A new approach to the characteristic polynomial of a random unitary matrix

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Outline

The Riemann Zeta function

This is a function defined for $\mathfrak{Re}(s) > 1$ by its Dirichlet series or its Eulerian product

$$\zeta(s) := \sum_{n \geqslant 1} \frac{1}{n^s} = \prod_{p \in \mathcal{P}} \frac{1}{1 - p^{-s}}, \quad \mathcal{P} := \{\text{prime numbers}\}$$

This function is extended to the whole complex plane using the functional equation

$$\boxed{\xi(s) = \xi(1-s)}, \quad \xi(s) := (s-1)\Gamma\left(\frac{s}{2} + 1\right) \pi^{-s/2} \zeta(s)$$

Conjecture (Riemann)

Denote the non trivial zeroes of ζ by $(\sigma_k + \mathrm{i}\gamma_k)_{k \geqslant 1}$. Then $\sigma_k = \frac{1}{2}$ for all k.

The Montgommery conjectures

First statistical study of the zeroes of ζ: the *pair correlation*.

Theorem (Montgommery, 1972)

Denote the non trivial zeroes of ζ *by* $(\sigma_k + \mathrm{i}\gamma_k)_{k\geqslant 1}$ *and set* $\widehat{\gamma}_k := \gamma_k \log\left(\frac{\gamma_k}{2\pi}\right)$ *(so that* $|\widehat{\gamma}_{k+1} - \widehat{\gamma}_k| \sim 1$*). Then, with* $\mathrm{sinc}(x) := \frac{\sin(x)}{x}$

$$\frac{1}{N} \sum_{1\leqslant k\neq \ell\leqslant N} \phi(\widehat{\gamma}_k - \widehat{\gamma}_\ell) \xrightarrow[N\to+\infty]{} \int_{\mathbb{R}} \phi(x)\left(1 - \mathrm{sinc}(\pi x)^2\right) dx$$

for all $\phi \in \mathcal{C}^\infty$ *with* $\mathrm{supp}(\mathcal{F}\phi) \subset [-1, 1]$*. (*$\mathcal{F}$ *= Fourier transform)*

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$$\frac{1}{N} \sum_{1 \leqslant k \neq \ell \leqslant N} \phi(\widehat{\gamma}_k - \widehat{\gamma}_\ell) \xrightarrow[N \to +\infty]{} \int_{\mathbb{R}} \phi(x)\left(1 - \mathrm{sinc}(\pi x)^2\right) dx$$

for all $\phi \in \mathcal{C}^\infty$ *with* $\mathrm{supp}(\mathcal{F}\phi) \subset [a, b]$ *for all* $a, b \in \mathbb{R}$*.*

The extension to $\mathrm{supp}(\mathcal{F}\phi) \subset [-1 - \varepsilon, 1 + \varepsilon]$ involves a quantitative version of the twin primes conjecture.

Outline

The Haar measure on $\mathcal{U}(N)$

We endow the space of unitary matrices $\mathcal{U}(N)$ which is a compact group with its normalised (probability) Haar measure $\mathbb{P}_N$. The probability space $(\mathcal{U}(N), \mathbb{P}_N)$ is called the **Circular Unitary Ensemble** ; in short $CUE(N)$.

Theorem (Weyl, 30')

Let f be a bounded class function (i.e. $f(V^{-1}UV) = f(U)$). Then, with an abuse of notation :

$$
\begin{aligned}
\mathbb{E}_N(f(U)) := \int_{\mathcal{U}(N)} f(U) d\mathbb{P}_N(U) &= \int_{\mathcal{U}(N)} f(V^{-1}\Theta V) d\mathbb{P}'_N(V, \Theta) \\
&\equiv \mathbb{E}_N(f(\theta_1, \dots, \theta_N)) \\
&= \frac{1}{(2\pi)^N N!} \int_{(-\pi,\pi]^N} f(\theta_1, \dots, \theta_N) \prod_{1 \leqslant k < \ell \leqslant N} \left| e^{\mathrm{i}\theta_\ell} - e^{\mathrm{i}\theta_k} \right|^2 d\theta_1 \dots d\theta_N \\
&=: \frac{1}{(2\pi)^N N!} \int_{(-\pi,\pi]^N} f(\theta_1, \dots, \theta_N) \left| \Delta(e^{i\theta_1}, \dots, e^{i\theta_N}) \right|^2 d\theta_1 \dots d\theta_N
\end{aligned}
$$

The Haar measure on $\mathcal{U}(N)$

The link with number theory is given by the following theorem of Dyson analogue to Montgommery's:

Theorem (Dyson, 1962)

Let $U \sim \mathbb{P}_N$ *with eigenangles* $(\theta_k)_{1 \leqslant k \leqslant N}$ *and set* $\widehat{\theta}_k := \theta_k \frac{N}{2\pi}$*. Then*

$$\mathbb{E}_N\left(\frac{1}{N}\sum_{1\leqslant k\neq \ell\leqslant N} \phi\left(\widehat{\theta}_k - \widehat{\theta}_\ell\right)\right) \xrightarrow[N\to+\infty]{} \int_{\mathbb{R}} \phi(x)\left(1-\operatorname{sinc}(\pi x)^2\right) dx$$

for all $\phi \in \mathcal{C}^0$ *with compact support, with* $\operatorname{sinc}(x) := \frac{\sin(x)}{x}$.

The **Dyson-Montgommery correspondance** was famously found during an afternoon tea...

Random matrices and number theory in a few dates

- 62 : Dyson.
- 72 : Montgommery.
- 73 : Montgommery-Dyson correspondance.
- 80ies-90ies : numerical tests by Odlyzko, conjectures of Conrey-Farmer and Conrey-Ghosh, computations of Bogomolny-Keating.
- 99 : Katz-Sarnak prove the Montgommery conjectures on finite fields.
- 2000 : Keating-Snaith : the moments conjecture.

J.P. Keating, N.C. Snaith, *Random matrix theory and* $\zeta(1/2 + it)$, Comm. Math. Phys., **214** (2000), p 57-85.

- 2003 : CFKRS formula (Conrey-Farmer-Keating-Rubinstein-Snaith)

Outline

Probabilistic study of ζ, 1: convergence in law/fluctuations

Theorem (Bohr-Jessen, 30')

Let U be a uniform random variable in $[0,1]$ and $(U_p)_{p\in\mathcal{P}}$ be a sequence of i.i.d. uniform random variables in $[0,1]$. Let $\alpha > \frac{1}{2}$. Then,

$$\zeta(\alpha + iT\,U) \xrightarrow[T\to+\infty]{\mathcal{L}} \prod_{p\in\mathcal{P}} \left(1 - \frac{e^{2i\pi U_p}}{p^\alpha}\right)^{-1} =: \mathcal{BJ}_\alpha$$

Theorem (Selberg, 50')

Let U be a uniform random variable in $[0,1]$. Then

$$\frac{\log\left|\zeta\left(\frac{1}{2} + iT\,U\right)\right|}{\sqrt{\frac{1}{2}\log\log T}} \xrightarrow[T\to+\infty]{\mathcal{L}} \mathscr{N}(0,1)$$

Once the fluctuations have been computed, the next step in a probabilistic problem is to compute the (precise) large deviations.

Probabilistic study of ζ, 2: large deviations/mod-* c.v.

Conjecture (J. Keating, N. Snaith (2000))

When $T \to +\infty$, locally uniformly in $\lambda \in \mathbb{R}$ (or $i\mathbb{R}$)

$$\mathbb{E}\left(e^{\lambda \log\left|\zeta\left(\frac{1}{2}+iT\,U\right)\right|}\right) = e^{\frac{\lambda^2}{2}\times\frac{1}{2}\log\log T}\Phi_M(\lambda)\Phi_A(\lambda)(1+o(1))$$

- Proven for $\lambda = 2$ (Hardy-Littlewood) and $\lambda = 4$ (Ingham). Equivalent conjecture for $\lambda = 6$ and $\lambda = 8$ by Conrey-Gosh and Conrey-Gonek.
- $\Phi_A(\lambda)$ is the *arithmetic factor*

$$\Phi_A(\lambda) = \prod_{p\in\mathcal{P}} {}_2F_1\left(\begin{matrix}\lambda,\lambda\\ 1\end{matrix}\,\middle|\,\frac{1}{p}\right)e^{-\lambda^2/p} = \prod_{p\in\mathcal{P}}\mathbb{E}\left(\left|1-\frac{e^{2i\pi U}}{p^{1/2}}\right|^{-\lambda}\right)e^{-\lambda^2/p}$$

- $\Phi_M(\lambda)$ is the *random matrix factor*

$$\Phi_M(\lambda) = \prod_{k\in\mathbb{N}^*} {}_2F_1\left(\begin{matrix}-\lambda,-\lambda\\ k\end{matrix}\,\middle|\,1\right)e^{-\lambda^2/k} = e^{C\lambda^2}\frac{G(1+\lambda)^2}{G(1+2\lambda)}$$

where G is Barnes' double Gamma function: $G(\lambda+1) = \Gamma(\lambda)G(\lambda)$.

Explanation of the conjecture: the characteristic polynomial

- Selberg's CLT was:

$$\frac{\log\left|\zeta\left(\frac{1}{2}+iTU\right)\right|}{\sqrt{\frac{1}{2}\log\log T}} \xrightarrow[T\to+\infty]{\mathcal{L}} \mathscr{N}(0,1)$$

- Keating and Snaith prove the analogous CLT for $Z_{U_N}(1) := \det(I_N - U_N)$:

$$\frac{\log\left|Z_{U_N}(1)\right|}{\sqrt{\frac{1}{2}\log N}} \xrightarrow[N\to+\infty]{\mathcal{L}} \mathscr{N}(0,1)$$

- The matrix factor comes from the mod-Gaussian convergence of $Z_{U_N}(1)$

$$\mathbb{E}_N\left(e^{\lambda\log|Z_U(1)|}\right) = e^{\frac{\lambda^2}{2}\times\frac{\log N}{2}}\Phi_M(\lambda)(1+o(1))$$

- They make the association $\boxed{N \longleftrightarrow \log T}$ to produce their conjecture.

The Keating-Snaith philosophy

It is a twofold philosophy :

- Take problems already solved in number theory and adapt them in random matrix theory to see if they are still valid.
- Take unsolved problems on ζ and conjecture the result by solving the analogous problem on Z_{U_N} as the computations are easier to handle.

Number theory	Random Matrix Theory
Selberg's CLT $$\frac{\log\left\|\zeta\left(\frac{1}{2}+iT\,U\right)\right\|}{\sqrt{\frac{1}{2}\log\log T}} \xrightarrow[T\to+\infty]{\mathcal{L}} \mathcal{N}(0,1)$$	Keating-Snaith's CLT $$\frac{\log\left\|Z_{U_N}(1)\right\|}{\sqrt{\frac{1}{2}\log N}} \xrightarrow[N\to+\infty]{\mathcal{L}} \mathcal{N}(0,1)$$
Moments conjecture $$\frac{\mathbb{E}\left(e^{\lambda\log\left\|\zeta\left(\frac{1}{2}+iT\,U\right)\right\|}\right)}{e^{\frac{\lambda^2}{2}\frac{\log\log T}{2}}} \xrightarrow[n\to+\infty]{} \Phi_A(\lambda)\Phi_M(\lambda)$$	Keating-Snaith mod-* cv. $$\frac{\mathbb{E}\left(e^{\lambda\log\left\|Z_{U_N}(1)\right\|}\right)}{e^{\frac{\lambda^2}{2}\frac{\log N}{2}}} \xrightarrow[n\to+\infty]{} \Phi_M(\lambda)$$

How to compute functionals of $Z_{U_N}(X)$?

1. analysis of Toeplitz and Fredholm determinants (Onsager, Szegö, Johansson, Deift-Its-Krasovsky, Borodin-Okounkov),
2. Orthogonal Polynomials on the Unit Circle (Killip-Nenciu, Najnudel),
3. mathematical physics (Kostov) and supersymmetry (Conrey-Farmer-Zirnbauer, Fyodorov),
4. integrable systems (Adler-Van Moerbeke, Forrester),
5. algebraic combinatorics (Biane, Dehaye, Féray),
6. representation theory and symmetric functions (Bump-Gamburd, Diaconis-Shahshahani, Dehaye, B.-A.),
7. probability theory (Bourgade-Hughes-Nikeghbali-Yor, B.-A.),
8. CFKRS formula (Conrey-Farmer-Keating-Rubinstein-Snaith, etc.),
9. Integrable/determinantal probability (Borodin-Strahov),
10. Weingarten calculus (Collins, Weingarten, Matsumoto-Novak),
11. Itô calculus (Bru, Katori-Tanemura), etc.

Outline

A selection of functionals (1/2)

In the framework of the Keating-Snaith philosophy, the **integer moments** of several functionals have been studied:

- Value in 1: $\mathbb{E}\left(|Z_{U_N}(1)|^{2k}\right)$ (Keating-Snaith),
- Joint products/ratios: $\mathbb{E}\left(\prod_{j=1}^{k} Z_{U_N}\left(e^{2i\pi\frac{x_j}{N}}\right)^{\alpha_j} \prod_{\ell=1}^{m} \overline{Z_{U_N}\left(e^{-2i\pi\frac{y_\ell}{N}}\right)^{\alpha'_\ell}}\right)$,
  $\alpha_j, \alpha'_\ell \in \{\pm 1\}$ (Bump-Gamburd, Conrey-Farmer-Keating-Rubinstein -Snaith, Conrey-Farmer-Zirnbauer, Conrey-Forrester-Snaith, Day, etc.),
- Joint derivatives in 1: $\mathbb{E}\left(|Z_{U_N}(1)|^{2(k-|\boldsymbol{h}|)} \prod_{r=1}^{m} |\partial^r Z_{U_N}(1)|^{2h_r}\right)$,
  $|\boldsymbol{h}| := \sum_{r\geqslant 1} h_r$, $h_i \in \mathbb{N}$, $\partial := \frac{d}{dz}$ (Conrey-Rubinstein-Snaith, Hughes-Keating-O'Connell, Dehaye, Winn, Riedtman, Assiotis-Keating-Warren),

A selection of functionals (2/2)

- Secular/Fourier coefficients: $\mathbb{E}\left(|\mathrm{sc}_j(U_N)|^{2k}\right)$ with $Z_{U_N}(X) := \sum_{j=0}^{N}(-X)^j\,\mathrm{sc}_j(U_N)$ (Haake-Kus-Sommers-Schomerus-Zyckowski, Diaconis-Gamburd, Forrester-Gamburd, Conrey-Gamburd),
- A functional in link with $\sum_{P\in\mathbb{F}_q[X]} d_k(P)^2$: $\mathbb{E}\left(\left|[X^m]\,Z_{U_N}(X)^k\right|^2\right)$ with $d_k(P) := \sum_{Q_1,\dots,Q_k} \mathbb{1}_{\{Q_1\cdots Q_k = P\}}$ (Keating-Rodgers-[Roditty-Gershon]-Rudnick),
- The truncated characteristic polynomial: $\mathbb{E}\left(|Z_{U_N,\ell}(r)|^{2k}\right)$, $r\in(0,1]$, $Z_{U_N,\ell}(X) := \sum_{j=0}^{\ell}(-X)^j\,\mathrm{sc}_j(U_N)$ (Conrey-Gamburd, Heap-Lindqvist).
- The "Moments of Moments":

$$\mathrm{MoM}_N(k,\beta) := \mathbb{E}\left(\left(\int_0^1 \left|Z_{U_N}(e^{2i\pi\theta})\right|^{2\beta} d\theta\right)^{2k}\right)$$

(Bailey-Keating, Assiotis-Keating),

Limiting behaviour: polynomial in N

We have $\boxed{\mathbb{E}(\mathcal{F}(Z_{U_N})) \underset{N\to+\infty}{\sim} \Psi_{\mathcal{F}} N^{d_{\mathcal{F}}}}$ in all the previous cases. For instance

- $\mathcal{KS}_1(k,N) := \mathbb{E}\left(\left|Z_{U_N}(1)\right|^{2k}\right) \sim \Psi_{\mathcal{KS}_1}(k) N^{k^2}$, with $\Psi_{\mathcal{KS}_1}(k) = \frac{G(1+k)^2}{G(1+2k)}$,
- $\mathcal{A}_N(X,Y) := \mathbb{E}\left(\prod_{j=1}^{k} Z_{U_N}\left(e^{2i\pi\frac{x_j}{N}}\right) \prod_{\ell=1}^{m} \overline{Z_{U_N}\left(e^{-2i\pi\frac{y_\ell}{N}}\right)}\right) \sim \Psi_{\mathcal{A}(X,Y)} N^{km}$,
- $\mathcal{D}_N(\boldsymbol{h}) := \mathbb{E}\left(\left|Z_{U_N}(1)\right|^{2(k-|\boldsymbol{h}|)} \prod_{r=1}^{m} \left|\partial^r Z_{U_N}(1)\right|^{2h_r}\right) \sim \Psi_{\mathcal{D}(\boldsymbol{h})} N^{k^2+2C(\boldsymbol{h})}$
- $\mathcal{S}_\rho(k,N) := \mathbb{E}\left(\left|\mathrm{sc}_{[\rho N]}(U_N)\right|^{2k}\right) \sim \Psi_{\mathcal{S}_\rho}(k) N^{(k-1)^2}$, $\rho \in (0,1)$,
- $\mathcal{J}_c(k,N) := \mathbb{E}\left(\left|\left[X^{[cN]}\right] Z_{U_N}(X)^k\right|^2\right) \sim \Psi_{\mathcal{J}_c}(k) N^{k^2-1}$, $c \in (0,k)$,
- $\mathcal{T}_\rho(k,N) := \mathbb{E}\left(\left|Z_{U_{N,[\rho N]}}(1)\right|^{2k}\right) \sim \Psi_{\mathcal{T}_\rho}(k) N^{k^2}$, $\rho \in (0,1)$,

  $\mathcal{T}_{\rho,r}(k,N) := \mathbb{E}\left(\left|Z_{U_{N,[\rho N]}}(r)\right|^{2k}\right) \sim \Psi_{\mathcal{T}_{\rho,r}}(k) N^{(k-1)^2}$, $r < 1$.

Example 1: $\mathbb{E}\left(|\partial Z_{U_N}(1)|^{2h}\right) \sim \Psi_{\mathcal{D}(0,1)}(h) N^{h^2+2h}$

J. B. Conrey, M. O. Rubinstein, N. C. Snaith, *Moments of the derivative of characteristic polynomials with an application to the Riemann Zeta Function*, Comm. Math. Phys. 267(3):611-629 (**2006**).

2 J.B. CONREY, M.O. RUBINSTEIN, AND N.C. SNAITH

Theorem 1. *For fixed k and $N \to \infty$ we have*

$$\int_{U(N)} |\Lambda_A'(1)|^{2k} dA_N \sim b_k N^{k^2+2k}, \tag{1.3}$$

where

$$b_k = (-1)^{k(k+1)/2} \sum_{h=0}^{k} \binom{k}{h} \left(\frac{d}{dx}\right)^{k+h} \left(e^{-x} x^{-k^2/2} \det_{k\times k} \left(I_{i+j-1}(2\sqrt{x}) \right) \right) \Big|_{x=0}, \tag{1.4}$$

and $I_\nu(z)$ denotes the modified Bessel function of the first kind.

We also have combinatorial description of b_k.

Theorem 3.

$$b_k = (-1)^{k(k+1)/2} \sum_{m\in P_O^{k+1}(2k)} \binom{2k}{m} \left(\frac{-1}{2}\right)^{m_0} \left(\prod_{i=1}^{k} \frac{1}{(2k-i+m_i)!}\right) \left(\prod_{1\le i<j\le k} (m_j - m_i + i - j)\right), \tag{1.7}$$

where $P_O^{k+1}(2k)$ denotes the set of partitions $m = (m_0, \ldots, m_k)$ of $2k$ into $k+1$ non-negative parts.

Example 2: $\mathbb{E}\left(\left|\left[X^{[cN]}\right] Z_{U_N}(X)^k\right|^2\right) \sim \Psi_{\mathcal{J}_c}(k) N^{k^2-1}$

J.P. Keating, B. Rodgers, E. Roditty-Gershon, Z. Rudnick, *Sums of divisor functions in $\mathbb{F}_q[t]$ and matrix integrals*, (**2017**).

Theorem 1.5. *Let $c := m/N$. Then for $c \in [0, k]$,*

$$I_k(m; N) = \gamma_k(c) N^{k^2-1} + O_k(N^{k^2-2}). \tag{1.36}$$

Theorem 1.6.

$$\gamma_k(c) = \sum_{0 \le \ell < c} \binom{k}{\ell}^2 (c-\ell)^{(k-\ell)^2+\ell^2-1} g_{k,\ell}(c-\ell) \tag{1.38}$$

where $g_{k,\ell}(c-\ell)$ are (complicated) polynomials in $c-\ell$.

$$g_{k,\ell}(c-\ell) = \frac{(-1)^\ell}{(k!)^2} \frac{1}{(2\pi i)^{2k}} \oint \dots \oint J_\ell((c-\ell)\vec{v}) \tag{4.34}$$

$$e^{-(v_{k+1}+\ldots+v_{2k})} \prod_{\substack{1\le t\le \ell,\ k+\ell+1\le q\le 2k \\ \text{or} \\ \ell+1\le t\le k,\ k+1\le q\le k+\ell}} (v_q - v_t) \prod_{\substack{1\le i<j\le \ell \\ \text{or} \\ k+\ell+1\le i<j\le 2k \\ \text{or} \\ \ell+1\le i<j\le k \\ \text{or} \\ k+1\le i<j\le k+\ell}} (v_j - v_i)^2 \prod_{j=1}^{2k} \frac{dv_j}{v_j^k}$$

with

$$J_\ell(v_1, \dots, v_{2k}) = \int_{\substack{\sum x_{t,q}=1 \\ x_{tq}\ge 0}} e^{\sum x_{tq}(\epsilon_q-\epsilon_t)(v_q-v_t)} \prod dx_{tq} \tag{4.33}$$

The prefactor $g_{k,\ell}(c-\ell)$ depends polynomially on $c-\ell$, because to compute it we need to compute derivatives of $J_\ell((c-\ell)\vec{v})$ at $\vec{v}=0$, which are clearly polynomial in $(c-\ell)$.

Example 3: $\mathbb{E}\left(\left(\oint_{\mathbb{U}} |Z_{U_N}(z)|^{2\beta} \frac{d^*z}{z}\right)^k\right) \sim \Psi_{\mathfrak{MoM}(\beta)}(k) N^{(k\beta)^2 - k + 1}$

E. C. Bailey, J. P. Keating, *On the moments of the moments of the characteristic polynomials of random unitary matrices*, CMP (**2018**).

Lemma 3.6. $$\mathrm{MoM}_N(k,\beta) \sim \gamma_{k,\beta} N^{k^2\beta^2 - k + 1}$$

where $$\gamma_{k,\beta} = \sum_{l_1,\ldots,l_{k-1}=0}^{2\beta} c_{k,\beta;\underline{l}} \left((k-1)\beta - \sum_{j=1}^{k-1} l_j\right)^{|B_{k,\beta;\underline{l}}| - \binom{k}{2}} P_{k,\beta}(l_1,\ldots,l_{k-1}),$$

$c_{k,\beta;\underline{l}}$ *is some constant depending on* k, β, $\underline{l}$, *and*

$$P_{k,\beta}(l_1,\ldots,l_{k-1}) = \frac{(-1)^{g(k,\beta;\underline{l})}}{(2\pi i)^{2k\beta}((k\beta)!)^2}$$

$$\times \int_{\Gamma_0} \cdots \int_{\Gamma_0} \frac{e^{-\sum_{m=k\beta+1}^{2k\beta} v_m} \prod_{\substack{m<n \\ \alpha_m=\alpha_n}} (v_n - v_m)^2}{\prod_{\substack{m \le k\beta < n \\ \alpha_m = \alpha_n}} (v_n - v_m) \prod_{m=1}^{2k\beta} v_m^{2\beta}} \Psi_{k,\beta;\underline{l}}\left(\left((k-1)\beta - \sum_{j=1}^{k-1} l_j\right)\underline{v}\right) \prod_{m=1}^{2k\beta} dv_m,$$

with $g(k,\beta;\underline{l})$ *given by* (70), *and* $\Psi_{k,\beta;\underline{l}}(\underline{v})$ *as defined in Lemma* 3.5

$$\Psi_{k,\beta;\underline{l}}(\underline{v}) = \int_{\underline{y}=(y_{m,n})_{(m,n)\in B_{k,\beta;\underline{l}}}\ (\tilde{\ddagger})} \cdots \int \exp\left(\sum y_{m,n}(v_n - v_m)^{\pm}\right) \prod dy_{m,n},$$

and $(\tilde{\ddagger})$ denotes the normalised version of the constraints $(\ddagger')$ (since we need only consider

Outline

Reminder

The secular/Fourier coefficients are the coefficients of $Z_{U_N}(X)$ in the basis $(1, X, \dots, X^N)$:

$$Z_{U_N}(X) = \sum_{k=0}^{N} (-1)^k \operatorname{sc}_k(U_N) X^k \quad \Leftrightarrow \quad \operatorname{sc}_k(U_N) = \frac{(-1)^k}{2i\pi} \oint_{\mathbb{U}} Z_{U_N}(z) z^{-k} \frac{dz}{z}$$

For instance, $\operatorname{sc}_1(U_N) = \operatorname{tr}(U_N)$ and $\operatorname{sc}_N(U_N) = \det(U_N)$. More generally,

$$\operatorname{sc}_k(U_N) = \operatorname{tr}\left(\wedge^k U_N\right) = e_k(e^{i\theta_1}, \dots, e^{i\theta_N}), \qquad \{\theta_i\} = \textit{eigenangles of } U_N$$

where

$$e_k(x_1, \dots, x_N) := \sum_{1 \leqslant j_1 < \cdots < j_k \leqslant N} x_{j_1} \cdots x_{j_k}$$

are the elementary symmetric functions.

Motivations: the Diaconis-Gamburd open question

- The two first articles (and a recent one) to study $\mathrm{sc}_k(U_N)$ were:

F. Haake, M. Kus, H.-J. Sommers, H. Schomerus, K. Zyckowski, *Secular determinants of random unitary matrices (**1996**).*

P. Diaconis, A. Gamburd, *Random matrices, magic squares and matching polynomials (**2004**).*

J. Najnudel, E. Paquette, N. Simm, *Secular coefficients and the holomorphic multiplicative chaos (**2020**).*

- Diaconis and Gamburd ask the following question :

> *It is natural to ask for limiting distribution as k grows with N. For example what is the limiting distribution of the $[N/2]$ secular coefficient ? On the one hand, the formula [linking coefficients and traces of powers of U_N] suggests it is a complex sum of weakly dependent random variables, so perhaps normal. On the other hand, the formulas $\mathbb{E}(\mathrm{sc}_k(U_N)) = 0$, $\mathbb{E}\left(|\mathrm{sc}_k(U_N)|^2\right) = 1$ hold for all k making normality questionable.*

Integer moments of the mid-coefficients

Theorem (B.-A. (2020, arXiv:2011.02465))

Let $\rho \in (0,1)$ *and* $k \geqslant 2$. *Then, when* $N \to \infty$, *one has*

$$\mathbb{E}\left(\left| \mathrm{sc}_{[\rho N]}(U_N) \right|^{2k} \right) \sim I_\rho(k) N^{(k-1)^2}$$

where

$$I_\rho(k) := \frac{(2\pi)^{k(k-1)}}{k!} \int_{\mathbb{R}^{k-1}} \Phi_{I_\rho}(x_2, \dots, x_k) \Delta(0, x_2, \dots, x_k)^2 \prod_{j=1}^{k-1} dx_j$$

with

$$\Phi_{I_\rho}(x_2, \dots, x_k) := e^{i\pi(k-2)\sum_j x_j} h_{\rho,\infty}(0, x_2, \dots, x_k)^k h_{1-\rho,\infty}(0, x_2, \dots, x_k)^k$$

$$h_{\rho,\infty}(x_1, \dots, x_k) := \int_{\mathbb{R}} e^{-i\pi\alpha(k-2)} \prod_{j=1}^{k} \mathrm{sinc}(\pi(\rho x_j - \alpha)) \, d\alpha, \quad \mathrm{sinc}(x) := \frac{\sin(x)}{x}$$

Outline

Ideas of proof (1/5) Tensorisation (universal first step)

- Start from the formula (with $[x^n]\sum_\ell a_\ell x^\ell =: a_n$)

$$(-1)^j \operatorname{sc}_m(U_N) = [x^m]\, Z_{U_N}(x) := \frac{1}{2i\pi}\oint_{\mathbb{U}} Z_{U_N}(z) z^{-m}\frac{dz}{z}$$

- Tensorising, one thus gets

$$\mathbb{E}\left(|\operatorname{sc}_m(U_N)|^{2k}\right) = \mathbb{E}\left(\prod_{\ell=1}^k [x_\ell^m]\, Z_{U_N}(x_\ell) \prod_{j=1}^k \left[y_j^{-m}\right] \overline{Z_{U_N}(y_j)}\right)$$

$$= \left[x_1^m \dots x_k^m y_1^{-m} \dots y_k^{-m}\right] \mathbb{E}\left(\prod_{\ell=1}^k Z_{U_N}(x_\ell)\, \overline{Z_{U_N}(y_\ell)}\right)$$

- The interesting quantity becomes

$$\mathcal{A}_N(X, Y) := \mathbb{E}\left(\prod_{\ell=1}^k Z_{U_N}(x_\ell)\, \overline{Z_{U_N}(y_\ell)}\right)$$

The $\mathcal{A}_N(X, Y)$ are the ***autocorrelations*** *of the characteristic polynomial*

Ideas of proof (2/5) Formuli for the autocorrelations

- Several formulas are available in the literature for autocorrelations. For instance, Ackemann and Vernizzi/Fyodorov show that

$$\mathcal{A}_N(X, Y) = \frac{\det(K_{N+k}(x_j, \overline{y_\ell}))_{1 \leqslant j, \ell \leqslant k}}{\Delta(x_1, \dots, x_k)\,\overline{\Delta(y_1, \dots, y_k)}}, \quad K_N(x, y) := \sum_{\ell=0}^{N} (xy^{-1})^\ell$$

- Other formula are given by Borodin-Olshanski-Strahof, Moens-Van der Jeugt, Bump-Gamburd, Conrey-Farmer-Zirnbauer, Conrey-Forrester-Snaith, Conrey-Farmer-Keating-Rubinstein-Snaith, Day, etc.

- The useful formula is given by Bump-Gamburd

$$\mathcal{A}_N(X, Y) = \prod_{j=1}^{k} y_j^{-N} \times s_{N^k}(x_1, \dots, x_k, y_1^{-1}, \dots, y_k^{-1})$$

where $N^k := (N, \dots, N)$ (k times) and for $\lambda = (\lambda_1, \dots, \lambda_k)$, $s_\lambda(X) = \det(x_i^{\lambda_j + k - j})_{i,j \leqslant k} / \Delta(x_1, \dots, x_k)$ is a ***Schur function***.

Ideas of proof (3/5) Representation of $s_{N^k}(X, Y) \equiv s_{N^k}(T)$

- The **key formula** is the following representation of the Schur function:

$$\boxed{s_{N^k}(t_1, \dots, t_\ell) = \frac{1}{k!} \oint_{\mathbb{U}^k} \prod_{j=1}^{k} u_j^{-N} \times h_{Nk}(t_i u_j)_{\substack{1 \leqslant i \leqslant \ell \\ 1 \leqslant j \leqslant k}} \left|\Delta(u_1, \dots, u_k)\right|^2 \prod_{j=1}^{k} \frac{du_j}{2i\pi u_j}}$$

where $h_N(a_1, \dots, a_r)$ is the ***complete homogeneous function***:

$$h_N(a_1, \dots, a_r) = \sum_{1 \leqslant i_1 \leqslant \dots \leqslant i_r \leqslant N} a_{i_1} \dots a_{i_r} = \left[s^N\right] \prod_{j=1}^{r} \frac{1}{1 - s a_j}$$

$$= \left[s^N\right] \prod_{j=1}^{r} \frac{1 - s^{N+1} a_j^{N+1}}{1 - s a_j}$$

Ideas of proof (4/5) Hyperplane concentration

- Critical step in the massaging: **loss of dimension**/concentration of the integrand on a hyperplane.

  Using $h_N(\lambda X) = \lambda^N h_N(X)$ and $\Delta(\lambda X) = \lambda^{k(k-1)/2}\Delta(X)$, one has

$$s_{N^k}(t_1, \ldots, t_\ell) = \frac{1}{k!} \oint_{\mathbb{U}^k} u_1^{Nk} h_{Nk}(t_m u_j u_1^{-1})_{\substack{1\leqslant m\leqslant \ell \\ 1\leqslant j\leqslant k}} \left| \Delta\left(1, \frac{u_2}{u_1}, \ldots, \frac{u_k}{u_1}\right) \right|^2 \prod_{j=1}^{k} \left(u_1 \frac{u_j}{u_1}\right)^{-N} \frac{d^* u_j}{u_j}$$

$$=: \frac{1}{k!} \oint_{\mathbb{U}^{k-1}} h_{Nk}\left[(1+V)T\right] \left|\Delta\left[1+V\right]\right|^2 V^{-N} \frac{d^* V}{V}$$

- This step is the key ingredient that allows to keep the form intact until the end without further analysis (and the key difference with the CFKRS massaging).

Ideas of proof (5/5) : $\left[X^{[\rho N]} Y^{[(1-\rho)N]}\right] s_{N^k}(X, Y)$ rescaled

- Set $x_k = e^{i\theta_k/N}$, $y_k = e^{i\alpha_k/N}$, $u_k = e^{i\varphi_k/N}$ in the previous formuli.
- One has (locally uniformly in $\mathbb{R}$)

$$\frac{1}{N^{r-1}} h_{[\rho N]}(e^{i\frac{b_1}{N}}, \ldots, e^{i\frac{b_r}{N}}) \xrightarrow[N\to+\infty]{} \int_{\mathbb{R}} e^{-i\pi s(r-2)} \prod_{j=1}^{r} \operatorname{sinc}(\pi(\rho b_j - s))\, ds$$

$$=: h_{\rho,\infty}(b_1, \ldots, b_r)$$

- The Vandermonde term in the limit comes from the rescaling ($b_1 = 0$)

$$\frac{1}{N^{r(r-1)/2}} \Delta(e^{2i\pi\frac{b_1}{N}}, \ldots, e^{2i\pi\frac{b_r}{N}}) \xrightarrow[N\to+\infty]{} (2i\pi)^{\frac{k(k-1)}{2}} \Delta(b_1, \ldots, b_r)$$

- It remains to collect the powers of N that come from the different changes of variables and limits.
- To prove properly the result, one needs additional estimates to apply dominated convergence (comes from a probabilistic representation of h_N).

Outline

Discussion : duality

- The method consists in writing

$$s_{k^N}[T] = \int_{\mathcal{U}_N} \det(U)^{-k} \prod_{j=1}^{2k} \det(I + t_j U) dU$$

$$= \int_{\mathcal{U}_k} \det(U)^{-N} \prod_{j=1}^{2k} \det(I - t_j U)^{-1} dU$$

$$= \omega(s_{N^k}[T])$$

This is a *duality formula* in the theory of symmetric function (ω is the fundamental involution that exchanges the h_k's and the e_k's).

- The key formula

$$s_{N^k}[\mathcal{A}] = \frac{1}{k!} \int_{\mathbb{U}^{k-1}} V^{-N} h_{Nk}[(1+V)\mathcal{A}] \left|\Delta[1+V]\right|^2 \frac{1}{(2i\pi)^{k-1}} \frac{dV}{V}$$

can be specialised with any type of “abstract” alphabet.

Discussion : RKHS

- Its origin can also be understood with the concept of *Reproducing Kernel Hilbert Space* (RKHS) of symmetric functions.

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9. Let $C(x,y)=\prod_{i,j}(1-x_i y_j)^{-1}$. Then for all $f\in\Lambda$ we have

$$\langle C(x,y), f(x)\rangle = f(y)$$

where the scalar product is taken in the x variables. (By linearity, it is enough to prove this when $f=s_\lambda$, and then it follows from (4.3) and (4.8).)

In other words, $C(x,y)$ is a 'reproducing kernel' for the scalar product.

- Here, $C(X,Y) := \prod_{i,j}\frac{1}{1-x_i y_j}$ is the Cauchy kernel. Specifying the scalar product as in [Diaconis and Shashahani], one gets

$$s_\lambda[\mathcal{A}] = \frac{1}{\ell!}\oint_{\mathbb{U}^\ell} s_\lambda\left[U^{-1}\right] C(U,\mathcal{A})\,|\Delta(U)|^2 \frac{dU}{(2i\pi)^\ell U}\ , \quad \forall\,\ell \geqslant \ell(\lambda)$$

- For $\lambda = N^k$ and $\ell = k$, one has $s_\lambda(U) = \det(U)^N \equiv \prod_{j=1}^k u_j^N$ and one finds the Fourier coefficient form.

Discussion : Restricted RKHS

- A last trick allows to pass from

$$s_\lambda[\mathcal{A}] = \frac{1}{\ell!} \oint_{\mathbb{U}^\ell} s_\lambda\left[U^{-1}\right] C(U, \mathcal{A}) \left|\Delta(U)\right|^2 \frac{dU}{(2i\pi)^\ell U} , \quad \forall \ell \geqslant \ell(\lambda)$$

to

$$s_\lambda[\mathcal{A}] = \frac{1}{\ell!} \oint_{\mathbb{U}^\ell} s_\lambda\left[U^{-1}\right] h_{|\lambda|}[\mathcal{A}U] \left|\Delta(U)\right|^2 \frac{dU}{(2i\pi)^\ell U} , \quad \forall \ell \geqslant \ell(\lambda)$$

- One uses

$$C(X, Y) = H[XY] = \sum_{n \geqslant 0} h_n[XY] = \sum_{n \geqslant 0} \sum_{|\mu| = n} s_\mu(X) s_\mu(Y)$$

and the orthogonality of the Schur functions allows to keep only the sum for $n = |\lambda|$.

Discussion : The CFKRS formula (1/3)

- The CFKRS formula reads

$$s_{N^k}(e^{-\alpha_1}, \dots, e^{-\alpha_k}, e^{-\alpha_{k+1}}, \dots, e^{-\alpha_{2k}})$$
$$= \frac{(-1)^{\binom{2k}{2}}}{(k!)^2}$$
$$\times \oint_{\mathbb{U}^{2k}} e^{-N(z_{k+1}+\cdots+z_{2k})} \prod_{\substack{1 \leqslant \ell \leqslant k, \\ k+1 \leqslant r \leqslant 2k}} \mathcal{Z}(z_r - z_\ell) \prod_{1 \leqslant i,j \leqslant 2k} \frac{1}{1 - \alpha_j z_i^{-1}} |\Delta(Z)|^2 \frac{d^*Z}{Z}$$

with $\mathcal{Z}(x) := \dfrac{1}{1 - e^{-x}}$ and $d^*Z := \dfrac{dZ}{(2i\pi)^n}$

- The proof starts with the formula

$$s_{N^k}(X, Y) = \sum_{\sigma \in \Xi_k} \sigma \cdot \left(\prod_{i=1}^{k} y_i^N C(X, Y^{-1}) \right)$$

with $X := (x_1, \dots, x_k)$, $Y := (y_{k+1}, \dots, y_{2k})$ and $\Xi_k \subset \mathfrak{S}_k$.

Discussion : The CFKRS formula (2/3)

- One then applies the RKHS formula $f(a) = \langle f(u), C(u,a)\rangle_{u\in\mathcal{U}_N}$ with the Cauchy kernel in $2k$ variables (with $p_1(A) := \sum_{a\in A} a$)

$$
\begin{aligned}
s_{N^k}\left(e^X, e^Y\right) &= \left\langle \sum_{\sigma\in\Xi_k} \sigma\cdot\left(e^{Np_1(U_+)}C\left(e^{U_-}, e^{\varepsilon U_+}\right)\right), C(X\cup Y, U)\right\rangle_{U\in\mathcal{U}_{2k}} \\
&= |\Xi_k| \left\langle e^{Np_1(U_+)}C\left(e^{U_-}, e^{-U_+}\right), C(X\cup Y, U)\right\rangle_{U\in\mathcal{U}_{2k}} \\
&= \frac{\binom{2k}{k}}{(2k)!}\oint_{\mathbb{U}^{2k}} e^{Np_1(U_+)}C\left(e^{U_-}, e^{-U_+}\right)C\left(X\cup Y, U^{-1}\right)|\Delta(U)|^2\frac{d^*U}{U}
\end{aligned}
$$

with $U := \{u_1, \dots, u_{2k}\} = U_-\cup U_+$ and $U_- := \{u_1, \dots, u_k\}$.

- Several drawbacks : one does not see the scaling of the Schur function $s_\lambda(tX) = t^{|\lambda|}s_\lambda(X)$, one has $2k$ integrals (v.s. k in the key formula) and one has a mixture of exponential coordinates and linear ones. This last drawback prevents from using a (restricted) polynomial reproducing kernel.

Discussion : The CFKRS formula (3/3)

- As a result, one needs to perform several additional steps before getting the result.
- Ultimately, the massaging ends up with the rescaling of $h_N(e^{X/N})$ as in our methodology :

4.4.2. *A lemma on geometric sums.* Let $V = \operatorname{diag}(v_1, \dots, v_d)$ be a diagonal $d \times d$ matrix, and M a large parameter. We want to compute the asymptotic behaviour of

$$\operatorname{tr} \operatorname{Sym}^M \exp(\frac{1}{M}V) = \sum_{\substack{k_1+\ldots+k_d=M \\ k_1,\ldots,k_d\geq 0}} \exp(\frac{1}{M}\sum_{j=1}^{d} k_j v_j) \tag{4.37}$$

This is the coefficient of x^M in the power series expansion of

$$\det(I - x\exp(\frac{1}{M}V))^{-1} = \frac{1}{\prod_{j=1}^{d}(1-e^{v_j/M}x)}$$

Lemma 4.12. *As* $M \to \infty$,

$$\operatorname{tr} \operatorname{Sym}^M \exp(\frac{1}{M}V) = M^{d-1} \iint\limits_{\substack{x_1+\ldots+x_d=1 \\ x_j\geq 0}} e^{\sum x_j v_j} dx_1 \ldots dx_d + O(M^{d-2})$$

From [Keating-Rodgers-(Roditty-Gershon)-Rudnick]

Outline

Conclusion

- Main concepts to exchange N integrals with k (fixed) integrals : duality, reproducing kernel methodology.
- All previously described functionals can be written by means of the secular coefficients with a probabilistic writing ("randomisation"). For example $\partial Z_{U_N}(1) = \sum_{k=1}^{N} k\, \mathrm{sc}_k(U_N) = \frac{N(N-1)}{2}\mathbb{E}(\mathrm{sc}_{X_N}(U_N)|U_N)$ with $\mathbb{P}(X_n = k) = \frac{2k}{N(N-1)}\mathbb{1}_{\{1\leqslant k\leqslant N\}}$.
- The method is general : there exists a function $\Phi_{\mathcal{S}} : \mathbb{R}^{k-1} \to \mathbb{C}$ s.t. all integer moments of the previous functionals of Z_{U_N} display a behaviour of the form $\mathcal{S}(k)N^{d(k)}$ with $\mathcal{S}(k) = \frac{(2\pi)^{k(k-1)}}{k!}\int_{\mathbb{R}^{k-1}} \Phi_{\mathcal{S}}\Delta^2$.
- All expressions with $\Delta(X)^2$ for $X \in \mathbb{R}^{k-1}$ have an equivalent expression with probability and Hankel determinants.

Conclusion

$\mathbb{E}\left(\lvert Z_{U_N}(1)\rvert^{2k}\right)$	N^{k^2}	$a_k \int_{\mathbb{R}^{k-1}} \Phi_{\widetilde{L}_1}(0,\boldsymbol{x})\Delta(0,\boldsymbol{x})^2 d\boldsymbol{x}$
$\mathbb{E}\left(\lvert \mathrm{sc}_{[\rho N]}(U_N)\rvert^{2k}\right)$	$N^{(k-1)^2}$	$a_k \int_{\mathbb{R}^{k-1}} \Phi_{\mathcal{SC}_\rho^{(k)}}(0,\boldsymbol{x})\Delta(0,\boldsymbol{x})^2 d\boldsymbol{x}$
$\mathbb{E}\left(\left\lvert \sum_{\substack{1\leqslant j_1,\dots,j_k\leqslant N \\ j_1+\dots+j_k=[cN]}} \mathrm{sc}_{j_1}(U_N)\dots \mathrm{sc}_{j_k}(U_N)\right\rvert^2\right)$	N^{k^2-1}	$a_k \int_{\mathbb{R}^{k-1}} \Phi_{\mathcal{J}_c}(0,\boldsymbol{x})\Delta(0,\boldsymbol{x})^2 d\boldsymbol{x}$
$\mathbb{E}\left(\prod_{j=1}^{k} Z_{U_N}\left(e^{2i\pi\frac{x_j}{N}}\right)\overline{\prod_{\ell=1}^{m} Z_{U_N}\left(e^{-2i\pi\frac{y_\ell}{N}}\right)}\right)$	N^{km}	$a_m \int_{\mathbb{R}^{m-1}} \Phi_{\mathcal{A}(X,Y)}(0,\boldsymbol{t})\Delta(0,\boldsymbol{t})^2 d\boldsymbol{t}$
$\mathbb{E}\left(\lvert Z_{U_N}(1)\rvert^{2(k-\lvert\boldsymbol{h}\rvert)}\prod_{r=1}^{m}\lvert\partial^r Z_{U_N}(1)\rvert^{2h_r}\right)$	$N^{k^2+2C(\boldsymbol{h})}$	$a_k \int_{\mathbb{R}^k} \Phi_{\mathcal{D}_\infty(k;\boldsymbol{h})}(\boldsymbol{x})\Delta(0,\boldsymbol{x})^2 d\boldsymbol{x}$
$\mathbb{E}\left(\lvert Z_{U_N,[\rho N]}(1)\rvert^{2k}\right)$	N^{k^2}	$a_k \int_{\mathbb{R}^k} \Phi_{\mathcal{ZT}_\rho^{(k)}}(\boldsymbol{x})\Delta(0,\boldsymbol{x})^2 d\boldsymbol{x}$
$\mathbb{E}\left(\lvert \lambda^{-\rho N} Z_{U_N,[\rho N]}(\lambda)\rvert^{2k}\right), \lambda > 1$	$N^{(k-1)^2}$	${}_2F_1\left(\begin{smallmatrix}k,k\\1\end{smallmatrix}\middle\vert\lambda^{-2}\right)\mathcal{SC}_\rho^{(k)}$
$\mathbb{E}\left(\det(U_N)^{-k}\frac{\prod_{j=1}^{\ell} Z_{U_N}\left(e^{2i\pi x_j/N}\right)}{\prod_{r=1}^{m} Z_{U_N}\left(e^{2i\pi y_r/N}\right)}\right)$	$N^{k(\ell-m-k)}$	$a_k \int_{\mathbb{R}^{k-1}} \Phi_{\mathcal{R}(X,Y)}(0,\boldsymbol{t})\Delta(0,\boldsymbol{t})^2 d\boldsymbol{t}$
$\mathbb{E}\left(\left(\oint_{\mathbb{U}} \lvert Z_{U_N}(z)\rvert^{2\beta}\frac{d^*z}{z}\right)^k\right)$	$N^{(k\beta)^2+1-k}$	$a_{k\beta}\int_{\mathbb{R}^{k\beta-1}}\Phi_{\mathfrak{M}\mathfrak{o}\mathfrak{M}_+(k,\beta)}(0,\boldsymbol{x})$ $\Delta(0,\boldsymbol{x})^2 d\boldsymbol{x}$
[illegible]	[illegible]	[illegible]

Thank you for your attention.