak{A}_2 \in \mathcal{K}_{\text{partition}}\} = \mathcal{K}^{=2}_{\text{partition}} \times \mathcal{K}^{=1}_{\text{partition}}$.
3. $\mathcal{K}_{\text{square}} \overset{\text{def.}}{=} \{\mathfrak{A} \underset{mod}{\times} \mathfrak{A} \mid \mathfrak{A} \in \mathcal{K}_{\text{partition}}\} = \{\mathfrak{A}^{=2} \times \mathfrak{A}^{=1} \mid \mathfrak{A} \in \mathcal{K}_{\text{partition}}\}$.

The corollary of Mostowski's theorem:

Proposition

Let $\mathfrak{L}$ be a finite RFOL language with or without formal equality $\doteq$.

1. If $\mathfrak{A}_1$ and $\mathfrak{A}_2$ are structures for the language $\mathfrak{L}$ such that Th($\mathfrak{A}_1$) and Th($\mathfrak{A}_2$) are decidable, then Th($\mathfrak{A}_1 \times \mathfrak{A}_2$) is also decidable.
2. If $\mathcal{K}_1$ and $\mathcal{K}_2$ are classes of structures for language $\mathfrak{L}$ such that Th($\mathcal{K}_1$) and Th($\mathcal{K}_2$) are decidable, then Th($\mathcal{K}_1 \times \mathcal{K}_2$) is also decidable.

Theorem

The theory of $\mathcal{K}_{\text{rectangle}}$ is decidable.

Proof.

It is well known that $\mathcal{K}_{\text{partition}}$ has a decidable theory.
Therefore, by Corollary, so do $\mathcal{K}^{=2}_{\text{partition}}$ and $\mathcal{K}^{=1}_{\text{partition}}$. As a result from applying Proposition part 2 we have that $\mathcal{K}^{=2}_{\text{partition}} \times \mathcal{K}^{=1}_{\text{partition}}$ has a decidable theory which by Proposition means that $\mathcal{K}_{\text{rectangle}}$ has a decidable theory. ■

In a similar manner we can prove that:

Theorem

The theory of $\mathcal{K}_{\text{square}}$ is decidable.

Proposition

Let $\mathfrak{A}$ be a structure for the language $\mathfrak{L}(R, \doteq)$ and let $k \in \omega$. Then the $\mathcal{D}uplicator$ has a winning strategy for $G_k(\mathfrak{A}, \mathfrak{A}^{=2})$ and $G_k(\mathfrak{A}, \mathfrak{A}^{=1})$.

Proposition

$\mathcal{K}^{=2}_{\text{partition}}$ and $\mathcal{K}^{=1}_{\text{partition}}$ have **FMP**.

Lemma

Let $\mathfrak{A}_1, \mathfrak{A}_2, \mathfrak{B}_1$ and $\mathfrak{B}_2$ be structures for $\mathfrak{L}$ and $k \in \omega$ be a natural number.
If the $\mathfrak{A}_1 \equiv_k \mathfrak{B}_1$ and $\mathfrak{A}_2 \equiv_k \mathfrak{B}_2$, then $\mathfrak{A}_1 \times \mathfrak{A}_2 \equiv_k \mathfrak{B}_1 \times \mathfrak{B}_2$.

Th($\mathcal{K}_{\text{rectangle}}$) and Th($\mathcal{K}_{\text{square}}$) are decidable, 11

Theorem

$\mathcal{K}_{\text{rectangle}}$ has **FMP**.

Similarly, we can prove that:

Theorem

$\mathcal{K}_{\text{square}}$ has **FMP**.

Theorem

Th($\mathcal{K}_{\text{rectangle}}$), Th($\mathcal{K}^{\text{uni}}_{\text{rectangle}}$), Th($\mathcal{K}^{\text{fin}}_{\text{rectangle}}$), Th($\mathcal{K}_{\text{square}}$), Th($\mathcal{K}^{\text{uni}}_{\text{square}}$), Th($\mathcal{K}^{\text{fin}}_{\text{square}}$) are decidable.

Modal definability problem w.r.t. Th($\mathcal{K}_{\text{rectangle}}$) and Th($\mathcal{K}_{\text{square}}$)

Let by $\mathcal{K}$ we mean any of the classes $\mathcal{K}_{\text{rectangle}}$ or $\mathcal{K}_{\text{square}}$.

Theorem: Reducibility $\mathcal{K}$

The problem of deciding the validity of sentences in $\mathcal{K}$ is reducible to the modal definability problem w.r.t. $\mathcal{K}$.

Theorem: Reducibility $\mathcal{K}^{\text{fin}}$

The problem of deciding the validity of sentences in $\mathcal{K}^{\text{fin}}$ is reducible to the modal definability problem w.r.t. $\mathcal{K}^{\text{fin}}$.

Summary

Summary and conjectures

The main results of this work can be summarized in the following table:

Classes and status of validity in them		
Classes of structures	Arbitrary cardinality	Finite cardinality
$\mathcal{K}_{\text{commute}}$, $\mathcal{K}^{\text{uni}}_{\text{commute}}$	undecidable	undecidable
$\mathcal{K}^{R_1 \leq n}_{\text{commute}}$, $(\mathcal{K}^{R_1 \leq n}_{\text{commute}})^{\text{uni}}$	decidable	decidable
$\mathcal{K}^{R_1 \leq n, R_2 < \omega}_{\text{commute}}$, $(\mathcal{K}^{R_1 \leq n, R_2 < \omega}_{\text{commute}})^{\text{uni}}$	decidable	decidable
$\mathcal{K}^{R_1 \leq n, R_2 \leq m}_{\text{commute}}$, $(\mathcal{K}^{R_1 \leq n, R_2 \leq m}_{\text{commute}})^{\text{uni}}$	decidable	decidable
$\mathcal{K}_{\text{rectangle}}$, $\mathcal{K}^{\text{uni}}_{\text{rectangle}}$	decidable	decidable
$\mathcal{K}_{\text{square}}$, $\mathcal{K}^{\text{uni}}_{\text{square}}$	decidable	decidable

We conjecture that the modal definability problem w.r.t. all proven decidable classes is decidable.

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