On the exceptional series and its siblings

Series of representations

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11 January 2024

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Introduction

Finite groups, simple Lie algebras and reductive algebraic groups are all *rigid*; so we can't construct families depending on a continuous parameter.
Let V be an (irreducible) representation. The category of *invariant tensors* has objects 0,1,2,... and the morphisms are

$$\mathrm{Hom}(\otimes^n V, \otimes^m V)$$

These categories have additional structure e.g. tensor product, symmetry,... We can construct families of categories with these structures depending on one, or more, continuous parameters.

This is well-known for V the defining representation of a classical series

$$\mathrm{SL}(n), \mathrm{GL}(n), \mathrm{SO}(n), \mathrm{Sp}(2n), \mathfrak{S}_n$$

where n becomes a formal parameter.

- $\mathrm{SL}(n)$,$\mathrm{GL}(n)$ Schur-Weyl duality
- $\mathrm{SO}(n)$,$\mathrm{Sp}(2n)$ Brauer category
- $\mathfrak{S}_n$ Partition category

and the first two cases have quantum analogues.
What about exceptional simple Lie algebras?

Exceptional series

The exceptional series is a finite sequence of Lie algebras parametrised by $m \in \mathbb{Q}$.

m	-3/2	-4/3	-1	-2/3	0	1	2	4	8
	$\mathfrak{osp}(1\|2)$	A_1	A_2	G_2	D_4	F_4	E_6	E_7	E_8

These are the simple Lie algebras with no primitive quartic Casimir. Equivalently, the simple Lie algebras for which 4 is not an exponent.

Decompositions

Let L be a Lie algebra on the exceptional series and consider L as a representation of the algebraic group $\mathrm{Aut}(L)$. Then, for $m \geqslant -1$, we have the decompositions

$$\bigwedge^2 L(\theta) \cong L(\theta) \oplus L(\mu) \qquad S^2 L(\theta) \cong L(0) \oplus L(2\theta) \oplus L(\nu)$$

where θ is the highest root.

Casimirs

The values of the Casimir are computed using

$$C(\lambda) = \langle \lambda, \lambda + 2\rho \rangle$$

The key observation is that the values of the Casimir can be interpolated by linear functions of m. The linear functions are

θ	μ	2θ	ν
$6m + 12$	$12m + 24$	$12m + 28$	$10m + 16$

Vogel plane

Let L be a simple Lie algebra considered as a representation of the algebraic group $\mathrm{Aut}(L)$.
The decomposition of $L \otimes L$ is

$$\bigwedge^2 L \cong L \oplus X_2$$

$$S^2 L \cong I \oplus Y(\alpha) \oplus Y(\beta) \oplus Y(\gamma)$$

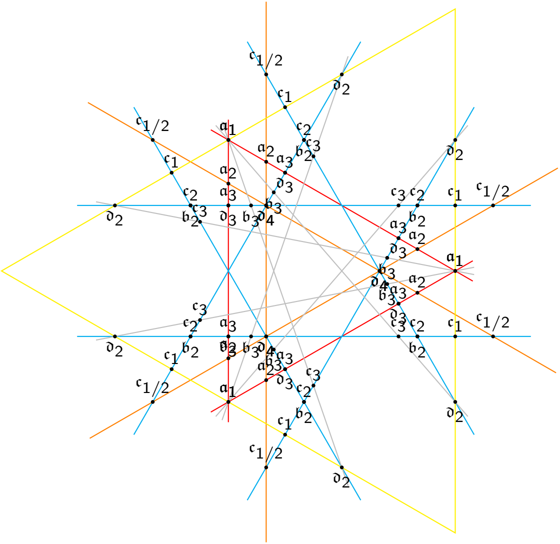
The Casimirs are

I	L	X_2	$Y(\alpha)$	$Y(\beta)$	$Y(\gamma)$
0	t	$2t$	$2t-\alpha$	$2t-\beta$	$2t-\gamma$

where $t = \alpha + \beta + \gamma = \check{h}$.

$$\dim(L) = \frac{(\alpha - 2t)(\beta - 2t)(\gamma - 2t)}{\alpha\beta\gamma}$$

Vogel plane



$\mathfrak{a}_1$
$\mathfrak{a}_2$
$\mathfrak{a}_3$
$\mathfrak{b}_2$
$\mathfrak{b}_3$
$\mathfrak{c}_{1/2}$
$\mathfrak{c}_1$
$\mathfrak{c}_2$
$\mathfrak{c}_3$
$\mathfrak{d}_2$
$\mathfrak{d}_3$
$\mathfrak{d}_4$

Magic square

The Freudenthal magic square is the following square of Lie algebras.

	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{O}$
$\mathbb{R}$	B_1^3	A_2^8	C_3^{21}	F_4^{52}
$\mathbb{C}$	A_2^8	$2A_2^{16}$	A_5^{35}	E_6^{78}
$\mathbb{H}$	C_3^{21}	A_5^{35}	D_6^{66}	E_7^{133}
$\mathbb{O}$	F_4^{52}	E_6^{78}	E_7^{133}	E_8^{248}

- The subscript is the *rank* of the Lie algebra.
- The superscript is the *dimension* of the Lie algebra

Dimensions

The following gives the dimension of the preferred representation and the dimension of the adjoint representation

	V	L
$\mathbb{R}$	$(3m+2)$	$\frac{3m(3m+2)}{(m+4)}$
$\mathbb{C}$	$(3m+3)$	$\frac{4(m+1)(3m+2)}{(m+4)}$
$\mathbb{H}$	$(6m+8)$	$\frac{3(3m+4)(2m+3)}{(m+4)}$
$\mathbb{O}$	$\frac{2(5m+8)(3m+7)}{(m+4)}$	$\frac{2(5m+8)(3m+7)}{(m+4)}$

Quaternion row

This gives Freudenthal triple sytems.

m	-2/3	0	1	2	4	8
$\mathfrak{g}$	A_1	$3A_1$	C_3	A_5	D_6	E_7
G	$\mathrm{SL}(2)$	$\mathfrak{S}_3 \rtimes \mathrm{SL}(2)$	$\mathrm{Sp}(6)$	$\mathfrak{S}_2 \rtimes \mathrm{SL}(6)/\mu_2$	$\mathrm{Spin}(12)$	E_7
λ	$3\omega_1$	$\omega_1 + \omega_2 + \omega_3$	ω_3	ω_3	ω_6	ω_7

m	-3	-8/3	-5/2
$\mathfrak{g}$	D_5	B_3	G_2
G	$\mathrm{SO}(10)$	$\mathrm{Spin}(7)$	G_2
ω	ω_1	ω_3	ω_1

Features

The features of a series are:

- members of the series are indexed by a point in a projective space
- shared Bratteli diagram (branching rules)
- shared Schur functors (e.g. symmetric and exterior powers)
- Casimirs are linear functions of homogeneous coordinates
- dimensions are rational functions

For a classical series the Bratteli diagram is known indefinitely and dimensions are polynomial functions.

Notation

$L(\lambda)$ is a highest weight module with highest weight λ.

- 0 is the zero weight so $L(0)$ is the trivial representation
- θ is the highest root so $L(\theta)$ is the adjoint representation

Property

The decomposition of $L(\lambda) \otimes L(\lambda)$ is

$$\bigwedge^2 L(\lambda) \cong L(\theta) \oplus L(\mu)$$

$$S^2 L(\lambda) \cong L(0) \oplus L(2\lambda)$$

The representation $L(\lambda)$ has an anti-symmetric quartic form.

Strategy

We find the quantum dimensions first and then find the dimensions by taking $q \to 1$.

- Interpolate Casimirs/eigenvalues
- Construct representation of braid group, B_3.
- Determine structure constants of algebra $A(2)$.
- (Optional) Take limit $q \to 1$.

Property

The decomposition of $L(\lambda) \otimes L(\lambda)$ is

$$\bigwedge^2 L(\lambda) \cong L(\theta) \oplus L(\mu)$$

$$S^2 L(\lambda) \cong L(0) \oplus L(2\lambda) \oplus L(\nu)$$

This includes the first ($\mathbb{R}$) line of the magic square ($\lambda = \theta$) and the fourth ($\mathbb{O}$) line ($\lambda = \nu$). The representation $L(\lambda)$ for the first ($\mathbb{R}$) line has an invariant symmetric cubic form and ahe representation $L(\lambda)$ for the fourth ($\mathbb{O}$) line has an invariant anti-symmetric cubic form

Exceptional symmetric spaces

Associated to a symmetric space is a Lie algebra with an involution. The $+1$-eigenspace is a Lie algebra, L, and the -1-eigenspace is an L-module, V.

$2/3$	1	$4/3$	$8/3$	5	8	10	12
EVIII	EV	E1	A1		BD1	FII	EIV
E_8	E_7	E_6	A_2	$OSp(1\|2)$	D_4	F_4	E_6
D_8	A_7^*	C_4	A_1	A_1	B_3	B_4	F_4
128	70	42	5		7	16	26

Property

The decomposition of $L(\lambda) \otimes L(\lambda)$ is

$$\bigwedge^2 L(\lambda) \cong L(\theta) \oplus L(\mu)$$

$$S^2 L(\lambda) \cong L(0) \oplus L(2\lambda) \oplus L(\nu_1) \oplus L(\nu_2)$$

This includes the Vogel plane, $\lambda = \theta$.
This includes the representations $L(2\omega_1)$ of $\mathrm{SO}(n)$ and the representations $L(\omega_2)$ of $\mathrm{Sp}(n)$. These are the infinite series of symmetric spaces AI and AII.
This includes the representations $L(\omega_1; \omega_1)$ of $(\mathrm{SO}(n) \times \mathrm{SO}(n)) \rtimes \mathfrak{S}_2$.

$$\bigwedge^2 L(\omega_1; \omega_1) \cong [L(\omega_2; 0) \oplus L(0; \omega_2)] \oplus [L(\omega_2; 2\omega_1) \oplus L(2\omega_1; \omega_2)]$$

$$S^2 L(\omega_1; \omega_1) \cong L(0; 0) \oplus L(2\omega_1; 2\omega_1) \oplus L(\omega_2; \omega_2) \oplus [L(2\omega_1; 0) \oplus L(0; 2\omega_1$$