On the uniqueness of maximal solvable extensions of nilpotent Lie algebras

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Introduction

From the classical theory of finite-dimensional Lie algebras, it is known that an arbitrary Lie algebra is decomposed into a semidirect sum of the solvable radical and its semisimple subalgebra (Levi's theorem). According to the Cartan-Killing theory, a semisimple Lie algebra can be represented as a direct sum of simple ideals, which are completely classied [1]. Thanks to Malcev's and Mubarakzjanov's results the study of non-nilpotent solvable Lie algebras is reduced to the study of nilpotent ones [2] [3]. Therefore, the study of finite-dimensional Lie algebras is focused on nilpotent algebras.

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[1] Jacobson N., Lie algebras, Interscience Publishers, Wiley, New York, 1962

[2] Malcev A.I., Solvable Lie algebras, Amer. Math. Soc. Translation, 1950

[3] Mubarakzyanov G.M. The classification of real structures of Lie algebra of order five. Izv. VUZ. Math., 1963

The research of solvable Lie algebras with some special types of nilradical comes from different problems in Physics and was the subject of various papers [4] [5] [6] [7] [8] [9] [10] [11] [12] and references given therein.

[4] Indecomposable Lie algebras with nontrivial Levi decomposition cannot have filiform radical, Int. Math. Forum, 2006

[5] Classification of Lie algebras with naturally graded quasi-filiform nilradicals, J. Geom. Phys., 2011

[6] Invariants of solvable Lie algebras with triangular nilradicals and diagonal nilindependent elements, Linear Alg. Appl., 2008

[7] Solvable Lie algebras with an $\mathbb{N}$-graded nilradical of maximal nilpotency degree and their invariants, J. Phys.: A., 2010

[8] Solvable Lie algebras with abelian nilradicals, J. Phys. A, 1994

[9] Classification of solvable Lie algebras with a given nilradical by means of solvable extensions of its subalgebras, Linear Alg. Appl., 2010

[10] A class of solvable Lie algebras and their Casimir invariants, J. Phys. A, 2005

[11] Solvable Lie algebras with triangular nilradicals, J. Phys. A, 1998

[12] Solvable Lie algebras with quasifiliform nilradicals, Commun. Algebra, 2008

Solvable Lie algebras with a given nilradical

Since the description of finite-dimensional nilpotent Lie algebras is an immense problem, usually they are studied under some additional restrictions. For instance in paper [13] the authors claim that they have found 24168 non-isomorphic 9-dimensional nilpotent algebras Lie algebras with a maximal Abelian ideal of dimension 7 alone. A realistic partial classification problem is to classify all solvable Lie algebras with a given nilradical of an arbitrary finite dimension n. So far this has been done for certain series of nilpotent Lie algebras, namely Abelian, Heisenberg and Borel nilradicals, as well as certain filiform and quasifiliform algebras.

[13] Gr. Tsagas, A. Kobotis, and T. Koukouvinos, Classification of nilpotent Lie algebras of dimension nine whose maximum abelian ideal is of dimension seven, Int. J. Comput. Math., 2000

Šnobl's Conjecture

The point here is that non-nilpotent solvable Lie algebras can be reconstructed using their nilradical and its special kind of derivations. The Mubarakzjanov's method of construction of solvable Lie algebras in terms of their nilradicals is effectively used.
In the paper of Šnobl [14] the following conjecture was stated.

Conjecture. Let $\mathfrak{n}$ be a complex nilpotent Lie algebra, not characteristically nilpotent. Let $\mathfrak{s}, \tilde{\mathfrak{s}}$ be solvable Lie algebras with the nilradical $\mathfrak{n}$ of maximal dimension in the sense that no such solvable algebra of larger dimension exists. Then, $\mathfrak{s}$ and $\tilde{\mathfrak{s}}$ are isomorphic.

[14] Šnobl L., On the structure of maximal solvable extensions and of Levi extensions of nilpotent Lie algebras, J. Phys. A: 2010

Nilpotent and Solvable Lie algebras

For a given Lie algebra $\mathcal{L}$, we define the lower central and derived series to the sequences of two-sided ideals defined recursively as follows:

$$\mathcal{L}^1 = \mathcal{L},\ \mathcal{L}^{k+1} = [\mathcal{L}^k, \mathcal{L}],\quad k \geq 1;\quad \mathcal{L}^{[1]} = \mathcal{L},\ \mathcal{L}^{[s+1]} = [\mathcal{L}^{[s]}, \mathcal{L}^{[s]}],\quad s \geq 1.$$

Definition

A Lie algebra $\mathcal{L}$ is said to be nilpotent (respectively, solvable), if there exists $n \in \mathbb{N}$ ($m \in \mathbb{N}$) such that $\mathcal{L}^n = 0$ (respectively, $\mathcal{L}^{[m]} = 0$).

Since the sum of finitely many nilpotent ideals is a nilpotent ideal, there exists the maximal nilpotent ideal called the nilradical of an algebra.

Torus of Lie algebra

Definition. An abelian subalgebra of $Der(\mathcal{N})$, which consists of semisimple linear transformations is called a torus on $\mathcal{N}$. A torus is said to be maximal if it is not strictly contained in any other torus.

Note that in the case of complex numbers field a semisimple endomorphism is diagonalizable.

It is known that any two maximal tori of an nilpotent Lie algebra are conjugate under an inner automorphism of the algebra [15]. Consequently, the dimensions of two maximal tori of an nilpotent Lie algebra are equal. Let $\mathcal{N}$ be a finite-dimensional nilpotent (non-characteristically nilpotent) Lie algebra. We denote by $rank(\mathcal{N})$ the dimension of a maximal torus $\mathcal{T}$. From [16] it is known that $rank(\mathcal{N}) \leq \dim(\mathcal{N}/\mathcal{N}^2)$, where $dim(\mathcal{N}/\mathcal{N}^2)$ means the number of generators of $\mathcal{N}$.

[15] Mostow G.D. Fully reducible subgroup of algebaic group, Amer. Math.,1956

[16] Meng D. J., Zhu L. S., Solvable complete Lie algebras I, Commun. Algebra, 1996

Root subspace decomposition

Let $\mathcal{T}$ be a maximal torus of an nilpotent Lie algebra $\mathcal{N}$. Then we have root subspaces decomposition

$$\mathcal{N} = \mathcal{N}_\alpha \oplus \mathcal{N}_\beta \oplus \cdots \oplus \mathcal{N}_\gamma,$$

where $\mathcal{N}_\alpha = \{n_\alpha \in \mathcal{N} \ : \ [n_\alpha, x] = \alpha(x) n_\alpha, \ \forall \ x \in \mathcal{T}\}$
and a root belongs to the dual space of $\mathcal{T}$. Moreover, $\mathcal{T}$ admits a basis $\{t_1, \dots, t_s\}$ which satisfies the condition:

$$t_i(n_{\alpha_j}) = \alpha_{i,j} n_{\alpha_j}, 1 \le i, j \le s,$$

where $\alpha_{i,i} \neq 0, \ \alpha_{i,j} = 0$ for $i \neq j$ and $n_{\alpha_j} \in \mathcal{N}_{\alpha_j}$.

Maximal solvable extension

Consider a solvable Lie algebra $\mathcal{R}_{\mathcal{T}} = \mathcal{N} \rtimes \mathcal{T}$ with the action of a maximal torus $\mathcal{T}$ on $\mathcal{N}$ as follows $[\mathcal{N}, \mathcal{T}] = \mathcal{T}(N)$.
In fact, the conjugacy of arbitrary two maximal tori $\mathcal{T}$ and $\mathcal{T}'$ under inner automorphism of $\mathcal{N}$ implies that algebras $\mathcal{R}_{\mathcal{T}}$ and $\mathcal{R}_{\mathcal{T}'}$ are isomorphic.

Definition

A solvable Lie algebra $\mathcal{R}$ with nilradical $\mathcal{N}$ is called a maximal solvable extension of the nilpotent Lie algebra $\mathcal{N}$, if $codim\mathcal{N}$ is maximal.

It is known that $codim\mathcal{N}$ does not exceed the number of generators of the nilradical [17].

[17] Mubarakzyanov G.M. The classification of real structures of Lie algebra of order five. Izv. VUZ. Math., 1963

Since the square of a solvable Lie algebra is nilpotent, any solvable Lie algebra has non-trivial nilradical. Therefore, for a solvable Lie algebra $\mathcal{R}$ we have a decomposition $\mathcal{R} = \mathcal{N} \oplus \mathcal{Q}$, where $\mathcal{N}$ is the nilradical of $\mathcal{R}$ and $\mathcal{Q}$ is the subspace complementary to $\mathcal{N}$.

Proposition

Let $\mathcal{R} = \mathcal{N} \oplus \mathcal{Q}$ be a maximal solvable extension of an nilpotent Lie algebra $\mathcal{N}$. Then $rank(\mathcal{N}) = \dim \mathcal{Q}$.

Examples of non-uniqueness of maximal solvable extensions

Let us present first an example which shows that in Šnobl's conjecture the condition of the ground field to be $\mathbb{C}$ is essential.

Example

Consider three-dimensional Heisenberg Lie algebra $H_1 : [e_2, e_3] = e_1$. *Up to isomorphism there exists a unique complex solvable five-dimensional Lie algebra with the nilradical* H_1 *and the table of multiplications*

$$[e_2, e_3] = e_1, \ [e_2, x_1] = e_2, \ [e_1, x_1] = e_1, \ [e_3, x_2] = e_3, \ [e_1, x_2] = e_1.$$

However, Muborakzyanov showed the existence of the following two non-isomorphic to each other real solvable Lie algebras:

$$g_1 : \ [e_2, e_3] = e_1, \ [e_1, x_1] = e_1, \ [e_2, x_1] = e_2, \ [e_2, x_2] = -e_2, \ [e_3, x_2] = e_3,$$

$$g_2 : \ [e_2, e_3] = e_1, \ [e_1, x_1] = 2e_1, \ [e_2, x_1] = e_2, \ [e_3, x_1] = e_3, \ [e_2, x_2] = -e_3, \ [e_3, x_2] = e_2.$$

Gorbatsevich's example

The existence of non-isomorphic two complex maximal solvable extensions of a given niradical that presented by Gorbatsevich [18], shows that Šnobl's conjecture is not true, in general. In fact, V. Gorbatsevich constructed two 9-dimensional complex solvable Lie algebras whose nilradical is the direct sum of the 7-dimensional characteristically nilpotent ideal $\mathcal{N}_7 = Span\{e_1, \dots, e_7\}$ (that means all derivations of $\mathcal{N}_7$ are nilpotent) and ideal $Span\{e_8\}$. Then the first algebra $\mathcal{R}_1$ is the direct sum of $\mathcal{N}_7$ and of 2-dimensional solvable ideal $Span\{x, e_8\}$, while the second algebra $\mathcal{R}_2$ is the solvable extension obtained by derivation $ad(x) + d$, where $d(e_2) = e_7$ is the outer derivation of $\mathcal{N}_7$. Direct computations leads that these solvable Lie extensions are non-isomorphic.

[18] Gorbatsevich V.V. On maximal extensions of nilpotent Lie algebras, Funct. Anal. Appl., 2022

Example 1. *Consider 9-dimensional nilpotent Lie algebra $\mathcal{N}_9$ with multiplications table:*

$$\begin{cases} [e_1, e_2] = e_3, \quad [e_1, e_3] = e_4, \quad [e_1, e_4] = e_5, \quad [e_1, e_6] = e_7, \quad [e_1, e_8] = e_9, \\ [e_2, e_3] = e_8, \quad [e_2, e_4] = e_9, \quad [e_2, e_5] = e_9, \quad [e_4, e_3] = e_9. \end{cases}$$

Direct computations lead to

$$Der(\mathcal{N}_9) : \begin{pmatrix} 0 & 0 & * & * & * & * & * & * & * \\ 0 & \alpha & \tau & \frac{1}{2}(\mu-\tau) & \delta & * & * & * & * \\ 0 & 0 & \alpha & \tau & \frac{1}{2}(\mu-\tau) & 0 & * & * & * \\ 0 & 0 & 0 & \alpha & \tau & 0 & 0 & 0 & * \\ 0 & 0 & 0 & 0 & \alpha & 0 & 0 & 0 & * \\ 0 & 0 & 0 & 0 & 0 & \beta & \nu & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & \beta & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2\alpha & \mu \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2\alpha \end{pmatrix}.$$

Obviously, $\mathcal{T} = Span\{t_1, t_2\}$ with

$$t_1 = diag(0,1,1,1,1,0,0,2,2), t_2 = diag(0,0,0,0,0,1,1,0,0)$$

forms a maximal torus of $\mathcal{N}_9$.

Straightforward computations lead that solvable Lie algebras

$$\mathcal{R}_1 = \mathcal{N} \rtimes \mathcal{T} \quad \text{and} \quad \mathcal{R}_2 = \mathcal{N} \rtimes \mathcal{T}'$$

with $\mathcal{T}' = Span\{t_1 + d, t_2\}$, where $d \in Der(\mathcal{N}_9)$ defined as

$$d(e_2) = 2e_3 - e_4,\ d(e_3) = 2e_4 - e_5,\ d(e_4) = 2e_5,\ d(e_i) = 0,\ i \neq 2,3,4$$

are non-isomorphic. Note that $\mathcal{T}'$ is abelian subalgebra of $Der(\mathcal{N}_9)$ and $t_1 + d$ is non-nilpotent derivation, which is non-diagonalizable on $\mathcal{N}_\alpha$.

d-locally diagonalizable nilpotent Lie algebras.

Based on the structures of maximal solvable extensions given in Gorbatsevich's example and Example 1, we give the notion of d-locally diagonalizable nilpotent Lie algebra.

Definition

An nilpotent Lie algebra $\mathcal{N}$ is called d-locally diagonalizable, if restrictions and projections of all non-nilpotent derivations of $\mathcal{N}$ on each root subspaces with respect to a certain maximal torus $\mathcal{T}$ are simultaneously diagonalizable.

Due to conjugacy of any two tori via inner automorphism one can assume that the definition of d-locally diagonalizability does not depend on the choice of torus.

Proposition

Let $\mathcal{N}$ be a d-locally diagonalizable nilpotent Lie algebra. Then there is no zero root with respect to a maximal torus $\mathcal{T}$ of $\mathcal{N}$.

However, there are non-locally diagonalizable nilpotent Lie algebras with respect to a maximal torus that admit both zero and non-zero roots.

Main result

Now we present the main result on the description of maximal solvable extensions of a d-locally diagonalizable nilpotent Lie algebra.

Main Theorem:

There exists a unique (up to isomorphism) complex maximal solvable extension of a locally diagonalizable nilpotent Lie algebra and it is isomorphic to the algebra $\mathcal{R}_{\mathcal{T}}$.

Remark

- *Example 1 as well as Gorbatsevich's example show the condition of locally diagonalizable in Main Theorem is essential.*
- *From Theorem we obtain the positive solution of Šnobls Conjecture in case of locally diagonalizable nilpotent Lie algebra.*

Proposition

Let $\mathcal{N}$ be an nilpotent Lie algebra such that

- *there exists a basis of $\mathcal{N}$ such that all derivations of $\mathcal{N}$ in this basis have upper-triangular matrix form;*
- *there exists a non-nilpotent derivation such that its restriction and projection on some root subspace $\mathcal{N}_\alpha$, where α is an non-zero primitive root, with respect to a maximal torus is non-diagonalizable;*

Then there exist at least two non-isomorphic maximal solvable extensions of $\mathcal{N}$.

Proposed Conjecture

Based on existing numerous examples and Proposition above we believe that the following criterion on uniqueness of maximal solvable extension is true.

Conjecture A complex maximal solvable extension of a finite-dimensional nilpotent Lie algebra is unique (up to isomorphic) if and only if its nilradical is d-locally diagonalizable. Moreover, it is isomorphic to an algebra $\mathcal{R}_{\mathcal{T}}$.

Completeness of maximal solvable extensions

Definition

A Lie algebra $\mathcal{L}$ is called complete if it is centerless and any derivation of $\mathcal{L}$ is inner.

Recall, the first group of cohomology describe the difference between the spaces of outer and inner derivations.

Proposition

A non-maximal solvable extension of an nilpotent Lie algebra admits an outer derivation.

Proposition

A maximal solvable extension of an nilpotent Lie algebra which is not d-locally diagonalizable admits an outer derivation.

Completeness of maximal solvable extensions

Now we present the criteria of completeness for complex solvable Lie algebras.

Theorem

A complex solvable Lie algebra is complete if and only if it is a maximal solvable extension of a d-locally diagonalizable nilpotent Lie algebra.

From Propositions and the proof of Theorem we conclude that a complex solvable Lie algebra admits only inner derivations if and only if it is a maximal solvable extension of a d-locally diagonalizable nilpotent Lie algebra.

The application of Main Theorem to the descriptions of maximal solvable extensions with d-locally diagonalizable nilradicals

Here we present the construction of a maximal solvable extension of a given nilpotent Lie algebra $\mathcal{N}$ of the type $\mathcal{R}_{\mathcal{T}}$ for some maximal torus of $\mathcal{N}$.

In general, $\mathcal{R}_{\mathcal{T}}$ does not describe all maximal solvable extensions of $\mathcal{N}$, while in the case of locally diagonalizable $\mathcal{N}$ thanks to Main Theorem we obtain the description of all maximal solvable extensions.

Assume that $\{e_1, \dots, e_n\}$ be a basis of $\mathcal{N}$ with $[e_i, e_j] = \sum\limits_{t=1}^{n} c_{i,j}^t e_t$ with $c_{i,j}^t$ in the ground field. For i, j, t such that $c_{i,j}^t \neq 0$ we consider the system of the linear equations

$$S_e : \quad \{\alpha_i + \alpha_j = \alpha_t$$

in the variables $\alpha_1, \dots, \alpha_n$ as i, j, t run from 1 to n.

We denote by $r\{e_1, \dots, e_n\}$ the rank of the system S_e. Setting $r\{\mathcal{N}\} = min\ r\{e_1, \dots, e_n\}$ as $\{e_1, \dots, e_n\}$ runs over all bases of $\mathcal{N}$, we note that for a nilpotent Lie algebra $\mathcal{N}$ over an algebraically closed field the equality $dim\mathcal{T} = dim\mathcal{N} - r\{\mathcal{N}\}$ holds true [19].
Consider the basis $\{(\alpha_{1,i}, \dots, \alpha_{n,i}) \mid 1 \leq i \leq s\}$ of fundamental solutions of the system S_e. Then the diagonal matrices

$$\{diag(\alpha_{1,i}, \alpha_{2,i}, \alpha_{3,i}, \dots, \alpha_{n-1,i}, \alpha_{n,i}) \mid 1 \leq i \leq s\}$$

forms a maximal torus of $\mathcal{N}$. Thus, the algebra $\mathcal{R}_{\mathcal{T}} = \mathcal{N} \rtimes \mathcal{T}$ is constructed.

[19] Leger G., Derivations of Lie algebras. III. Duke Math., 1963

Actually, there are several results on descriptions of maximal solvable extensions of locally diagonalizable nilpotent Lie algebras, which used the standard method (Mubarakzyanov's method). However, these results can be obtained due to Main Theorem just by construction of an algebra $\mathcal{R}_{\mathcal{T}}$ in the shortest way [20] [21] [22] [23] [24].

[20] Ancochea J.M., Campoamor-Stursberg R., Garcia Vergnolle L., Solvable Lie algebras with naturally graded nilradicals and their invariants, J. Phys. A: 2006

[21] Šnobl L., Winternitz P., Classification and Identification of Lie Algebras, CRM MONOGRAPH SERIES, Centre de Recherches Mathematiques Montreal, 2017

[22] Ndogmo J.C., Winternitz P., Solvable Lie algebras with abelian nilradicals, J. Phys. A, 1994

[23] Tremblay S., Winternitz P., Solvable Lie algebras with triangular nilradicals, J. Phys. A, 1998

[24] Wang Y., Lin J., Deng S., Solvable Lie algebras with quasifiliform nilradicals, Commun. Algebra, 2008

The most complete descriptions of solvable Lie algebras with various types of nilradicals are obtained till the present time is given in the book of Libor Šnobl and Pavel Winternitz **"CLASSIFICATION AND IDENTIFICATION OF LIE ALGEBRAS"**, Centre de Recherches Mathématiques Montréal, 2014, pp. 306.

Maximal solvable extensions of nilpotent Lie superalgebras

Recall the notion of Lie superalgebra. A vector space V is said to be $\mathbb{Z}_2$-graded if it admits a decomposition into a direct sum, $V = V_{\bar{0}} \oplus V_{\bar{1}}$, where $\bar{0}, \bar{1} \in \mathbb{Z}_2$. An element $x \in V$ is called *homogeneous of degree* $|x|$ if it is an element of $V_{|x|}$, $|x| \in \mathbb{Z}_2$. In particular, the elements of $V_{\bar{0}}$ (resp. $V_{\bar{1}}$) are also called *even* (resp. *odd*).

Definition

A Lie superalgebra is a $\mathbb{Z}_2$-graded vector space $\mathcal{L} = \mathcal{L}_{\bar{0}} \oplus \mathcal{L}_{\bar{1}}$ equipped with a bilinear bracket $[\ ,\ ]$, which is agreed with $\mathbb{Z}_2$-gradation and for arbitrary homogeneous elements x, y, z satisfies the conditions

1. $[x, y] = -(-1)^{|x||y|}[y, x]$,
2. $(-1)^{|x||z|}[x, [y, z]] + (-1)^{|x||y|}[y, [z, x]] + (-1)^{|y||z|}[z, [x, y]] = 0$ *(Jacobi superidentity).*

Evidently, $\mathcal{L}_{\bar{0}}$ is a Lie algebra and $\mathcal{L}_{\bar{1}}$ is a $\mathcal{L}_{\bar{0}}$-module. In addition, the Lie superalgebra structure also contains the symmetric pairing $S^2\mathcal{L}_{\bar{1}} \to \mathcal{L}_{\bar{0}}$.

Superderivations of Lie superalgebras

A superderivation d of degree $|d|, |d| \in \mathbb{Z}_2$, of a Lie superalgebra $\mathcal{L}$ is an endomorphism $d \in End(\mathcal{L})_{|d|}$ with the property

$$d([x,y]) = (-1)^{|d|\cdot|y|}[d(x), y] + [x, d(y)].$$

If we denote by $Der(\mathcal{L})_s$ the space of all superderivations of degree s, then

$$Der(\mathcal{L}) = Der(\mathcal{L})_0 \oplus Der(\mathcal{L})_1$$

forms the Lie superalgebra of superderivations of $\mathcal{L}$ with respect to the bracket

$$[d_i, d_j] = d_i \circ d_j - (-1)^{|d_i||d_j|} d_j \circ d_i,$$

where $Der(\mathcal{L})_0$ consists of even superderivations (or just derivation) and $Der(\mathcal{L})_1$ of odd superderivations.

Torus of Lie superalgebras

Now we give the definition of torus of an nilpotent Lie superalgebra, which plays an important role in the structure of solvable Lie superalgebras.

Definition

A torus on a Lie superalgebra $\mathcal{L}$ is a commutative subalgebra of $Der(\mathcal{L})$ consisting of semisimple endomorphisms. A torus is said to be maximal if it is not strictly contained in any other torus. We denote by $\mathcal{T}_{max}$ a maximal torus of a Lie superalgebra $\mathcal{L}$.

Note that since diagonalizability of odd superderivation leads to its nullity, one can assume that in the case of the complex numbers field a torus on a Lie superalgebra $\mathcal{L}$ is a commutative subalgebra of $Der(\mathcal{L})_0$ consisting of diagonalizable endomorphisms. In addition, choosing an appropriate basis of a Lie superalgebra we can bring the mutually commuting diagonalizable derivations into a diagonal form, simultaneously.

Torus of Lie superalgebras

Theorem

Any two maximal tori of a complex nilpotent Lie superalgebra $\mathcal{N}$ are conjugate under an inner automorphism from the radical of $[Der(\mathcal{N})_0, Der(\mathcal{N})_0]$.

From this theorem it follows that the dimensions of any two maximal tori of a nilpotent Lie superalgebra are equal.

Maximal solvable extensions of nilpotent Lie superalgebras

Consider a complex solvable (non-nilpotent) Lie superalgebra $\mathcal{R} = \mathcal{R}_0 \oplus \mathcal{R}_1$ such that $[\mathcal{R}_1, \mathcal{R}_1] \subseteq [\mathcal{R}_0, \mathcal{R}_0]$. This condition ensures fulfillment of an analogues of Lie's theorem for Lie superalgebras and as a consequence, the fact that the square of solvable Lie superalgebra lies in its nilradical. Thus, we have $\mathcal{R} = \mathcal{N} \oplus \mathcal{Q}$, where $\mathcal{N}$ is its nilradical and $\mathcal{Q}$ is a complementary subspace. It is known that $\mathcal{Q} \subseteq \mathcal{R}_0$ and $ad_{x|\mathcal{N}}$ is non-nilpotent operator for any $x \in \mathcal{Q}$.

Definition

An nilpotent Lie superalgebra $\mathcal{N}$ is called d-locally diagonalizable, if restrictions and projections of any non-nilpotent even superderivations of $\mathcal{N}$ on each root subspaces with respect to a certain maximal torus $\mathcal{T}$ are simultaneously diagonalizable.

Thanks to conjugacy of any two maximal tori the definition of d-locally diagonalizability does not depend on the choice of torus.

Theorem

Under the condition that an analogue of Lie's theorem holds there exists a unique (up to isomorphism) complex maximal solvable extension of a d-locally diagonalizable nilpotent Lie superalgebra and it is isomorphic to an algebra of the type $\mathcal{R}_{\mathcal{T}}$.

The essentialness of the property d-locally diagonalizability follows from Lie algebras part. The condition for the fulfillment of an analogue of Lie's theorem is essential, as well. It follows from the following example.

Example

Let $\mathcal{N} = \mathcal{N}_0 \oplus \mathcal{N}_1$ be a two-dimensional nilpotent Lie superalgebra with the following multiplication table $[y, y] = x$ with $\mathcal{N}_0 = Span\{x\}$ and $\mathcal{N}_1 = \{y\}$.

It is clear that $\mathcal{N}$ is d-locally diagonalizable nilpotent Lie superalgebra. The fulfillment of the analogue of Lie's theorem for a maximal extension of $\mathcal{N}$ leads to the unique solvable extension of $\mathcal{N}$, that is,

$$\mathcal{R}_{\mathcal{T}}: \quad \{[y, y] = x, \quad [x, z] = 2x, \quad [y, z] = y.$$

However, if we do not require the fulfillment of an analogue of Lie's theorem, then up to isomorphism there are two non-isomorphic maximal solvable extensions of $\mathcal{N}$:

$$\mathcal{R}_1: \quad \begin{cases} [y,y]=x, & [x,z_1]=2x, \quad [y,z_1]=y, \\ [z_2,z_1]=z_2, & [z_2,z_2]=\alpha x. \end{cases}$$

$$\mathcal{R}_2: \quad \begin{cases} [y,y]=x, & [x,z_1]=2x, \\ [y,z_1]=y, & [z_2,z_1]=\beta z_2, \end{cases}$$

where $\mathcal{Q}=Span\{z_1,z_2\}$ and parameters $\alpha,\beta\in\mathbb{C}$.

Let $\mathcal{N} = \mathcal{N}_0 \oplus \mathcal{N}_1$ be an nilpotent d-diagonalizable Lie superalgebras with $\mathcal{N}_0 = Span\{x_1, \ldots, x_n\}$, $\mathcal{N}_1 = Span\{y_1, \ldots, y_m\}$ and the table of multiplications:

$$\begin{cases} [x_i, x_j] = \sum\limits_{t=1}^{n} a_{i,j}^t x_t, \quad 1 \leq i, j \leq n, \\ [x_i, y_j] = \sum\limits_{p=1}^{m} b_{i,j}^p y_p, \quad 1 \leq i \leq n, \quad 1 \leq j \leq m \\ [y_i, y_j] = \sum\limits_{q=1}^{n} c_{i,j}^q x_q, \quad 1 \leq i, j \leq m. \end{cases}$$

For i, j, t such that $a_{i,j}^t \neq 0$, $b_{i,j}^p \neq 0$ and $c_{i,j}^q \neq 0$ we consider the system of the linear equations

$$S_{x,y}: \quad \begin{cases} \alpha_i + \alpha_j = \alpha_t, \\ \alpha_i + \beta_j = \beta_p, \\ \beta_i + \beta_j = \alpha_q, \end{cases}$$

in the variables $\alpha_1, \ldots, \alpha_n$ as i, j, t run from 1 to n and $\beta_1, \ldots, \beta_m$ as i, j, p run from 1 to m. We denote by $r\{x_1, \ldots, x_n, y_1, \ldots, y_m\}$ the rank of the system $S_{x,y}$.
Setting

$$r\{\mathcal{N}\} = min\ r\{x_1, \ldots, x_n, y_1, \ldots, y_m\}$$

as $\{x_1, \ldots, x_n, y_1, \ldots, y_m\}$ runs over all bases of $\mathcal{N}$. Similar to Lie algebras case for an nilpotent Lie superalgebra $\mathcal{N}$ over an algebraically closed field the equality $dim\mathcal{T} = dim\mathcal{N} - r\{\mathcal{N}\}$ holds true.

Consider the basis

$$\{(\alpha_{1,i},\ldots,\alpha_{n,i},\beta_{1,j},\ldots,\beta_{m,j}) \mid 1 \leq i \leq s,\ 1 \leq j \leq t\}$$

of fundamental solutions of the system $S_{x,y}$. Then the diagonal matrices

$$\{diag(\alpha_{1,i},\ldots,\alpha_{n,i},\beta_{1,j},\ldots,\beta_{m,j}) \mid 1 \leq i \leq s,\ 1 \leq j \leq t\}$$

forms a basis of a maximal torus of $\mathcal{N}$. Thus, having $\mathcal{N}$ and a maximal torus of $\mathcal{T}$ of $\mathcal{N}$ one can assume that superalgebra $\mathcal{R}_{\mathcal{T}} = \mathcal{N} \rtimes \mathcal{T}$ is constructed.

It should be noted that the description (up to isomorphism) of a maximal solvable extension, under the condition that an analogue of Lie's theorem holds true, of a given nilpotent Lie superalgebra $\mathcal{N}$ is carrying out by using an extension of Mubarakzjanov's method to solvable Lie superalgebras, which involve usually long enough computations. However, in the case of $\mathcal{N}$ is d-locally diagonalizable thanks to theorem on maximal extension of nilpoetnt Lie superalgebras we obtain the description can be obtained just by construction of an algebra $\mathcal{R}_{\mathcal{T}}$ (for instance Theorem 4.3 in [25], Theorem 4.1 in [26] and others).

[25] Camacho L.M., Fernandez-Barroso J.M., Navarro R.M., Solvable Lie and Leibniz superalgebras with a given nilradical, Forum Math., 2020

[26] Camacho L.M., Navarro R.M., Omirov B.A., On solvable Lie and Leibniz superalgebras with maximal codimension of nilradical, J. Algebra, 2022

In [27] the description of maximal solvable extensions of $\mathcal{N}$ is obtained.

Theorem

An arbitrary maximal solvable extension of the algebra $\mathcal{N}$ admits a basis such that the non-vanishing Leibniz brackets are become: $\mathcal{R}(a_{i,j}, b_{i,j}, c_{i,j})$:

$$\begin{cases} [e_i, e_1] = e_{i+1}, & 1 \le i \le n-4p-1, \\ [e_1, f_j] = f_{j+2p}, & 1 \le j \le 2p, \\ [e_i, x_j] = \sum\limits_{t=i+1}^{n-4p} a_{t-i+1,j} e_t, & 1 \le t \le n-4p, \ 1 \le j \le 2p, \\ [f_i, x_i] = f_i, & 1 \le i \le 2p, \\ [f_{2p+i}, x_i] = f_{2p+i}, & 1 \le i \le 2p, \\ [x_i, f_i] = -f_i, & 1 \le i \le 2p, \\ [x_i, f_j] = b_{i,j} f_{2p+j}, & 1 \le i \ne j \le 2p, \\ [x_i, x_j] = c_{i,j} e_{n-4p}, & 1 \le i, j \le 2p, \end{cases}$$

[27] Adashev J.Q., Camacho L.M., Omirov B.A., Solvable Leibniz algebras with naturally graded non-Lie p-filiform nilradicals whose maximal complemented space of its nilradical, Linear Multilinear A.,2021

Below the necessary and sufficient conditions of the existence of an isomorphism between two algebras of the family $\mathcal{R}(a_{i,j}, b_{i,j}, c_{i,j})$ are established.

Proposition

Two algebras $\mathcal{R}(a'_{i,j}, b'_{i,j}, c'_{i,j})$ and $\mathcal{R}(a_{i,j}, b_{i,j}, c_{i,j})$ are isomorphic if and only if there exists $A \in \mathbb{C} \setminus \{0\}$ such that

$$\begin{aligned} a'_{i,j} &= \frac{a_{i,j}}{A^{i-1}}, \quad 2 \le i \le n-4p+1,\ 1 \le j \le 2p, \\ b'_{i,j} &= \frac{b_{i,j}}{A}, \quad 1 \le i \ne j \le 2p, \\ c'_{i,j} &= \frac{c_{i,j}}{A^{n-4p}}, \quad 1 \le i,j \le 2p. \end{aligned}$$

From Proposition we conclude that there are infinitely many non-isomorphic maximal extensions of d-locally diagonalizable Leibniz algebra $\mathcal{N}$. Thus, the results on uniqueness of maximal extensions of d-locally diagonalizable nilpotent Lie (super)algebras no longer true for Leibniz algebras, in general. In fact, it depends on the left and right sides actions of a maximal torus of an nilpotent Leibniz algebra.

Further plans

Conjecture

A complex maximal solvable extension of an nilpotent Lie algebra is unique (up to isomorphism) if and only if the nilpotent algebra is locally diagonalizble.

It is a great interest to establish similar results for finite-dimensional Leibniz, n-Lie algebras and special classes of infinite-dimensional Lie algebras (like pro-solvable or residually solvable Lie algebras).

THANK YOU FOR YOUR ATTENTION !