Transfinite version of the Mittag-Leffler condition for the vanishing of the derived limit

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- The study of the right derived functors of the functor of limit was initiated in the works of Yeh, Milnor, Roos and Grothendieck.
- In particular, it was shown that for inverse sequences of abelian groups, $\lim^i = 0$ for $i > 1$, but the functor $\lim^1$ turned out to be non-trivial.
- Milnor emphasized the significant role of this functor in algebraic topology by introducing what is now known as Milnor exact sequence for homotopy groups.
- It was also proven that if the inverse sequence consists of epimorphisms, the derived limit is trivial.
- Grothendieck in [1] introduced the Mittag-Leffler condition for an inverse sequence, which generalizes the epimorphism condition, and proved that it also implies the vanishing of the derived limit.

- If all components of an inverse sequence are at most countable abelian groups, then the Mittag-Leffler condition becomes necessary and sufficient for the vanishing of the derived limit.
- However, for arbitrary inverse sequences of abelian groups, it is not a necessary condition.
- There arose a need to find a necessary and sufficient variant of this condition.

The same results can be proved for any abelian category with a generator G, small direct sums $(\mathrm{AB3})$ and exact small products $(\mathrm{AB4}^*)$.

Inverse sequences

Definition

An inverse sequence of abelian groups A is a couple consisting of two families $A = ((A_i), (f_i))$ indexed by natural numbers $i \in \omega$, where A_i is an abelian groups and $f_i : A_{i+1} \to A_i$ is a homomorphism.

$$A_1 \xleftarrow{f_1} A_2 \xleftarrow{f_2} A_3 \xleftarrow{f_3} A_4 \xleftarrow{f_4} A_5 \xleftarrow{f_5} \dots$$

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We call an inverse sequence A *null-sequence* if every f_i is a null-mapping.
We call an inverse sequence A *epimorphic* if every f_i is an epimorphism.

Definition

Let $A = ((A_i), (f_i))$ be an inverse sequence. The limit mapping of A is the homomorphism

$$\phi : \Pi_{i=1}^{\infty} A_i \to \Pi_{i=1}^{\infty} A_i$$

defined as follows:
$\phi(a_1, a_2, a_3, \dots) = (a_1 - f_1(a_2), a_2 - f_2(a_3), \dots)$.

Limit and derived limit

Suppose A is an inverse sequence and ϕ is the limit mapping of A.

Definition

Inverse limit $\lim A := Ker(\phi)$.

All the sequences $(a_1, a_2, a_3, ...)$, such that
$(a_1, a_2, a_3, ...) = (f(a_2), f(a_3), f(a_4), ...)$.

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Definition

Derived inverse limit $\lim^1 A := Coker(\phi)$.

If $\lim^1 A = 0$, then for any sequence $(b_1, b_2, b_3, \ldots)$ exists sequence $(a_1, a_2, a_3, \ldots)$ such that
$(a_1 - f(a_2), a_2 - f(a_3), \ldots) = (b_1, b_2, b_3, \ldots)$.

Connection of limits

Proposition

Suppose we have the short exact sequence of inverse sequence $A \hookrightarrow B \twoheadrightarrow C$, then there is an exact sequence of abelian groups:

$$0 \to \lim\ A \to \lim\ B \to \lim\ C \to \lim^1 A \to \lim^1 B \to \lim^1 C \to 0$$

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Example

$$A := \mathbb{Z} \xleftarrow{i} p\mathbb{Z} \xleftarrow{i} p^2\mathbb{Z} \xleftarrow{i} p^3\mathbb{Z} \xleftarrow{i} p^4\mathbb{Z} \xleftarrow{i} \dots$$

$$0 \to A \to \{\mathbb{Z}\} \to \{\mathbb{Z}/p^i\mathbb{Z}\} \to 0$$

$\lim^1 A = \hat{\mathbb{Z}}_p/\mathbb{Z}$

Mittag-Leffler condition

Definition

For any $i \in \mathbb{N}$ Let us define image filtration $I^\alpha(A_i)$ for an inverse sequence A recursively for any ordinal α:

- $I^0(A_i) := A_i$
- $I^{\alpha+1}(A_i) := f_i(I^\alpha(A_{i+1}))$
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We say that A satisfies the Mittag-Leffler condition if for every i filtration $I^k(A_i)$ stabilizes in a finite number of steps.

This condition is sufficient for the vanishing of $\lim^1 A$ but not necessary if groups are uncountable.

Definition

For an inverse sequence A let us define for every ordinal α

$$A^{\alpha} := A/I^{\alpha}(A)$$

For any limit ordinal λ the λ-*completion* of A is defined as

$$\widehat{A}^{\lambda} = \lim_{\beta<\lambda} A^{\beta}. \tag{1}$$

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A is called λ-*complete*, if the morphism $A \to \widehat{A}^\lambda$ is an epimorphism. Also this morphism induces a monomorphism

$$A^\lambda \rightarrowtail \widehat{A}^\lambda. \tag{2}$$

Preservation of the limit

Proposition

For any ordinal α there are isomorphisms

$$\lim A^\alpha = \lim \widehat{A}^\alpha = 0, \qquad \lim I^\alpha(A) \cong \lim A, \tag{3}$$

where the last isomorphism is induced by the monomorphism $I^\alpha(A) \rightarrowtail A$. Moreover, there is a short exact sequence

$$\lim{}^1 I^\alpha(A) \rightarrowtail \lim{}^1 A \twoheadrightarrow \lim{}^1 A^\alpha. \tag{4}$$

Length of the transfinite image filtration

Definition

The *length of the transfinite image filtration* of A is the the least ordinal $\mathrm{len}(A)$ such that for any $\alpha > \mathrm{len}(A)$ the monomorphism $I^{\alpha}(A) \rightarrowtail I^{\mathrm{len}(A)}(A)$ is an isomorphism.

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Proposition

For an inverse sequence A the ordinal $\mathrm{len}(A)$ is well defined. Moreover, in this case $I^{\mathrm{len}(A)}(A)$ is an epimorphic inverse sequence, and the canonical morphisms $I^{\mathrm{len}(A)}(A) \rightarrowtail A \twoheadrightarrow A^{\mathrm{len}(A)}$ induce isomorphisms

$$\lim{}^1 A \cong \lim{}^1 A^{\mathrm{len}(A)}, \qquad \lim A \cong \lim I^{\mathrm{len}(A)}(A). \tag{5}$$

Corollary

For an inverse sequence A the following statements are equivalent

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If S is an epimorphic inverse sequence and $\lim S = 0$, *then* $S = 0$.

Local sequences

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For a general category $\mathcal{C}$ and a morphism $\theta : c' \to c$ an object I of $\mathcal{C}$ is called θ-*local*, if the map $\theta^* : \mathcal{C}(c, I) \to \mathcal{C}(c', I)$ is a bijection. The class of θ-local objects is closed with respect to small limits [2, §1.5]. Further, we show that local inverse sequences are θ-local objects with respect to particular morphism θ in the category of inverse sequences. As a corollary, we obtain that the class of local inverse sequences is closed with respect to small limits.

Definition

For a natural number n we denote by $\mathbb{Z}(n)$ the inverse sequence such that $\mathbb{Z}(n)_i = \mathbb{Z}$ for $i \le n$, $\mathbb{Z}(n)_i = 0$ for $i > n$, and $f_i = 1_{\mathbb{Z}}$ for $i < n$.

$$\mathbb{Z}(n) = \mathbb{Z} \xleftarrow{1_{\mathbb{Z}}} \dots \xleftarrow{1_{\mathbb{Z}}} \mathbb{Z} \leftarrow 0 \leftarrow 0 \leftarrow \dots$$

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Lemma

For any inverse sequence A there is a natural isomorphism

$$\mathbf{Ab}^{(\omega^{op})}(\mathbb{Z}(n), A) \cong \mathbf{Ab}(\mathbb{Z}, A_n), \qquad \varphi \mapsto \varphi_n. \tag{6}$$

Generator of inverse sequences

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Let us define our morphism $\psi : \tilde{\mathbb{Z}} \to \tilde{\mathbb{Z}}$ such that $\psi(0, 0, \dots, 0, 1, 0, \dots) = (0, 0, \dots, 0, -1, 1, 0, \dots)$, that is, an element that has 1 in place n and all the other zeros, it sends to the element that has 1 in place of n and -1 in place of $n-1$.

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Proposition

An inverse sequence A is local if and only if it is a ψ-local object.

Small limits

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Proposition

The class of local inverse sequences is closed with respect to extensions by null-sequences.

Localization in the category $\mathcal{C}$, this is the endofunctor $L : \mathcal{C} \to \mathcal{C}$ for which there is a natural transformation $\eta : \mathbf{Id} \to L$ such that $\eta L = L\eta : L \to L^2$ is an isomorphism.

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In this case, the object $c \in \mathcal{C}$ is called *local* with respect to L if $\eta_c : c \to Lc$ is an isomorphism.

Corollary

In the category $\mathbf{Ab}^{(\omega^{op})}$ there is a localization for which local objects are exactly all local sequences.

Transfnite Mittag-Leffler condition

Theorem

Let A be an inverse sequence. Then the following statements are equivalent:

1. $\lim^1 A = 0$;
2. $\lim \mathrm{Coker}(A \to \widehat{A}^\lambda) = 0$ *for any limit ordinal* λ*;*
3. *for a limit ordinal* λ*, if cofinality of* λ *is countable, then* A *is* λ*-complete, if the cofinality of* λ *is uncountable, then* $\lim \mathrm{Coker}(A \to \widehat{A}^\lambda) = 0$.

A description of the class of local inverse sequences

Theorem

The class of local inverse sequences is the least class containing the zero inverse sequence and closed with respect to small limits and null-extensions.

Sequences defined by one abelian group.

Definition

Consider an inverse sequence $S(A)$ such that $S(A)_i = A$ and $f_i(a) = pa$.

$$S(A): \qquad A \xleftarrow{p\cdot} A \xleftarrow{p\cdot} A \xleftarrow{p\cdot} \dots \tag{9}$$

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Let us fix an ordinal α.

Definition

Denote by $\alpha^\diamond$ the set of all finite increasing sequences of ordinals $(\alpha_1, \dots, \alpha_n)$ such that $\alpha_1 < \dots < \alpha_n < \alpha$, $n \geq 1$. We endow $\alpha^\diamond$ by the deg-lex order: $(\alpha_1, \dots, \alpha_n) < (\alpha'_1, \dots, \alpha'_{n'})$ if and only if either $n < n'$, or $n = n'$ and there exists $1 \leq m \leq n$ such that $\alpha_m < \alpha'_m$ and $\alpha_i = \alpha'_i$ for any $1 \leq i < m$. It easy to check that it is a well order.

A local sequence with a long image filtration

Definition

Consider the direct product and the direct sum of the group of p-adic integers $\mathbb{Z}_p$ indexed by $\alpha^\diamond$

$$P_\alpha := \mathbb{Z}_p^{\prod \alpha^\diamond}, \qquad P'_\alpha := \mathbb{Z}_p^{\oplus \alpha^\diamond}. \tag{10}$$

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We denote by $(e_\sigma)_{\sigma \in \alpha^\diamond}$ the standard basis of P'_α over $\mathbb{Z}_p$. Consider a $\mathbb{Z}_p$-submodule of $R_\alpha \subseteq P'_\alpha$ generated by the elements of the form

$$r_{\alpha_1} := p e_{\alpha_1}, \qquad r_{\alpha_1,\dots,\alpha_n} := p e_{\alpha_1,\dots,\alpha_n} - e_{\alpha_2,\dots,\alpha_n}, \qquad n \geq 2, \tag{11}$$

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$$r_{\alpha_1} := pe_{\alpha_1}, \qquad r_{\alpha_1,\dots,\alpha_n} := pe_{\alpha_1,\dots,\alpha_n} - e_{\alpha_2,\dots,\alpha_n}, \qquad n \geq 2, \tag{11}$$

$$D_\alpha := P_\alpha / R_\alpha, \quad D'_\alpha := P'_\alpha / R_\alpha, \quad E_\alpha := D_\alpha / p^\alpha D_\alpha. \tag{12}$$

Theorem

For any ordinal α the inverse sequence $S(E_\alpha)$ is local and

$$\mathrm{len}(S(E_\alpha)) = \alpha. \quad (13)$$

Definition

Let κ be a regular cardinal. For a family of abelian groups $(A_x)_{x\in X}$ we define its κ-supported product as the subgroup $\prod_{x\in X}^{(\kappa)} A_x$ of the product $\prod_{x\in X} A_x$ consisting of elements whose support has cardinality $< \kappa$.

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Lemma

If J is a small category such that $|\mathrm{Ob}(J)|, |\mathrm{Mor}(J)| < \kappa$, then for any family of functors to the category of abelian groups $(A_x : J \to \mathrm{Ab})_{x\in X}$ there is a natural isomorphism

$$\prod^{(\kappa)}_{x\in X} \left(\lim_J A_x \right) \cong \lim_J \left(\prod^{(\kappa)}_{x\in X} A_{x,y} \right).$$

Lemma

If κ is uncountable cardinal and there is a family of inverse sequences $(S_x)_{x\in X}$, then there is a natural isomorphism

$$\lim{}^1\left(\prod\nolimits_{x\in X}^{(\kappa)} S_x\right) \cong \prod\nolimits_{x\in X}^{(\kappa)} \left(\lim{}^1 S_x\right).$$

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Corollary

Let κ be a regular uncountable cardinal and $(A_x)_{x\in X}$ be a family of local inverse sequences of abelian groups. Then the κ-supported product $\prod_{x\in X}^{(\kappa)} A_x$ is local.

Theorem

Let κ be a regular uncountable cardinal, $A = \prod_{\alpha<\kappa}^{(\kappa)} E_{\alpha+1}$ and $S = S(A)$. Then S is local, not κ-complete and $\mathrm{len}(S) = \kappa$.

Question

Is there a ω-complete inverse sequence of abelian groups with nonzero derived limit?

Alexander Grothendieck and J Dieudonné.
Ega iii, premiere partie.
Publ. Math. IHES, 11, 1961.

Assaf Libman.
Cardinality and nilpotency of localizations of groups and g-modules.
Israel Journal of Mathematics, 117:221–237, 2000.