Algebra and Logic Seminar
Institute of Mathematics and Informatics
Bulgarian Academy of Sciences

The Bulgarian Solitaire and Other Games on Partitions

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and

Dedicated to my friends
Ualbai Umirbaev and Leonid Makar-Limanov,
great mathematicians and remarkable persons.

In 2020 the L.N. Gumilyov Eurasian National University (Nur-Sultan, Kazakhstan) decided to organize International Scientific Conference “Algebra and Logic” Dedicated to the 60th anniversary of Professor U. Umirbaev and the 75th anniversary of Professor L. Makar-Limanov Nur-Sultan, May 11-15, 2020

- I was invited to give a talk;
- I prepared it;
- The organizers even booked my ticket;
- But the meeting was canceled because of the Covid-19 pandemic.

The present talk is based on the talk I planned to give in Nur-Sultan.

I shall be honest: Starting from 2015, this is the 7-th time I give a talk on the Bulgarian solitaire. But every time I add something new and take into account the mathematical level of the audience.

Question. Why I have chosen once again this topic for the present talk?
Answer. The son Sergei of Leonid Makar-Limanov discovered the Bulgarian solitaire when he was 12 years old.

- The Bulgarian solitaire and the mathematics around it (invited talk, in Bulgarian), The 44-th Spring Conf. of the Union of Bulgar. Mathematicians, SOK-Kamchia, April 2-6, 2015.
- Games on partitions, International Workshop Groups and Rings – Theory and Applications (GRiTA2015), Sofia, July 15-22, 2015.
- Games on Partitions: The Bulgarian Solitaire, Colloquium, Department of Mathematics, University of Ghent, November 18, 2016.
- The Bulgarian solitaire and the mathematics around it (invited talk, in Bulgarian), Second National Seminar in Computer Mathematics, Hisarya, November 10-12, 2017.
- The Bulgarian Solitaire as a Scientific Heritage, DiPP 2019: Digital Presentation and Preservation of Cultural and Scientific Heritage Burgas, September 26-28, 2019.
- The Bulgarian Solitaire – the Anticucumber of Mircea Crișan, Conference Dedicated to the 70-th Anniversary of IMAR "Simion Stoilow", Bucharest, October 4-5, 2019.

Starting with the conclusion

The (quite amusing) story presented in the talk is an example of this how an elementary mathematical game can inspirit serious mathematical investigations in combinatorics, graph theory, discrete dynamical systems, cellular automata, linear algebra, statistics, economical models. The topic has the advantage that most of the problems can be stated in a form which attracts young people to mathematical research. The talk is based on the paper

V. Drensky, *The Bulgarian solitaire and the mathematics around it*, Math. and Education in Math., Proc. of the 44-th Spring Conf. of the Union of Bulgar. Mathematicians, SOK-Kamchia, April 2-6, 2015, 79-91; arXiv:1503.00885v1 [math.CO].
http://www.math.bas.bg/smb/2015_PK/tom_2015/pdf/079-091.pdf

(where one can find detailed references) with additional information collected after its publishing.

A small experiment: search in Mathematical Reviews

Search: “Bulgarian” in the title

Results: 40 publications, 12 for “Bulgarian solitaire”.

The other publications are for Bulgarian mathematicians, Bulgarian conferences, Bulgarian dictionaries, etc.

Journals (1985-2020):

- Amer. Math. Monthly
- Math. Mag.
- Ars Combin.
- J. Combin. Theory Ser. A
- Random Structures Algorithms
- College Math. J.
- Ann. Sci. Math. Québec
- Integers
- Fibonacci Quart.
- Online J. Anal. Comb.

MSC: 00A, 05A, 05E, 11B, 11P, 60C, 60J, 90D, 91A

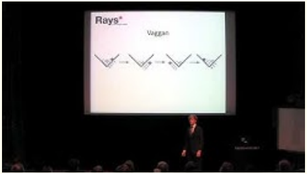
Languages

The Bulgarian solitaire is studied in papers, Ph.D. and M.Sci. theses, preprints, computer codes, works of secondary school students in:

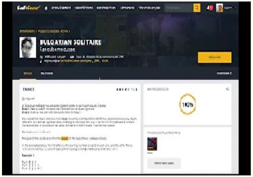
- Russian
- Bulgarian
- Swedish
- English
- French
- German
- Italian
- Portuguese
- There is a list of references in Chinese.

Youtube:

Movie: 5:32 min.



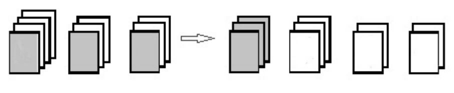
Lecture: 10:46 min.



Codingame: 24:13

The Bulgarian solitaire is a mathematical card game played by one person. A pack of n cards is divided into several decks (or “piles”). Each move consists of the removing of one card from each deck and collecting the removed cards to form a new deck. The game ends when the same position occurs twice.

For example, starting with a deck of 10 cards divided in three packs of size 4, 3, 3, we obtain a new pack of 3 cards and the number of cards in the old packs decreases to 3, 2, 2, respectively.



A deck of 10 cards is divided in three packs of size 4, 3, 3.

Let us "mathematatize" the problem!

It is more convenient to denote only the size of the packs, ordering the sizes in nonincreasing order. For example, starting from the position $(4,3,3)$, we have marked the new size of the new pack in red and have consecutively

$$(4,3,3) \Rightarrow (3,3,2,2) \Rightarrow (4,2,2,1,1) \Rightarrow (5,3,1,1)$$

(In the last step the two packs consisting of a single card disappear.) Then we continue

$$(5,3,1,1) \Rightarrow (4,4,2) \Rightarrow (3,3,3,1) \Rightarrow (4,2,2,2)$$

$$\Rightarrow (4,3,1,1,1) \Rightarrow (5,3,2) \Rightarrow (4,3,2,1) \Rightarrow (4,3,2,1).$$

In this way we obtain the stable position $(4,3,2,1)$.

Further mathematization of the problem

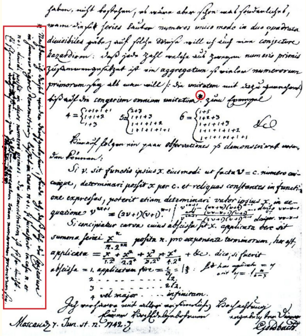
A finite sequence of nonnegative integers

$$\lambda = (\lambda_1, \ldots, \lambda_c), \quad \lambda_1 \geq \ldots \geq \lambda_c \geq 0, \quad \lambda_1 + \cdots + \lambda_c = n,$$

is called a *partition* of n. (The standard notation is $\lambda \vdash n$.) We denote by $\mathcal{P}(n)$ the set of all partitions of n.

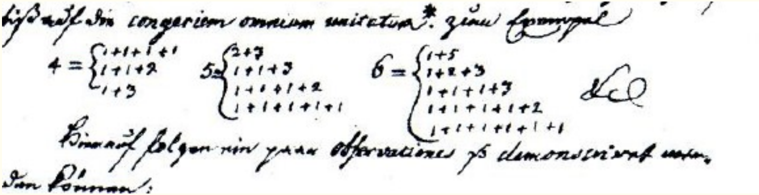
Partitions of integers are classical objects

In a letter to Euler of June 7, 1742 Goldbach states his famous conjecture that every integer greater than 2 is a sum of three primes. (In that time 1 was considered to be prime.)



Es scheinet wenigstens, dass eine jede Zahl, die grösser ist als 2, ein aggregatum trium numerorum primorum sey.

In the same letter Goldbach discusses the problem for presenting the positive integers as partitions in prime parts.


congeriem omnium unitatum*

$$4=\begin{cases}1+1+1+1\\1+1+2\\1+3\end{cases}\qquad 5=\begin{cases}2+3\\1+1+3\\1+1+1+2\\1+1+1+1+1\end{cases}\qquad 6=\begin{cases}1+5\\1+2+3\\1+1+1+3\\1+1+1+1+2\\1+1+1+1+1+1\end{cases}$$

&c.

The Bulgarian solitaire on partitions

In the language of partitions, we have an operator $B : \mathcal{P}(n) \to \mathcal{P}(n)$ which sends the partition $\lambda = (\lambda_1, \dots, \lambda_c)$ with $\lambda_c > 0$ to the partition $\mathcal{B}(\lambda) = (c, \lambda_1 - 1, \dots, \lambda_c - 1)$. If $\lambda_i - 1 > c \geq \lambda_{i+1} - 1$, then we assume that $\mathcal{B}(\lambda) = (\lambda_1 - 1, \dots, \lambda_i - 1, c, \lambda_{i+1} - 1, \dots, \lambda_c - 1)$.
Starting with a partition λ, the game stops when for the first time $\mathcal{B}^{t_1}(\lambda) = \mathcal{B}^{t_2}(\lambda)$ for two different nonnegative integers t_1 and t_2.

In the above example

$$\lambda = (4,3,3) \stackrel{\mathcal{B}}{\to} (3,3,2,2) \stackrel{\mathcal{B}}{\to} (4,2,2,1,1) \stackrel{\mathcal{B}}{\to} (5,3,1,1)$$

$$\stackrel{\mathcal{B}}{\to} (4,4,2) \stackrel{\mathcal{B}}{\to} (3,3,3,1) \stackrel{\mathcal{B}}{\to} (4,2,2,2)$$

$$\stackrel{\mathcal{B}}{\to} (4,3,1,1,1) \stackrel{\mathcal{B}}{\to} (5,3,2) \stackrel{\mathcal{B}}{\to} (3,4,2,1) = (4,3,2,1) \stackrel{\mathcal{B}}{\to} (4,3,2,1).$$

$$\mathcal{B}^9(\lambda) = \mathcal{B}^{10}(\lambda).$$

For $\lambda = (3, 2, 1, 1) \vdash 7$:

$$\lambda = (3, 2, 1, 1) \stackrel{\mathcal{B}}{\to} (4, 2, 1) \stackrel{\mathcal{B}}{\to} (3, 3, 1)$$

$$\stackrel{\mathcal{B}}{\to} (3, 2, 2) \stackrel{\mathcal{B}}{\to} (3, 2, 1, 1) \stackrel{\mathcal{B}}{\to} \mathcal{B}^5(\lambda) = \mathcal{B}(\lambda) = (4, 2, 1).$$

The beginning (for the speaker)

In 2005 I received the following message from the Swedish mathematician Henrik Eriksson:

Dear Vesselin,
I am writing a paper for Amer. Math. Monthly about Bulgarian Solitaire. Twenty-five years ago, my best friend Gert Almkvist visited your Academy of Science. He believes that you told him this problem, which he passed on to me and other western mathematicians. When Martin Gardner wrote about it in Scientific American August 1983, it became world famous and papers are still being written about it.

I would like to trace the origins of Bulgarian Solitaire, so I would be grateful if you could answer these questions.

- Were you indeed the one who told it to Almkvist? (Or was it perhaps Angel Popov?)
- If so, who told you the problem? Is it originally Russian? Almkvist thinks so.
- Who invented the name Bulgarian Solitaire?

Best regards,
Henrik Eriksson, KTH, Stockholm

My answer of November 9, 2005

Dear Henrik,
I think that there was a Russian visitor in our Institute (maybe from St Peterburg) who told us the problem. Two of my colleagues found the solution. In that time Gert Almkvist was in Sofia, visiting Luchezar Avramov and sharing the office with Angel Popov. I do not remember who of us told Gert the problem. In that time many of us (in particular both Angel and me) were interested in problems in elementary mathematics and worked with good secondary school students, preparing the Bulgarian olympic team for the Mathematical Olympiad.
Maybe you can ask two of the Bulgarian mathematicians who, as I think, solved the problem. One of them is Pencho Petrushev and the other is Borislav Boyanov.
Yours Vesselin

I connected Henrik Eriksson with Borislav Bojanov who is one of the main protagonists of the Bulgarian solitaire. Below we follow the story according to the answer of Borislav Bojanov to Henrk Eriksson.

The story

The problem was brought to Bulgaria by the famous number theorist Anatolii Karatsuba from the Steklov Mathematical Institute in Moscow. In May 1980 he visited the Institute of Mathematics and Informatics at the Bulgarian Academy of Sciences in Sofia. Once, after his lecture at the Seminar of Approximation Theory, he told his Bulgarian colleagues the story of the problem.

Konstantin Oskolkov, in that time professor at the Steklov Institute, was traveling from Moscow to Leningrad (now Saint Petersburg) in the night, by the fastest train in the Soviet Union, the so called “Red Arrow”. There was another man in his compartment and they started a conversation. When the other man learned that Konstantin Oskolkov is a mathematician, he showed him the card game.

Surprisingly, it turns out that if the pack consists of $n = 1 + 2 + \cdots + k$ cards, after several moves one reaches the stable position of k piles consisting of $1, 2, \dots, k$ cards, respectively. (The legend claims that the game was illustrated with several experiments with 15 cards.)

Returning back to Moscow, Konstantin Oskolkov told the problem to the people of the Department of Number Theory at the Steklov Institute. Anatolii Karatsuba described this moment in the following way. “When Genadii Arkhipov (professor in Number Theory who liked very much nice problems) learned about the problem, his face took a Satanic expression, he ran to his office, closed the door and did not came out until he solved the problem.”

My Bulgarian colleague Borislav Bojanov also liked very much nice problems. He went home, waited until the children went to the bed and then started to think about it. Around midnight he found a solution and was very happy. The next day he shared the solution with some of his colleagues. Pencho Petrushev said that he also had a solution. Milko Petkov who was an editor of the Bulgarian high school mathematical journal "Obuchenieto po matematika" ("Education in Mathematics") published the problem (as a nonoriginal problem) in the section "Competition Problems" in the issue 5 of 1980. Since no student submitted a solution, in 1981 the Editorial Board of the journal decided to publish the solution of Borislav Bojanov

B. Bojanov, *Problem Solution 4* (Bulgarian), Obuchenieto po matematica **24** (1981), No. 5, 59-60.

The problem in the Russain “Kvant”

Approximately in the same time the problem was published by S. Limanov and A. L. Toom in the issue 11 of 1980 of the Russian mathematical journal “Kvant”. The speaker contacted them and they told him the following.

The story in "Kvant"

In 1980 Sergei Makar-Limanov, the 12 years old son of Leonid Makar-Limanov was playing with cards, and asked his father what would happen if we have n piles of cards, the i-th pile having m_i cards and make the moves described above in the description of the Bulgarian solitaire. Then Leonid Makar-Limanov was the one who thought that it was a good idea to publish the problem in "Kvant".

Sergei Makar-Limanov in April 2018

In 1994 Sergei Makar-Limanov defended at Stanford his Ph.D. Thesis "Tight Contact Structures on Solid Tori" under the supervision of Yakov Eliashberg. Now he is in the computer business.

The story in "Kvant" continues

In that time Andrei Toom was writing a book in collaboration with Vasilyev, Gutenmakher and Rabbot. One of his co-authors told him the game. In any case all they had done with it was some experimentation with real objects (playing cards) but found nothing theoretical around it. In a few days Toom found the geometric interpretation in the form of a triangle which led him to that formula $k(k+1)/2$. His solution appeared also in 1981:

A. L. Toom, *Problem Solution M655* (Russian), Kvant **12** (1981), No. 7, 28-30.

Later it was also included in the book

N. B. Vasilyev, V. L. Gutenmakher, Zh. M. Rabbot, A. L. Toom, *Mathematical Olympiads by Correspondence* (Russian), "Nauka", Moscow, 1986.

The solution of Toom contains also some analysis of the general case of an arbitrary number of cards. It seems that the solutions of Bojanov and Toom are the first published solutions of the Bulgarian solitaire.

The Swedish connection

In that time the Swedish mathematician Gert Almkvist from the University of Lund visited the Department of Algebra at the Institute of Mathematics and Informatics in Sofia. When he learned the problem he brought it to Sweden and told it to his colleagues including his friend Henrik Eriksson from the Royal Institute of Technology in Stockholm. In that time Eriksson already had some experience in the popularization of mathematics as a scriptwriter and a speaker in the educational TV and radio. In 1981 he wrote the paper

H. Eriksson, *Bulgarisk patiens* (Swedish), Elementa **64** (1981), No. 4, 186-188.

where he also presented a solution for the puzzle and gave it the name *Bulgarian solitaire* (*Bulgarisk patiens* in Swedish).

Later Eriksson visited the USA and spread the puzzle there. Jørgen Brandt from the Aarhus University, Denmark, also learnt about the problem but without its name, and in 1982 published another solution

J. Brandt, *Cycles of partitions*, Proc. Amer. Math. Soc. **85** (1982), No. 3, 483-486.

where he also analyzed the general case. (Brandt starts his paper with "The problem to be discussed in the following has been circulating for some time.")

In 1982 Donald Knuth used the Bulgarian solitaire to start his Programming and Problem-Solving Seminar in Stanford. Finally, with the help of Ron Graham the problem reached Martin Gardner who included it in his paper

M. Gardner, *Bulgarian solitaire and other seemingly endless tasks*, Sci. Amer. **249** (1983), 12-21.

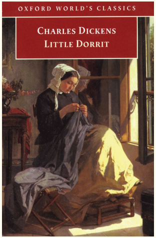
The paper by Gardner was the starting point of the popularity of the Bulgarian solitaire among mathematicians all over the world and was the main source of references for many years. For already 35 years the Bulgarian solitaire and its generalizations continue to inspire new research in combinatorics, game theory, probability, computer science, and to be an object of intensive study in research and teaching literature.

Henrik Eriksson never wrote his paper on the Bulgarian solitaire. But five years later the story appeared in the paper by Brian Hopkins

B. Hopkins, *30 years of Bulgarian solitaire*, College Math. J. **43** (2012), No. 2, 135-140.

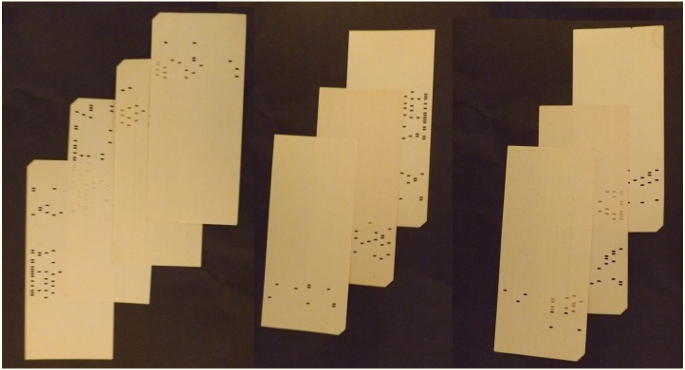
Finally the real story of the Bulgarian solitaire finally reached the large audience.

In the first publications on the Bulgarian solitaire (by Bojanov, Toom, and Eriksson) the main problem is stated in three different ways:
In the Bulgarian version there are $k(k+1)/2$ balls grouped in m piles.
In the Russian version a clerk from the Circumlocution Office rearranges piles of volumes of Encyclopædia Britannica.



(The Circumlocution Office is a place of endless confusion in *Little Dorrit* by Charles Dickens.)
The Swedish text handles packs of cards.

It is interesting how Donald Knuth advices his students: *And you can save money by using old punched cards that computer centers are throwing away, since the cards don't have to have any distinguishing marks.*



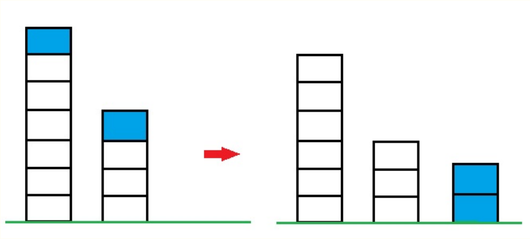
In 2003 the problem appears in Australian Mathematics Challenge in a version with a herd of goats, see
E. Barbeau, *Education notes*, Can. Math. Soc. Notes **36** (2004), No. 5, 21-22.

And recently my colleagues Evgenia Sendova and Petar Kenderov demonstrated the Bulgarian solitaire to mathematics teachers and inspectors with clementines.

Another version of the Bulgarian Solitaire

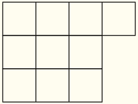
King Kong rearranges the city skyline.
He removes the top floor from every skyscraper

and then makes a new skyscraper out of these floors.



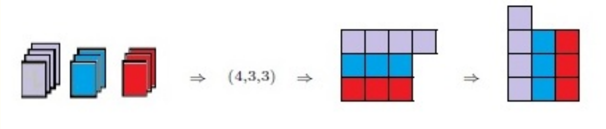
King Kong removes the top floor from two skypescrapers and makes a new skyscraper.

The partition $\lambda = (\lambda_1, \dots, \lambda_c)$ is visualized by its *Young diagram* $[\lambda]$ consisting of boxes arranged in left-justified rows, with λ_i boxes in the i-th row. For example, the Young diagram of the partition $\lambda = (4, 3, 3) \vdash 10$ is

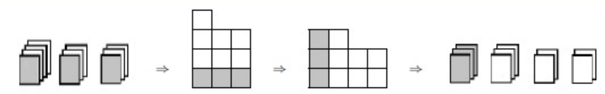


$$[\lambda] = [4, 3, 3].$$

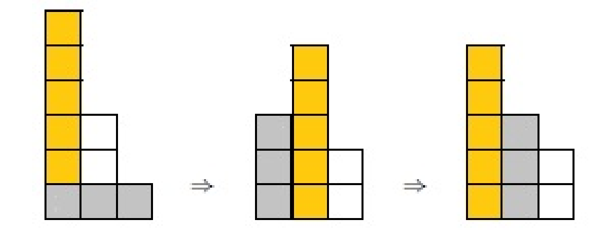
For our purposes it is more convenient to rotate the Young diagram on 90°, when the height of each row is equal to the number of cards in the corresponding pack.



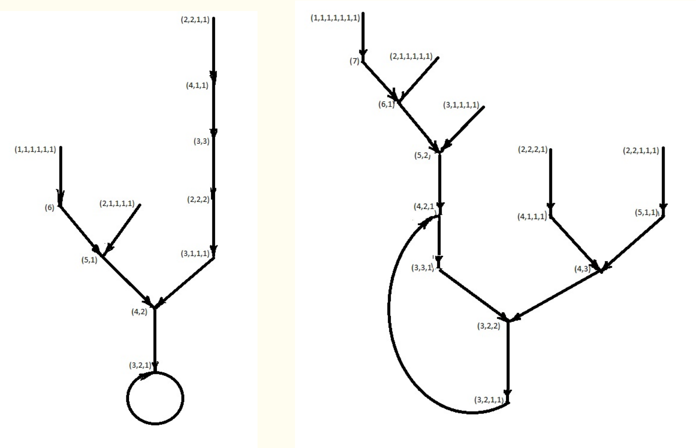
Then the move in the Bulgarian solitaire consists of removing the bottom row of the (rotated) Young diagram and adding it as a column.



As we mentioned in the beginning of the talk, in the language of partitions, we start with a partition $\lambda = (\lambda_1, \dots, \lambda_c)$ with $\lambda_c > 0$ and obtain the partition $\mathcal{B}(\lambda) = (c, \lambda_1 - 1, \dots, \lambda_c - 1)$. Clearly, if $\lambda_i - 1 > c \geq \lambda_{i+1} - 1$, as in the figure below, then we assume that
$\mathcal{B}(\lambda) = (\lambda_1 - 1, \dots, \lambda_i - 1, c, \lambda_{i+1} - 1, \dots, \lambda_c - 1)$.



This is a typical example of a *discrete dynamical system*. We consider the set $\mathcal{P}(n)$ of all partitions of n and the operator $B : \mathcal{P}(n) \to \mathcal{P}(n)$ which in a period of time changes the state of the system, the partition λ, to the new state, the partition $\mathcal{B}(\lambda)$. Hence $\mathcal{B}$ plays the role of the *updating function of the system*. The main problem is, starting with the initial state $\lambda^{(0)}$ to determine the state of the system $\lambda^{(t)} = \mathcal{B}(\lambda^{(t-1)})$ which it will reach after some interval of time t. Since we have a finite number of states $\mathcal{P}(n)$ only, we may associate to the discrete dynamical system its oriented graph with vertices the partitions λ of n and oriented edges $(\lambda, \mathcal{B}(\lambda))$.



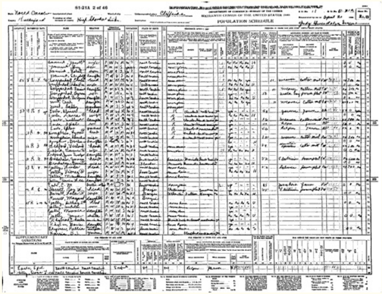
The graph of the Bulgarian solitaire for $n = 6$ and $n = 7$.

Theorem

When the total number $n = k(k+1)/2$ of cards is triangular, the Bulgarian solitaire will converge into piles of size $1, 2, \dots, k$.

We shall give an idea for the solution of the Bulgarian solitaire by Toom from 1981 modified in the spirit of the exposition by Brandt from 1982. We shall use the brilliant visualization of the Bulgarian solitaire, *the cradle model*, suggested by Anders Björner.

There are also several other solutions, using different arguments, e.g., the inductive proof of Meštrović from 2010. There are also suggestions to use spreadsheets to analyze the cases of arbitrary (nontriangular) small n.
S. Sugden, *Spreadsheets and Bulgarian goats*, Internat. J. Math. Education in Science and Technology **43** (2012), No. 7, 953-963.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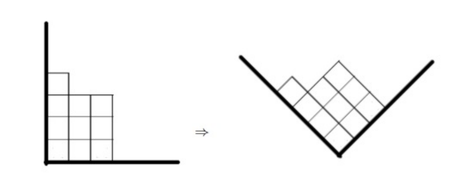
POPULATION SCHEDULE
SUPPLEMENTARY QUESTIONS

Idea of the proof

Let $\lambda = (\lambda_1, \dots, \lambda_c) \vdash n$, $\lambda_c > 0$, be the partition corresponding to the given collection of card packs. We turn the Young diagram $[\lambda]$ counter-clockwise by 45°, and further consider this 45°-turn interpretation of $[\lambda]$.



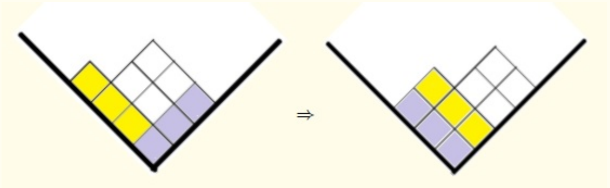
$$\lambda = (4, 3, 3)$$

Assuming that the boxes of $[\lambda]$ are material points with the same mass m, we consider the potential energy of the system

$$U(\lambda) = mg \sum h_{ij},$$

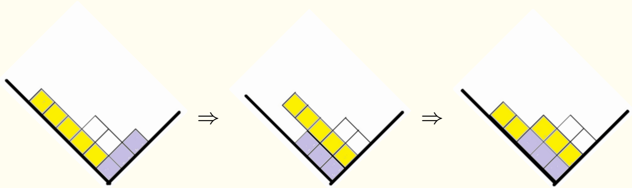
where $g \approx 9,8$ m/s^2 is the free fall acceleration on Earth, the sum runs on all boxes of $[\lambda]$, and h_{ij} is the height of the center of the box with coordinates (i, j) corresponding to the j-th card of the i-th pack. Clearly, h_{ij} is proportional to $i + j$ and we may assume that it is equal to $i + j$ units.

The first move of the Bulgarian solitaire removes the c boxes of the bottom row of $[\lambda]$ and adds them as the first column. In this way, the box $(j,1) \in [\lambda]$ becomes the box $(1,j) \in [\mathcal{B}(\lambda)]$. Obviously, in the 45°-turn interpretation the potential energy of these c boxes of $[\lambda]$ does not change. The other $n-c$ boxes of $[\lambda]$ move one step to the right, from the position (i,j), $j>1$, to the position $(i+1,j-1)$. Hence they also preserve their potential energy. Therefore, if $c \geq \lambda_1 - 1$, the move of the Bulgarian solitaire forces the boxes to cycle on the same level and preserves the potential energy of the Young diagram.



$$\mathcal{B}(4,3,3) = (3,3,2,2)$$

If $c < \lambda_1 - 1$, then by the gravity the excessive boxes of the second pile of the Young diagram of $\mathcal{B}(\lambda)$ will fall down southwest and the potential energy of the Young diagram will decrease.

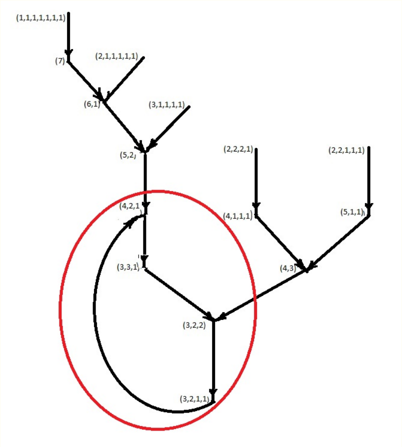


$$\mathcal{B}(6,3,1) = (3,5,2) = (5,3,2).$$

Since the partitions of n are a finite number, we shall reach the position when the moves do not decrease the height of the boxes and the potential energy of the system. With some additional easy combinatorial arguments we can show that the minimal potential energy is reached in the stable position $(k, k-1, \ldots, 2, 1)$ only.

The general case of any n

Consider the oriented graph associated with the set $\mathcal{P}(n)$ of all partitions of n with vertices the partitions $\lambda \in \mathcal{P}(n)$ and oriented edges $(\lambda, \mathcal{B}(\lambda))$. Clearly, the graph consists of several components and, starting from any vertex of a given component, the multiple application of the operator $\mathcal{B}$ defines a cycle which is unique for the component. For example the cycle of the unique component of the graph of $\mathcal{P}(7)$ is

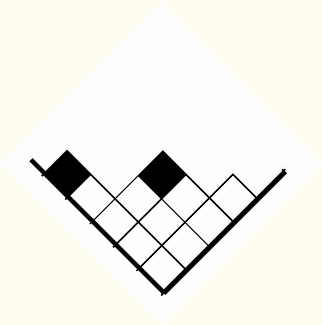


$$\lambda = (4,2,1) \xrightarrow{\mathcal{B}} (3,3,1) \xrightarrow{\mathcal{B}} (3,2,2)$$

$$\xrightarrow{\mathcal{B}} (3,2,1,1) \xrightarrow{\mathcal{B}} \mathcal{B}^4(\lambda) = \lambda = (4,2,1).$$

Theorem

Let n *have the form* $n = (k-1)k/2 + r$, $0 < r \leq k$, *and let* $\lambda \in \mathcal{P}(n)$. *Then in the interpretation of the cradle model the solitaire will converge with a cycle of partitions which consists of the triangular partition as bottom and* r *surplus blocks cycling above.*



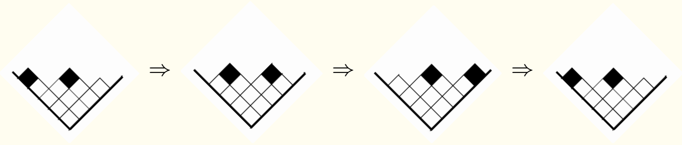
The number of the components of the oriented graph associated with the partitions of n is equal to the number of necklaces consisting of r black beads and $k-r$ white beads, where the symmetry group of the necklace is the cyclic group of order k. It is

$$C(n) = \frac{1}{k} \sum_{d|(r,k)} \varphi(d) \binom{k/d}{r/d},$$

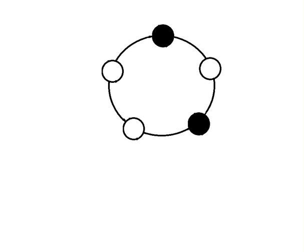
where (r,k) is the greatest common divisor of r and k and $\varphi(d)$ is the Euler φ-function, i.e., the number of positive integers $\leq d$ and relatively prime to d.

(The number of necklaces with r black and $k-r$ white beads can be obtained applying the Pólya enumeration theorem.)

1-to-1 correspondence between cycles and necklaces



$$\Updownarrow$$



The longest path for triangular numbers

For triangular $n = k(k+1)/2$, already Toom in 1981 raised the problem *to determine the longest path in the graph of* $\mathcal{P}(n)$ *to reach the stable partition* $\sigma = (k, k-1, \dots, 2, 1)$. He showed that, *starting from the partition* $\tau = (k-1, k-1, k-2, k-3, k-4, \dots, 3, 2, 1, 1)$, *the minimal* s *with the property* $\sigma = \mathcal{B}^s(\tau)$ *is* $s = k(k-1)$. Knuth checked the equality

$$\sigma = (k, k-1, \dots, 2, 1) = \mathcal{B}^{k(k-1)}(\lambda), \quad \lambda \in \mathcal{P}(k(k+1)/2),$$

for $k \leq 5$. He asked his students to write a computer program to check it for $k \leq 10$ *and conjectured that this holds for any* k.

The conjecture of Knuth was proved by Igusa and Bentz in 1985-1987. Bentz also established the following interesting property of the partition τ considered by Toom: *The partitions* $\mathcal{B}^i(\tau)$ *and* $\mathcal{B}^{k(k-1)-i-1}(\tau)$ *are conjugate for* $i = 0, 1, 2, \ldots, k(k-1) - 1$. This means that the related Young diagrams are obtained by reflection with respect to the bisectrix from the origin of the first quadrant of the coordinate plane, i.e., the lengths of the columns of one diagram are equal to the lengths of the rows of the other. The general case of an arbitrary n was studied by Etienne (1991).

How specific is the graph $\mathcal{P}(n)$?

By the classical result of G. H. Hardy and Ramanujan in 1918 and independently by J. V. Uspensky in 1920, the asymptotic estimate for the number of partitions of n is

$$p(n) = |\mathcal{P}(n)| \approx \frac{1}{4n\sqrt{3}} \exp\left(\pi\sqrt{\frac{2n}{3}}\right).$$

Let $\mathcal{T}(N)$ is the set of all labeled trees with N vertices. Another classical theorem

A. Cayley, *A theorem on trees*, Quart. J. of Pure and Applied Math. **23** (1889), 376-378.

gives that

$$t(N) = |\mathcal{T}(N)| = N^{N-2}.$$

For triangular $n = k(k+1)/2$ the graph $\mathcal{P}(n)$ of all partitions of n is an oriented labeled tree with the root the partition $\sigma = (k, k-1, \dots, 2, 1)$ planted on the loop around σ. For a while we remove this loop and consider the obtained tree. The theorem of Igusa and Bentz gives that the longest path in $\mathcal{P}(n)$ is of length $k(k-1) \approx 2n$.
One of the fundamental result in the statistics of trees is in
A. Rényi, G. Szekeres, *On the height of trees*, J. Aust. Math. Soc. **7** (1967), 497-507.
Let $\mathcal{H}(N)$ be the height over the root of a labeled random tree with N vertices, i.e., of a tree selected at random from the set of N^{N-2} elements of $\mathcal{T}(N)$ with uniform probability distribution. Then the expectation value of $\mathcal{H}(N)$ is $\sqrt{2N\pi} \approx 2.50663\sqrt{N}$.

Hence for $N = p(n) = |\mathcal{P}(n)|$ and $n = k(k+1)/2$ the expectation value of the height over the root of a labeled random tree with N vertices

$$\mathcal{H}(N) = \sqrt{2N\pi} \approx \frac{1}{\sqrt[4]{3}}\sqrt{\frac{\pi}{2n}}\exp\left(\pi\sqrt{\frac{n}{6}}\right)$$

is much larger than the height $k(k-1) \approx 2n$ of the tree of the partitions $\mathcal{P}(n)$ of n. Therefore, $\mathcal{P}(n)$ consists of many branches of small length.

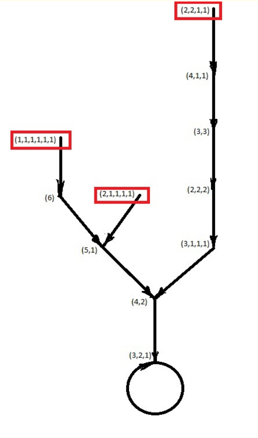
Gardens of Eden

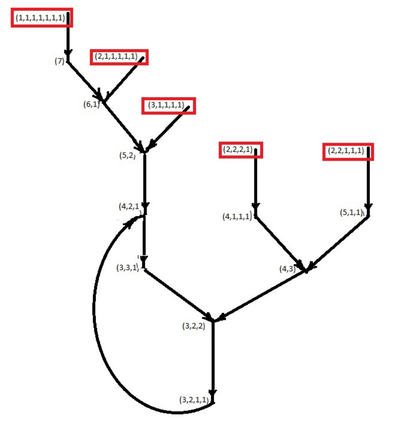
In the theory of cellular automata, a *Garden of Eden configuration* is a configuration that cannot appear on the lattice after one time step, no matter what the initial configuration. In other words, these are the configurations with no predecessors. The terminology comes from the foundational paper

E. F. Moore, *Machine models of self-reproduction*, Proc. Sympos. Appl. Math. **14** (1962), 17-33.

by analogy with the concept of the Garden of Eden which, following Semitic religions, was created out of nowhere. (And one cannot enter there, one can only get out.) In 2006-2010 Hopkins (alone and with co-authors) studied the *Garden of Eden partitions* (*GE-partitions*) defined by the property that they do not belong to the image $\mathcal{B}(\mathcal{P}(n))$. It has turned out that *each cycle in the oriented graph of* $\mathcal{P}(n)$ *can be reached from a GE-partition*. We shall mention only the following easy property.

A partition $\lambda = (\lambda_1, \ldots, \lambda_s) \vdash n$, $\lambda_s > 0$, *is a GE-partition if and only if* $\lambda_1 < s - 1$.





Gardens of Eden for $n = 6$ and $n = 7$.

The adjacency matrix of $\mathcal{P}(n)$

There are two matrices related with any graph Γ with N vertices $v_1, \dots, v_N$ and E edges:

- the incidence $E \times N$ matrix with rows labeled by the edges and columns labeled by the vertices;
- the adjacency $N \times N$ matrix $A(\Gamma)$ with rows and columns labeled by the vertices and with entries

$$a_{ij} = \begin{cases} 1, & \text{if } (v_j, v_i) \text{ is an edge;} \\ 0, & \text{otherwise.} \end{cases}$$

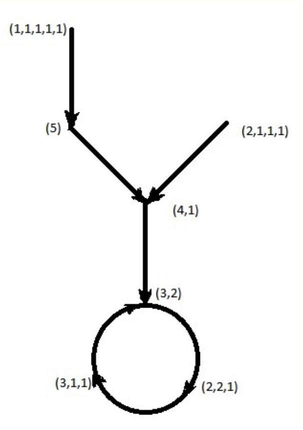
The adjacency matrix $A(\Gamma)$ has the remarkable property that the (i,j)-entry of $A^m(\Gamma)$ is equal to the number of paths of length m from v_j to v_i.

The adjacency matrix of a simple nonoriented graph is symmetric (and hence diagonalizable) with zero diagonal and was intensively studied.

Problem

What is the Jordan normal form of the adjacency matrix $A(\mathcal{P}(n))$ of the graph $\mathcal{P}(n)$ of the partitions of n for n arbitrary?

Each column of $A(\mathcal{P}(n))$ has one entry equal to 1 and all other entries are equal to 0.



The graph of $\mathcal{P}(5)$

$A(\mathcal{P}(5))$	(5)	$(4,1)$	$(3,2)$	$(3,1^2)$	$(2^2,1)$	$(2,1^3)$	(1^5)
(5)	0	0	0	0	0	0	1
$(4,1)$	1	0	0	0	0	1	0
$(3,2)$	0	1	0	1	0	0	0
$(3,1^2)$	0	0	0	0	1	0	0
$(2^2,1)$	0	0	1	0	0	0	0
$(2,1^3)$	0	0	0	0	0	0	0
(1^5)	0	0	0	0	0	0	0

Matrices with entries 1 and 0 only with the property that each column has exactly one entry equal to 1 were considered in
D. A. Cardon, B. Tuckfield, *The Jordan canonical form for a class of zero-one matrices*, Linear Algebra Appl. **435** (2011), No. 11, 2942-2954.

The main motivation there was related with the Collatz conjecture: Let

$$f(n) = \begin{cases} 3n+1, & \text{if } n \text{ is odd;} \\ \frac{n}{2}, & \text{if } n \text{ is even.} \end{cases}$$

The conjecture states that for each $k \in \mathbb{N}$, the sequence $k, f(k), (f \circ f)(k), (f \circ f \circ f)(k), \ldots$ contains the number 1.

Recent progress on the Collatz conjecture

Terence Tao posted the preprint

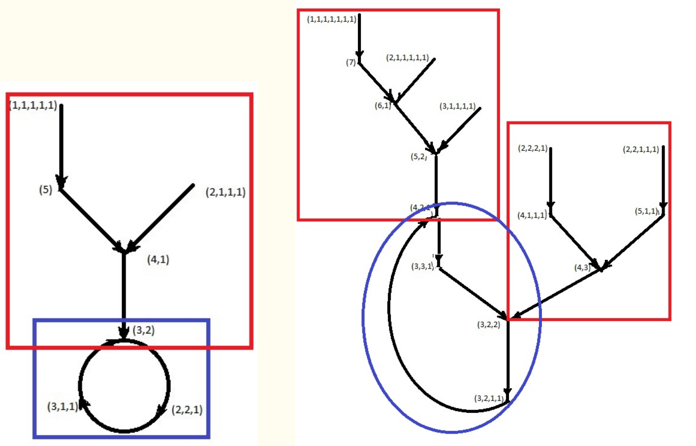
T. Tao, *Almost all orbits of the Collatz map attain almost bounded values*, arXiv:1909.03562v4 [math.PR]. (Latest version posted on September 18, 2021

He proved that for almost all positive integers k (in the sense of logarithmic density) the minimal element $\mathrm{Col}_{\mathrm{min}}(k)$ in the orbit

$$\{k, f(k), (f \circ f)(k), (f \circ f \circ f)(k), \dots\}$$

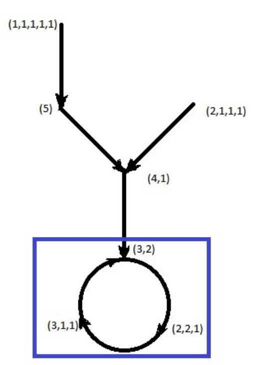
satisfies the inequality $\mathrm{Col}_{\mathrm{min}}(k) < f(k)$. This for instance implies$\mathrm{Col}_{\mathrm{min}}(k) < \log\log\log\log k$ for almost all k.

Every matrix of this kind corresponds to the adjacency matrix of an oriented graph Γ similar to the graph of $\mathcal{P}(n)$. The graph consists of a forest planted to cycles.

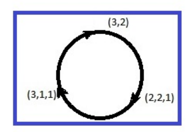


Graphs of $\mathcal{P}(5)$ and $\mathcal{P}(7)$.

Each cycle of length k corresponds to a diagonal $k \times k$ block of the Jordan normal form of the matrix $A(\Gamma)$ with the k-th roots of 1 on the diagonal.



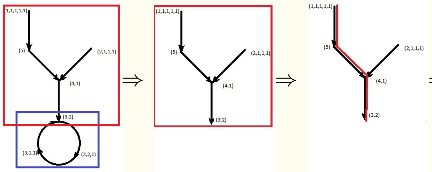
$\Rightarrow$



$$\Rightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & \varepsilon & 0 \\ 0 & 0 & \varepsilon^2 \end{pmatrix}$$

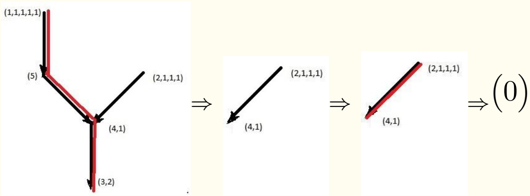
$$\varepsilon^3 = 1,\ \varepsilon \neq 1$$

We remove the cycles and consider the longest path in the obtained forest. This path corresponds to a Jordan block with zero diagonal of size equal to the length of the path.

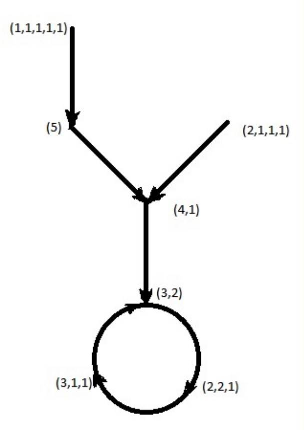


$$\Rightarrow \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

We remove the longest path of the forest and continue in the same way until we remove all the paths. The result will be the Jordan normal form of the matrix.



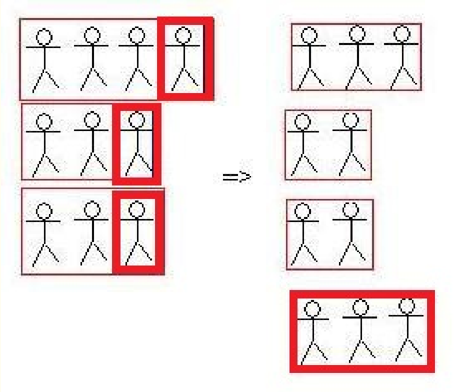
The result for $A(\mathcal{P}(5))$ is



$$\Rightarrow \begin{pmatrix} 1 & & & & & & \\ & \varepsilon & & & & & \\ & & \varepsilon^2 & & & & \\ & & & 0 & 1 & 0 & \\ & & & 0 & 0 & 1 & \\ & & & 0 & 0 & 0 & \\ & & & & & & 0 \end{pmatrix}$$

Real life interpretation of the Bulgarian solitaire

The Bulgarian solitaire reflects the following situation from the real life. *Consider a company consisting of a number of departments. The Board of Directors decides to create a new department, but does not want to increase the total number of employees. So, the Board takes a member from the existing departments and move the person to the new department.* If we assume that the number of cards in the piles is equal to the number of persons in the departments, the Bulgarian solitaire corresponds to the "greediest" case, when the new department is formed by taking a person from each department of the company.



Generalizations

- Austrian solitaire
- Carolina solitaire
- Montreal solitaire
- Red-green Bulgarian solitaire
- Three-dimensional Bulgarian solitaire
- Multiplayer Bulgarian solitaire
- Solitaires with several players sitting around a circular table
- Dual Bulgarian solitaire
- Random Bulgarian solitaire of Popov
- Stochastic Bulgarian solitaire of Kimmo Eriksson, Markus Jonsson, and Jonas Sjöstrand
- Generalized Bulgarian solitaire (2020)

Austrian solitaire (1985)

It has the following economic interpretation. *A company has several machines. Each machine has, when new, a life of exactly* L *years. Each year for each machine on line the company deposits* $1/L$ *of its cost into the bank as a sinking fund. Then it buys as many new machines as it can afford, and the remaining funds are left in the bank until next year.*

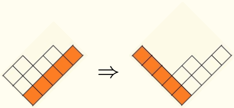
Dual Bulgarian solitaire

The piles are ordered in nonincreasing order. In each move the largest pile is removed and its cards are distributed to the remaining piles one by one, from larger to smaller, with any excessive blocks forming piles of size 1. For example, the partition $(4,4,3,2,2,1,1)$ goes to $(5,4,3,3,1,1)=(4+1,3+1,2+1,2+1,1+0,1+0)$, and $(6,6,3,2,1)$ goes to $(7,4,3,2,1,1)=(6+1,3+1,2+1,1+1,0+1,0+1)$.

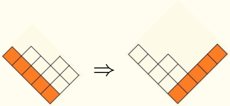
In the regular Bulgarian solitaire: *We may assume that each part of a partition corresponds to the income of a citizen or a company. Then the Government collects the same taxes from each person and each company and uses the collected money to make a new company.*
In the dual game: *One applies the principles of Robin Hood: taking from the rich and giving to the poor.* It can be shown that *if Robin Hood continues to take from the rich and give to the poor, then the distribution of fortune in his community will become close to triangular.*

Explanation

In the cradle model, the dual Bulgarian solitaire is equivalent to the regular one and can be obtained from it by transposition.



The ordinary solitaire $(3,3,2,2,1) \Rightarrow (5,2,2,1,1)$



The dual solitaire $(3,3,2,2,1)^t = (5,4,2) \Rightarrow (5,3,1,1,1) = (5,2,2,1,1)^t$

It turns out that *the dual Bulgarian solitaire corresponds to the old African game Owari which consists of cyclically ordered pits that are filled with pebbles. In a sowing move all the pebbles are taken out of one pit and distributed one by one in subsequent pits. Repeated sowing will give rise to recurrent states of the Owari.*

Instead of conclusion: The anticucumber of Mircea Crișan

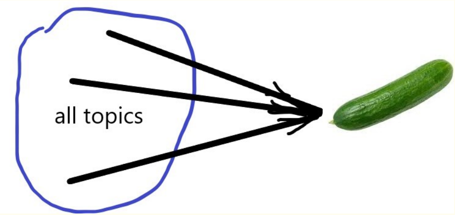
Every Bulgarian and many East Europeans born in the 1950s and 1960s remember the cucumber of Mircea Crișan.

In Romanian – Castravetele

In German – Die Gurke

In Bulgarian with Sorina Dan – Краставицата

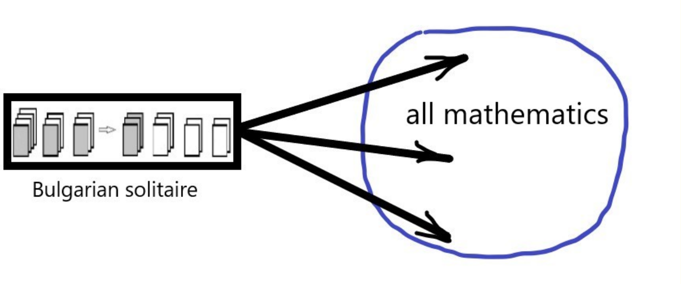
The story is about a student who during the exam tries from any topic to reach the cucumber, which ... and contains 98 % water.



Including the Suez canal which ... contains 98 % water.

The Bulgarian solitaire is in the opposite direction

From it one can reach almost all areas of mathematics.



The anticucumber of Mircea Crișan

My contributions

- I am almost sure that I told the Bulgarian solitaire to Gert Almkvist in 1980;
- I connected Henrik Eriksson with Borislav Boyanov in 2005;
- I wrote *The Bulgarian solitaire and the mathematics around it* in 2015;
- All the time I am trying to popularize the Bulgarian solitaire.

Acknowledgements for my efforts

- My paper was read > 1000 times (by researchgate.net: 1080 reads on February 10, 2022);
- The paper inspired a short movie posted in Youtube;
- I received the following letter from Henrik Eriksson:

From: Henrik Eriksson [he@kth.se]
Sent: Monday, March 30, 2015 3:59 PM
To: Vesselin Drensky
Subject: Bulgarian solitaire

Високо-уважаеми професор и член на почитаната Българска Академия на Науките,

or rather, Dear Vesselin,

I cannot thank you enough for your wonderful paper on Bulgarian solitaire! It is so exciting, well written and illustrated, incredibly well researched, and generous as concerns the involvement of me and my circle (my best friend Gert, my adviser Bjorner, my son Kimmo, my nephew Jonas and my protégés, Anton and Emil).

Your paper also freed me of my bad conscience for not writing up the story ten years ago. I am forever yours

Henrik Eriksson

THANK YOU VERY MUCH FOR YOUR ATTENTION!