Noncommutative Invariants of Dihedral Groups

Boyan Kostadinov
Institute of Mathematics and Informatics,
Bulgarian Academy of Sciences

Joint work with Vesselin Drensky

Algebra and Logic Seminar
(Sofia, Bulgaria, 17th November 2023)

The speaker is supported by National Programme "Young Scientists and Post Doctoral Researchers – 2" of the Ministry of Education and Science of Republic of Bulgaria and partially supported by the Bulgarian National Science Foundation, contract KP-06 N 32/1.

Based on the paper

- Vesselin Drensky, Boyan Kostadinov, Noncommutative invariants of dihedral groups, arXiv: 2311.09380 [math.RA].

Definition 1

A vector space R is called an *algebra* over a field K (or a K-*algebra*) if R is equipped with a binary operation $*$ (i.e., mapping $* : (R, R) \to R$), called *multiplication*, such that for any $a, b, c \in R$ and any $\alpha \in K$

$$(a + b) * c = a * c + b * c,$$

$$a * (b + c) = a * b + a * c,$$

$$\alpha(a * b) = (\alpha a) * b = a * (\alpha b).$$

Usually, we denote the multiplication of R by $\cdot$ (and write ab instead of $a \cdot b$), by $\times$, etc. Initially we do not require $1 \in R$, the associativity of R, etc.

I. Definitions – Subalgebra & Ideal

The subspace S of the algebra R is called a *subalgebra* if it is closed with respect to the multiplication, i.e. $s_1, s_2 \in S$ implies $s_1 * s_2 \in S$.

The subalgebra I of R is called a *left ideal* of R if $RI \subseteq I$ (i.e., $r * i \in I$ for all $r \in R$, $i \in I$). Similarly one defines a *right ideal* and a *two-sided ideal* (or simply an *ideal*) ($=$ left $+$ right ideal in the same time, notation $I \triangleleft R$).

Definition 2

Let R be an algebra over K.

- R is *associative* if $(a * b) * c = a * (b * c)$ for every $a, b, c \in R$;
- R is *commutative* if $a * b = b * a,\ a, b \in R$;
- R is a *Lie algebra* if for every $a, b, c \in R$:

$$a * a = 0, \text{ the anticommutative law},$$

$$(a * b) * c + (b * c) * a + (c * a) * b = 0, \text{ the Jacoby identity.}$$

Definition 3

Let $\mathfrak{B}$ be a class of algebras and let $F \in \mathfrak{B}$ be an algebra generated by a set X. The algebra F is called a *free algebra in the class* $\mathfrak{B}$, *freely generated by the set* X, if for any algebra $R \in \mathfrak{B}$, every mapping $X \longrightarrow R$ can be extended to a homomorphism $F \longrightarrow R$.

Definition 4

Let M be a vector space, let R be an associative algebra and let

$$\rho : R \longrightarrow \mathrm{End}_K(M)$$

be an algebra homomorphism (such that $\rho(1) = \mathrm{id}$). Then ρ is called a *representation of R in M* and M is a *left R-module*. Similarly one defines a *right R-module* assuming that the linear operators of M act from the right.

Definition 5

Let $\{f_i(x_1, \ldots, x_{n_i}) \in K\langle X\rangle \mid i \in I\}$ be a set of polynomials in the free associative algebra $K\langle X\rangle$. The class $\mathfrak{B}$ of all associative algebras satisfying the polynomial identities $f_i = 0,\ i \in I$, is called the *variety (of associative algebras) defined (or determined) by the system of polynomial identities* $\{f_i \mid i \in I\}$. The variety $\mathfrak{W}$ is called a *subvariety of* $\mathfrak{B}$ if $\mathfrak{W} \subseteq \mathfrak{B}$.

For a background on varieties of algebras see e.g.

- V. Drensky, Free Algebras and PI-Algebras, Graduate Course in Algebra, Springer-Verlag Singapore, 1999.
- Y. Bahturin, Identical Relations in Lie Algebras, 2nd edition, De Gruyter Expositions in Mathematics 68. Berlin: De Gruyter, 2021.

I. Definitions – Hilbert (Poincaré) Series

A Hilbert (or Poincaré) series of a graded vector space $W = \sum_{i \geq 0} W^{(i)}$ with finite dimensional homogeneous components $W^{(i)}$ (where $\dim W^{(i)} < +\infty$) is the formal power series

$$H(W, t) = \mathrm{Hilb}(W, t) = \sum_{i \geq 0} \dim(W^{(i)}) t^i.$$

Definition 6

The commutator ideal F' of the algebra F is given by $F' = F[F,F]F$ and its elements are of the form

$$\sum \alpha_{abc} x_1^{a_1} \cdots x_m^{a_m} [x_{b_1}, x_{b_2}] x_1^{c_1} \cdots x_m^{c_m}, \ \alpha_{abc} \in K$$

where all x's for each a_j, b_k and c_j ($k = 1, 2$ and $j = 1, \ldots, m$) belong to the algebra F.

Let K be a field of characteristic 0. We consider the case $K = \mathbb{C}$ although the results can be restated for any K of characteristic 0.

In classical commutative invariant theory the general linear group $GL_d(\mathbb{C})$ acts on the d-dimensional complex vector space V_d with basis $\{v_1, \ldots, v_d\}$ and this action induces an action on the algebra of polynomial functions $\mathbb{C}[X_d] = \mathbb{C}[x_1, \ldots, x_d]$.
For a subgroup G of $GL_d(\mathbb{C})$ the algebra of G-invariants $\mathbb{C}[X_d]^G$ consists of all polynomials which are fixed under the action of G.

II. Introduction – Noncommutative Invariant Theory

In one of the main branches of noncommutative invariant theory one replaces the algebra $\mathbb{C}[X_d]$ with an algebra which shares some of the important properties of $\mathbb{C}[X_d]$. Among the candidates for such an algebra are the free associative algebra $\mathbb{C}\langle X_d\rangle = \mathbb{C}\langle x_1, \ldots, x_d\rangle$, the free Lie algebra L_d or the d-generated relatively free algebra of a variety of (associative, Lie, Jordan or other nonassociative) algebras, $d \geq 2$. In this case it is more convenient to assume that $GL_d(\mathbb{C})$ acts directly on the vector space $\mathbb{C}X_d$ with basis X_d instead of on V_d.

Going back to classical invariant theory this means that $GL_d(\mathbb{C})$ acts on the symmetric algebra $S(X_d)$ of $\mathbb{C}X_d$ instead on the algebra of polynomial functions $\mathbb{C}[X_d]$.

Let $\mathfrak{V}$ be a variety of unitary associative algebras over a field K of characteristic 0 and let $A_d(\mathfrak{V})$ be the relatively free algebra of $\mathfrak{V}$ freely generated by X_d, $d \geq 2$. The general linear group $GL_d(K)$ acts canonically on the vector space KX_d and this action is extended diagonally on the whole algebra $A_d(\mathfrak{V})$:

$$g(f(x_1,\dots,x_d)) = f(g(x_1),\dots,g(x_d)),\ f \in A_d(\mathfrak{V}), g \in GL_d(K).$$

For a subgroup G of $GL_d(K)$ the algebra of G-invariants is

$$A_d(\mathfrak{V})^G = \{f(X_d) \in A_d(\mathfrak{V}) \mid g(f(X_d)) = f(X_d) \text{ for all } g \in G\}.$$

Theorem 7

Let $\mathfrak{V}$ be a variety of unitary associative algebras. The following conditions on $\mathfrak{V}$ are equivalent. If some of them is satisfied for some $d_0 \geq 2$, then all of them hold for all $d \geq 2$:
(i) *The algebra $A_d(\mathfrak{V})^G$ is finitely generated for every finite subgroup G of $GL_d(K)$.*
(ii) *The variety $\mathfrak{V}$ satisfies the polynomial identity*

$$[x_1, x_2, \dots, x_2]x_3^n[x_4, x_5, \dots, x_5] = 0$$

for sufficiently long commutators and n large enough.

The commutator of z_1 and z_2 is $[z_1, z_2] = z_1 z_2 - z_2 z_1 = z_1 \operatorname{ad}(z_2)$ and the longer commutators are left normed:

$$[z_1, \dots, z_{n-1}, z_n] = [[z_1, \dots, z_{n-1}], z_n], \ n \geq 3.$$

Definition 8

$$[[x_1, x_2], [x_3, x_4]] = 0 \text{ (the } \textit{metabelian identity}\text{)},$$

where x_1, x_2, x_3 and x_4 are algebra elements.

Definition 9

The metabelian variety of Lie algebras $\mathfrak{A}^2$ is defined by the polynomial identity $[[x_1, x_2], [x_3, x_4]] = 0$.

Since the free metabelian Lie algebra $L_d(\mathfrak{A}^2)$ is naturally embedded into the free metabelian associative algebra $A_d(\mathfrak{M})$, when $K = \mathbb{C}$ the results for the algebra of invariants $A_2(\mathfrak{M})^{D_{2n}}$ allow easily to obtain a minimal set of generators of $L_2'(\mathfrak{A}^2)^{D_{2n}}$ as a $\mathbb{C}[x, y]^{D_{2n}}$-module. As in the associative case, we compute the Hilbert series of the algebra $L_2(\mathfrak{A}^2)^{D_{2n}}$.

Theorem 10

For any finite subgroups G of $GL_d(\mathbb{C})$ the algebra of invariants $\mathbb{C}[X_d]^G$ is finitely generated and has a homogeneous system of generators of degree $\leq |G|$.

- E. Noether, Der Endlichkeitssatz der Invarianten endlicher Gruppen, Math. Ann. 77 (1916), 89-92. Reprinted in "Gesammelte Abhandlungen. Collected Papers", Springer-Verlag, Berlin–Heidelberg–New York–Tokyo, 1983, 181-184.

Theorem 11

Every ideal of $K[X_d]$ is finitely generated.

- D. Hilbert, Über die Theorie der algebraischen Formen, Math. Ann. 36 (1890), 473-534; reprinted in “Gesammelte Abhandlungen, Band II, Algebra, Invariantentheorie, Geometrie”, Zweite Auflage, Springer-Verlag, Berlin-Heidelberg- New York, 1970, 199-257.

III. Preliminaries – The Molien formula

The Molien formula gives the Hilbert series of the algebra of invariants of the finite subgroup G of $GL_d(K)$:

$$H\left(K[X_d]^G, t\right) = \frac{1}{|G|} \sum_{g \in G} \frac{1}{det(1 - gt)}.$$

- T. Molien, Über die Invarianten der linearen Substitutionsgruppen, Sitz. König Preuss. Akad. Wiss (1897), No. 52, 1152-1156.

A generalisation of the Molien formula is established by Formanek.

Theorem 12

Let G be a finite subgroup of $GL_d(K)$ and let $\xi_1(g), \ldots, \xi_d(g)$ be the eigenvalues of $g \in G$. If $\mathfrak{B}$ is a variety of algebras and $H(F_d(\mathfrak{B}), t_1, \cdots, t_d)$ is the Hilbert series of $F_d(\mathfrak{B})$ considered as a $\mathbb{Z}^d$-graded vector space, then the Hilbert series of the algebra $F_d(\mathfrak{B})^G$ is

$$H\left(F_d(\mathfrak{B})^G, t\right) = \frac{1}{|G|} \sum_{g \in G} H(F_d(\mathfrak{B}), \xi_1(g)t, \ldots, \xi_d(g)t).$$

- Formanek, E.: Noncommutative invariant theory. In: Montgomery, S. (ed.) Group Actions on Rings. Contemp. Math. 43, 87119. American Mathematical Society, Providence (1985).

An element $g \in GL_d(\mathbb{C})$ of finite order is a (pseudo) reflection if it has an eigenvalue 1 of multiplicity $d-1$ and another eigenvalue of multiplicity 1 which is a root of unity. The group $G < GL_d(\mathbb{C})$ is called a reflection group if it is generated by reflections.

Theorem 13

The following properties of the finite subgroup G of $GL_d(\mathbb{C})$ are equivalent:

1. *G is a finite reflection group;*
2. *$\mathbb{C}[X_d]$ is a free graded module over $\mathbb{C}[X_d]^G$ with a finite basis;*
3. *$\mathbb{C}[X_d]^G$ is generated by d algebraically independent homogeneous elements.*

- T.A. Springer, Invariant Theory, Lect. Notes in Math. 585, Springer-Verlag, Berlin–Heidelberg–New York, 1977.

III. Preliminaries – “Stronger” Theorem of Chevalley-Shephard-Todd

For our purposes we need a form of the Chevalley-Shephard-Todd theorem that is stronger than the one in the original papers:

- C. Chevalley, Invariants of finite groups generated by reflections, Amer. J. Math. 67 (1955), 778-782.
- G.C. Shephard, J.A. Todd, Finite unitary reflection groups, Canad. J. Math. 6 (1954), 274-304.

III. Preliminaries – "Stronger" Theorem of Chevalley-Shephard-Todd (continued)

Corollary 14

Let G be a reflection group and let $\mathbb{C}[X_d]^G = \mathbb{C}[f_1, \dots, f_n]$, where f_i is homogeneous of degree d_i. Then up to order the integers d_i are uniquely determined by G. The order $|G|$ of G is equal to $\prod_{i=1}^{n} d_i$ and the number of reflections in G is equal to $\sum_{i=1}^{n}(d_i - 1)$.

- T.A. Springer, Invariant Theory, Lect. Notes in Math. 585, Springer-Verlag, Berlin–Heidelberg–New York, 1977.

Proposition 1

If G is a reflection group then the free $\mathbb{C}[X_d]^G$-module $\mathbb{C}[X_d]$ is freely generated by $|G|$ homogeneous elements.

- J.E. Humphreys, Reflection Groups and Coxeter Groups, Cambridge Studies in Advanced Mathematics, 29, Cambridge, Cambridge University Press, 1992.

The dihedral group D_{2n}, $n \geq 3$, acts on the two dimensional real vector space xOy as the group of symmetries of the regular n-gon. It is generated by a rotation by angle $\frac{2\pi}{n}$ around the origin and a reflection with respect to the axis Ox.

We change the coordinate system, as introduced by Riemann:

$$u = x + iy, v = x - iy.$$

Then D_{2n} is generated by the rotation $\rho : u \longrightarrow \xi u,\ v \longrightarrow \bar{\xi} u$, where $\xi = e^{\frac{2\pi i}{n}}$, and the reflection $\tau : u \longleftrightarrow v$.

Proposition 2

The algebra of invariants $\mathbb{C}[u, v]^{D_{2n}}$ is generated by uv and $u^n + v^n$.

Proposition 3

The polynomial algebra $\mathbb{C}[u, v]$ is a free $\mathbb{C}[u, v]^{D_{2n}}$-module with free generators

$$W = (1, u, u^2, \ldots, u^n, v, v^2, \ldots, v^{n-1}).$$

Due to the algebra isomorphism below the next corollary follows

$$\mathbb{C}[u_1, v_1, u_2, v_2] \cong \mathbb{C}[u_1, v_1] \otimes_{\mathbb{C}} \mathbb{C}[u_2, v_2].$$

Corollary 15

Let D_{2n} act on $\mathbb{C}[u_1, v_1]$ and $\mathbb{C}[u_2, v_2]$ in the same way as on $\mathbb{C}[u, v]$. Then $\mathbb{C}[u_1, v_1, u_2, v_2]$ is a free $\mathbb{C}[u_1, v_1]^{D_{2n}} \otimes_{\mathbb{C}} \mathbb{C}[u_2, v_2]^{D_{2n}}$-module freely generated by

$$u_1^a u_2^c, u_1^a v_2^d, v_1^b u_2^c, v_1^b v_2^d, \; 0 \leq a, c \leq n, 1 \leq b, d \leq n-1.$$

Proposition 4

The free metabelian algebra $A_d(\mathfrak{M})$ *has a basis consisting of all*

$$x_1^{a_1}, \ldots, x_d^{a_d},\ x_1^{a_1} \cdots x_d^{a_d}[x_{i_1}, \ldots, x_{i_n}],$$

where

$$a_1, \ldots, a_d \geq 0,\ \ i_1 > i_2 \leq \cdots \leq i_n.$$

It satisfies the identities

$$x_{\varphi(1)} \cdots x_{\varphi(m)}[x_{m+1}, x_{m+2}, x_{\psi(m+3)}, \ldots, x_{\psi(m+n)}]$$

$$= x_1 \cdots x_m[x_{m+1}, x_{m+2}, x_{m+3}, \ldots, x_{m+n}],\ m \geq 0, n \geq 3,$$

where φ *and* ψ *are permutations of* $\{1, \ldots, m\}$ *and* $\{m+3, \ldots, m+n\}$*, respectively.*

Corollary 16

(i) *In the special case of* $d = 2$ *the algebra* $A_2(\mathfrak{M})$ *has a basis with respect to the set* $\{u, v\}$

$$u, v,\ u^a v^b [v, u] u^c v^d,\ a, b, c, d \geq 0.$$

(ii) *The commutator ideal* $A_2'(\mathfrak{M})$ *is a free* $\mathbb{C}[u_1, v_1, u_2, v_2]$*-module generated by* $[v, u]$*, where* $\mathbb{C}[u_1, v_1]$ *and* $\mathbb{C}[u_2, v_2]$ *act on* $A_2'(\mathfrak{M})$ *by multiplication by* u *and* v *from the left and the right, respectively.*

III. Preliminaries – Properties of Free Metabelian Lie Algebras

As in the associative case, the structure of free metabelian Lie algebra $L_d(\mathfrak{A}^2)$ is also well known. It has a basis

$$x_1, \ldots, x_d, \ [x_{i_1}, \ldots, x_{i_n}], \ i_1 > i_2 \leq \cdots \leq i_n.$$

Clearly, $L_d(\mathfrak{A}^2)$ may be considered as the Lie subalgebra of $A_d(\mathfrak{M})$ generated with respect to the commutator operation by $x_1, \ldots, x_d$. In the special case $d = 2$ the basis of $L_2(\mathfrak{A}^2)$ consists of

$$u, v, \ [v, u]\mathrm{ad}^a(u)\mathrm{ad}^b(v), \ a, b \geq 0,$$

and $L_d'(\mathfrak{A}^2)$ is a free $\mathbb{C}[u, v]$-module generated by $[v, u]$ with respect to the action

$$f(u, v) : w \longrightarrow wf(\mathrm{ad}(u), \mathrm{ad}(v)), \quad w \in L_d'(\mathfrak{A}^2), f(u, v) \in \mathbb{C}[u, v].$$

IV. Main Results – Associative Case: Basis of A_2' as Module of Dihedral Invariants

Let us define an isomorphism of $\mathbb{C}[u_1, v_1, u_2, v_2]$-modules

$$\nu : A_2'(\mathfrak{M}) \longrightarrow \mathbb{C}[u_1, v_1, u_2, v_2] \text{ by } \nu(u^a v^b [v, u] u^c v^d) = u_1^a v_1^b u_2^c v_2^d.$$

Proposition 5

Let D_{2n} act on $\mathbb{C}[u_1, v_1]$ and $\mathbb{C}[u_2, v_2]$ in the same way as on $\mathbb{C}[u, v]$. Then $A_2'(\mathfrak{M})$ is a free $\mathbb{C}[u_1, v_1]^{D_{2n}} \otimes_{\mathbb{C}} \mathbb{C}[u_2, v_2]^{D_{2n}}$-module freely generated by

$$u^a [v, u] u^c, u^a [v, u] v^d, v^b [v, u] u^c, v^b [v, u] v^d,$$

$$0 \leq a, c \leq n, 1 \leq b, d \leq n - 1.$$

Proposition 6

(i) *A polynomial* $f(u,v)\in A_2'(\mathfrak{M})$ *belongs to* $A_2'(\mathfrak{M})^{D_{2n}}$ *iff it is fixed under the rotation* ϱ *and reflection* τ *which generate* D_{2n}*, i.e.*

$$f(\xi u,\bar{\xi}v)=f(u,v), f(v,u)=f(u,v),\ \text{where } \xi=e^{\frac{2\pi i}{n}}.$$

The polynomial $h(u_1,v_1,u_2,v_2)=\nu(f(u,v))\in\mathbb{C}[u_1,v_1,u_2,v_2]$ *belongs to* $\mathbb{C}[u_1,v_1,u_2,v_2]^{D_{2n}}$ *iff*

$$\varrho(h(u_1,v_1,u_2,v_2))=h(\xi u_1,\bar{\xi}v_1,\xi u_2,\bar{\xi}v_2)=h(u_1,v_1,u_2,v_2),$$

$$\tau(h(u_1,v_1,u_2,v_2))=h(v_1,u_1,v_2,u_2)=-h(u_1,v_1,u_2,v_2).$$

(ii) *The* $\mathbb{C}[u_1,v_1]^{D_{2n}}\otimes_{\mathbb{C}}\mathbb{C}[u_2,v_2]^{D_{2n}}$*-module* $A_2'(\mathfrak{M})^{D_{2n}}$ *is free with a set of free generators*

$$u^a[v,u]u^{n-a}-v^a[v,u]v^{n-a},\ a=0,1,\dots,n,\ u^n[v,u]u^n-v^n[v,u]v^n,$$

$$u^a[v,u]v^a-v^a[v,u]u^a,\ a=1,\dots,n-1.$$

Theorem 17

(i) *For $n \geq 3$ the algebra $A_2(\mathfrak{M})^{D_{2n}}$ is generated by $uv + vu$, $u^n + v^n$ and*

$$u^a[v,u]u^{n-a} - v^a[v,u]v^{n-a},\ a = 0,1,\dots,n,\ u^n[v,u]u^n - v^n[v,u]v^n,$$

$$u^a[v,u]v^a - v^a[v,u]u^a,\ a = 1,\dots,n-1.$$

(ii) *The Hilbert series of $A_2(\mathfrak{M})^{D_{2n}}$ is*

$$H(A_2(\mathfrak{M})^{D_{2n}}, t) = \frac{1}{(1-t^2)(1-t^n)}$$

$$+\frac{1}{(1-t^2)^2(1-t^n)^2}\left((n+1)t^{n+2} + \frac{t^4(1-t^{2n})}{1-t^2}\right).$$

IV. Main Results – Associative Case: Generating Set of $A_2(\mathfrak{M})^{D_{2n}}$

Remark 1

The generating set of the algebra $A_2(\mathfrak{M})^{D_{2n}}$ is not minimal because the commutator $[uv + vu, u^n + v^n]$ is of degree $n + 2$ and can be expressed as a linear combination of the $n + 1$ generators

$$u^a[v, u]u^{n-a} - v^a[v, u]v^{n-a},\ a = 0, 1, \ldots, n.$$

It is easy to see that if we remove one of these generators, we shall obtain a minimal generating set of the algebra $A_2(\mathfrak{M})^{D_{2n}}$.

IV. Main Results – Lie Case: Basis of L'_2 as a Module of Dihedral Invariants

In the Lie case our considerations are similar, but are much simpler because the commutator ideal $L'_2(\mathfrak{A}^2)$ is a free $\mathbb{C}[u, v]$-module generated by $[v, u]$ and hence we obtain the following Proposition.

Proposition 7

The commutator ideal $L'_2(\mathfrak{A}^2)$ is a free $\mathbb{C}[u, v]^{D_{2n}}$-module freely generated by

$$[v, u]\mathrm{ad}^a(u),\ a = 0, 1, \dots, n,\ [v, u]\mathrm{ad}^b(v),\ b = 1, \dots, n - 1.$$

Theorem 18

(i) *The* $\mathbb{C}[u,v]^{D_{2n}}$*-module* $L_2'(\mathfrak{A}^2)^{D_{2n}}$ *is freely generated by*

$$[v,u](\mathrm{ad}^n(u) - \mathrm{ad}^n(v)).$$

(ii) *The Hilbert series of* $L_2(\mathfrak{A}^2)^{D_{2n}}$ *is*

$$H(L_2(\mathfrak{A}^2)^{D_{2n}}, t) = \frac{t^{n+2}}{(1-t^2)(1-t^n)}.$$

THANK YOU VERY MUCH FOR YOUR ATTENTION!