Ideals of Q-algebras

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Doklad za Godishnata Otchetnata Sesiya 01/12/2023

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- also called: propositional logic, statement logic, sentential calculus

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- x is called **involution** if $x^{**} = x$

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- Ideals: $\{0\}$, $\{0, a\}$, X

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Let $(X; *, 0)$ be Q-algebra.
a) If $|X| = 2$ then $G(X)$ is an ideal.
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Problem

Characterization of all Q-algebras X such that $G(X)$ is an ideal.

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Theorem (A & K, 2023)

For $a, b, c \in G(X)$, it holds:
a) $a \neq b \Rightarrow ab \notin \{0, a, b\}$
b) $ab = ba$
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Theorem

a) $|X|$ *odd* $\Rightarrow G(X) \neq X$.
b) *If $G(X)$ is an ideal then $G(X)$ is an abelian Group.*
c) *$G(X) = X$ if and only if $(X; *)$ is an abelian Group.*

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Example

$|X| = 4$: $G(X)$ is an ideal if and only if $G(X) = \{0\}$ or $G(X) = X$ or $G(X) = \{0, a\}$ $(a \in X \setminus \{0\})$ with $ba \neq ca$, whenever $b \neq c$.

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- If $G(X) = X$ ($G(X)$ is ideal!!) then $|X| = 2^k$ $(k \in \mathbb{N})$ and $X \cong \mathbb{Z}_{2^k}$.
- If X_1 and X_2 are Q-algebras with $G(X_1) = X_1$ and $G(X_2) = X_2$ then $X_1 \cong X_2$.