On an Aspect of Second Quantum Revolution

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The 2022 Nobel Prize in Physics has been awarded to Alain Aspect, John Clauser, and Anton Zeilinger for their work in Quantum Theory. Mass media called this event part of second quantum revolution which includes mainly quantum computing and other super-technologies. Here we discuss Alain Aspect's version of Einstein-Podolsky-Rosen thought experiment and show that there exists an internal dependence of the simultaneous measurements made by the two pairs of linear polarizers operated in each leg of the apparatus during this experiment. The corresponding Shannon-Kolmogorov information flow (or, noise?) linking a polarizer from one leg to a polarizer from the other leg is proportional to the absolute value of this function of dependence. It turns out that if Bell's inequality is violated, then the experiment performed at one leg is informationally dependent on the experiment at the other leg.

Postulate 1 (Principle of superposition). Any physical system (electron, hydrogen atom, etc.) can be associated with an unitary space $\mathcal{H}$ with inner product $\langle\phi|\psi\rangle$ which we assume to be linear in the second slot and anti-linear in the first slot. The corresponding norm $\|\psi\| = \langle\psi|\psi\rangle^{\frac{1}{2}}$, $\psi \in \mathcal{H}$, makes $\mathcal{H}$ a Hilbert space. The non-zero vectors ψ are called *state vectors*, or, ψ*-functions* of the quantum system. It is supposed that the ray $\mathbb{C}\psi$ carries the whole information of the state of the system at a fixed moment of time and is represented by a unit (or, *normalized*) vector ψ, $\|\psi\| = 1$. Chronologically, the physicists started with ψ-functions and the statement that if ψ_1 and ψ_2 are ψ-functions of the quantum system, then any linear combination $\lambda_1\psi_1 + \lambda_2\psi_2$, $\lambda_i \in \mathbb{C}$, is also a ψ-function of this system ($\mathbb{C}$-linearity of $\mathcal{H}$).

Brief Description of the Postulates of Quantum Mechanics

Postulate 2 (Observables). Any physical quantity of the system (position, velocity, energy, etc.) is represented by a self-adjoint linear operator A on $\mathcal{H}$ which is said to be *observable* of this system. The spectre $\mathrm{Spec}(A)$ of A (consisting of real numbers) coincides with the set of all values of this physical quantity obtained after measurement: its *eigenvalues*. The corresponding *eigenvectors* are normalized ψ-functions (that is, states of the physical system) ψ_i such that under state ψ_i the physical quantity assumes as value the corresponding eigenvalue $\lambda_i \in \mathrm{Spec}(A)$ with probability 1. We fix an orthonormal coordinate system $\Psi = (\psi_i)_{i=1}^n$ of $\mathcal{H}$, consisting of eigenvectors. Let ψ be the *initial* (that is, the actual) state of the system. In terms of the coordinate system Ψ we have $\psi = \sum_{i=1}^n \langle \psi_i | \psi \rangle \psi_i$ and the square $|\langle \psi_i | \psi \rangle|^2$ is the probability that after the measurement of the quantity we obtain the value λ_i. We define the sample space

$$S(\psi; A) = \begin{pmatrix} 1 & 2 & \dots & n \\ \downarrow & \downarrow & \dots & \downarrow \\ |\langle\psi_1|\psi\rangle|^2 & |\langle\psi_2|\psi\rangle|^2 & \dots & |\langle\psi_n|\psi\rangle|^2 \end{pmatrix}. \quad (1)$$

The image of the sequence $(\lambda_1, \lambda_2, \dots, \lambda_n)$ of eigenvalues of A is $\mathrm{Spec}(A)$ and for any $\lambda \in \mathrm{Spec}(A)$ we denote by $U_\lambda \subset [n]$ its inverse image. The linear operator $P_{U_\lambda}(\psi) = \sum_{i \in U_\lambda} \langle\psi_i|\psi\rangle\psi_i$ is the projection of $\mathcal{H}$ onto the eigenspace $\mathcal{H}_\lambda$ corresponding to λ and $\Pr(U_\lambda) = \|P_{U_\lambda}(\psi)\|^2$. Moreover, we obtain the *spectral decomposition of* A: $A = \sum_{\lambda \in \mathrm{Spec}(A)} \lambda P_{U_\lambda}$.
We consider the operator A as a random variable $A\colon [n] \to \mathbb{R}$ on the sample space $S(\psi; A)$ with probability distribution

Brief Description of the Postulates of Quantum Mechanics

$p_A(\lambda) = \|P_{U_\lambda}(\psi)\|^2$ for $\lambda \in \text{Spec}\,(A)$ and $p_A(\lambda') = 0$ for $\lambda' \notin \text{Spec}(A)$. Then we write $\Pr(A = \lambda) = \|P_{U_\lambda}(\psi)\|^2 = \langle P_{U_\lambda}(\psi)|\psi\rangle = \langle \psi|P_{U_\lambda}(\psi)\rangle$ and identify the event U_λ in the sample space $S(\psi; A)$ with the "event" $A = \lambda$. Furthermore, immediately after the measurement the state of the system is changed and becomes $\psi' = \frac{P_{U_\lambda}(\psi)}{\|P_{U_\lambda}(\psi)\|}$. This phenomenon is said to be *reduction of the wave packet*, or, *wave function collapse*. The commutation $AB = BA$ is equivalent to existence of an orthonormal basis Ψ of $\mathcal{H}$ consisting of eigenvectors of both A and B. Moreover, if $A = \sum_{\lambda \in \text{Spec}(A)} \lambda P_{U_\lambda}$ and $B = \sum_{\mu \in \text{Spec}(B)} \mu P_{V_\lambda}$ are the corresponding spectral decompositions, then $AB = BA$ is equivalent to commutation of the projections: $P_{U_\lambda} \circ P_{V_\mu} = P_{V_\mu} \circ P_{U_\lambda}$.

Brief Description of the Postulates of Quantum Mechanics

Under these two equivalent statements we have $P_{U_\lambda} \circ P_{V_\mu} = P_{V_\mu} \circ P_{U_\lambda} = P_{U_\lambda \cap V_\mu}$ and $AB = BA = \sum_{(\lambda,\mu) \in \mathrm{Spec}(A) \times \mathrm{Spec}(B)} \lambda\mu P_{U_\lambda \cap V_\mu}$. Thus, we can define the "event" $A = \lambda$ and $B = \mu$ by identifying it with the event $U_\lambda \cap V_\mu$ from the sample space $S(\psi; A)$. In particular, we can define without ambiguity the logical operation "and": $A = \lambda$ "and" $B = \mu$ being $(A = \lambda$ and $B = \mu) = (B = \mu$ and $A = \lambda)$. After performing these two measurements A and B, the initial state ψ of the physical system collapses to $\psi'' = \frac{P_{U_\lambda}(P_{V_\mu}(\psi))}{\|P_{U_\lambda}(P_{V_\mu}(\psi))\|} = \frac{P_{U_\lambda \cap V_\mu}(\psi)}{\|P_{U_\lambda \cap V_\mu}(\psi)\|}$, regardless of their order. Two measurements are said to be *simultaneous* if the corresponding self-adjunct operators A and B commute.

Brief Description of the Postulates of Quantum Mechanics

In the case of non-commuting operators A and B one observes the phenomenon of *complementarity* - the idea being introduced by Niels Bohr. In particular, we can not define the logical operation "and": $A = \lambda$ "and" $B = \mu$. Moreover, in general, the collapse of the initial state ψ of the physical system depends on the order of the measurements A and B: $\frac{P_{U_\lambda}(P_{V_\mu}(\psi))}{\|P_{U_\lambda}(P_{V_\mu}(\psi))\|} \neq \frac{P_{V_\mu}(P_{U_\lambda}(\psi))}{\|P_{V_\mu}(P_{U_\lambda}(\psi))\|}$. The *expected value of the random variable A at the state ψ* is defined to be $\mathcal{E}(A)(\psi) = \sum_{\lambda \in \mathrm{Spec}(A)} \lambda \langle \psi | P_{U_\lambda}(\psi) \rangle$ and we have $\mathcal{E}(A)(\psi) = \langle \psi | A(\psi) \rangle$. The *standard deviation of the random variable A at state ψ* is $(\Delta A)(\psi) = \sqrt{\mathcal{E}((A - \mathcal{E}(A)(\psi))^2)(\psi)}$.

Heisenberg Uncertainty Principle

As a consequence of Cauchy-Schwarz inequality one has

$$\Delta A(\psi)\Delta B(\psi) \geq \frac{1}{2}|\langle [A,B](\psi)|\psi\rangle|.$$

Thus, if $[A,B] \neq 0$, then, in general, the standard deviations of the random variables A and B at state ψ can not be simultaneously very small. It is also said that the physical quantities which correspond to A and B at state ψ have no *simultaneous reality*.

Brief Description of the Postulates of Quantum Mechanics and Entanglement

Postulate 3 (Composite systems) If the physical system with state space $\mathcal{H}$ can be decomposed into two sub-systems 1 and 2 with state spaces $\mathcal{H}_1$ and $\mathcal{H}_2$, then $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$.
A state $\psi \in \mathcal{H}$ is said to be *separable* if there exit states $\psi_1 \in \mathcal{H}_1$ and $\psi_2 \in \mathcal{H}_2$ such that $\psi = \psi_1 \otimes \psi_2$. Otherwise ψ is called *entangled state*. If ψ is separable and the corresponding ψ_1 and ψ_2 are normalized, then ψ is normalized and ψ_1 and ψ_2 are unique up to phase factors $e^{\alpha \mathrm{i}}$ and $e^{-\alpha \mathrm{i}}$, respectively. If the state ψ is entangled, then the state ψ does not defines uniquely the states $\psi_j^{(i)}$ in any one of the representations $\psi = \sum_{i=1}^m \psi_1^{(i)} \otimes \psi_2^{(i)}$, $m \geq 2$.This is one of the reasons why entangled states play a key role in quantum information theory. The special entangled state $\psi = \frac{1}{\sqrt{2}}(\psi_1 \otimes \psi_2 - \psi_2 \otimes \psi_1)$ is called *singlet state*.

Einstein-Podolsky-Rosen Thought Experiment

The EPR paper A. Einstein, B. Podolsky, N. Rosen, Can Quantum-Mechanical Description of Physical Reality Be Considered Complete?, Physical Review, 47,(1935), 777-780, begins with specification of the meaning of some words:
"Element of physical reality": It exists if and only if it is determined by experiments and measurements.
"Complete theory": For any element of physical reality there exists a corresponding physical concept (not contradictory mathematical counterpart) in the theory.
"Reality of a physical quantity": If one can predict with certainty its value without disturbing the physical system, then this quantity has a counterpart in reality. Later the last thesis is called *realism*. In other words, the value of the quantity must in some sense already exists as a counterpart in the physical reality before the measurement and is revealed by this measurement.

Einstein-Podolsky-Rosen (EPR) Thought Experiment

Quantum mechanics (QM) is supposed to be a complete theory: its physical reality consists of quantum systems which have their unitary spaces $\mathcal{H}$ of states (synonym: wave functions) furnished with self-adjoined linear operators A representing the measurements, as their mathematical counterparts in QM. The wave functions are supposed to carry the whole information about the system at this state.

In the case of non-commuting operators A and B one observes the phenomenon of *complementarity* - the idea being introduced by Niels Bohr. For example, the operators A and B of momentum and coordinate of a particle having a single degree of freedom do not commute. If ψ is an eigenfunction of A corresponding to eigenvalue λ, that is, if $A\psi = \lambda\psi$, then the momentum has with certainty value λ whenever the particle is in state ψ.

Einstein-Podolsky-Rosen (EPR) Thought Experiment

Thus, the momentum has a counterpart in reality. On the other hand, $B\psi \neq \mu\psi$ for all values $\mu \in \mathbb{R}$, hence the coordinate has not counterpart in reality for this state ψ. If we apply measurement B in order to get the value of the coordinate, the eigenfunction ψ of A collapses into an eigenfunction of B, which, in general, is not an eigenfunction of A and now the momentum has no counterpart in reality. The usual conclusion from this in QM is that when the momentum is known, the coordinate has no physical reality. It is also said that momentum and coordinate have no *simultaneous reality*.

In their thought experiment, EPR consider a composite system $1+2$ consisting of two subsystems 1 and 2 (two particles) which interact from the time $t=0$ to $t=T$ and after which time there is no longer any interaction between the two particles.

Einstein-Podolsky-Rosen (EPR) Thought Experiment

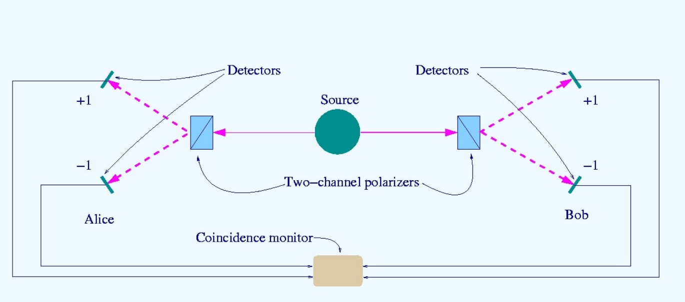
They also suppose that the states of both parts are known before the interaction. The state Ψ of the composite system $1+2$ can be calculated for any $t>0$, and, hence, for any $t>T$ via Schrodinger equation. Because of entanglement, using Ψ one can not calculate the state in which either one of the systems is left after interaction. The only way this can be done is to apply further measurements. Thus, one applies the operator A_1 of momentum on the first particle and Ψ collapses to $u_p \otimes \varphi_p$ where u_p is an eigenfunction of A_1 corresponding to eigenvalue p and φ_p is an eigenfunction of the operator A_2 of momentum of the second particle, corresponding to the eigenvalue $-p$. Thus, the wave function φ_p fixes the reality of the second particle.

Einstein-Podolsky-Rosen (EPR) Thought Experiment

After that one applies the operator B_1 of the coordinate of the first particle and Ψ collapses to $v_x \otimes \psi_x$ where v_x is an eigenfunction of B_1 corresponding to eigenvalue $x_1 = x$ (x_1 is the parameter of particle 1) and ψ_x is and eigenfunction of the operator B_2 of the coordinate of the second particle 2, corresponding to the eigenvalue $x + x_2$ where x_2 is the parameter of particle 2. In particular, the wave function ψ_x fixes the reality of the second particle. Since the operators A_2 and B_2 do not commute, the momentum and the coordinate of the second particle have no simultaneous reality, which is a contradiction because after the interaction the second particle has one reality!

Aspect's Version of EPR experiment

In Aspect optical version of EPR, he considers two photons prepared in the singlet state, which move in opposite directions.



In 1964, Irish physicist John Stewart Bell deduced under the assumptions of "local causality" and "realism" that if measurements are performed *independently* on the two separated particles of an entangled pair, then the assumption that the outcomes depend upon "hidden variables" within each half implies a mathematical constraint on how the outcomes on the two measurements are correlated. This constraint would later be named the *Bell inequality*. Bell then showed that quantum physics predicts correlations that violate this inequality. Consequently, the only way that hidden variables could explain the predictions of quantum physics is if they are either "nonlocal", which is to say that somehow the two particles are able to influence one another instantaneously no matter how widely they ever become separated, or "non-realistic", that is, the values of the quantities do not exist before the measurement.

Aspect's Experiment and the Internal Dependence

An important condition under which a Bell's inequality holds is that the measurement at the one leg of the apparatus does not affect (that is, it is *independent* of) the measurement at the other leg. Alain Aspect witty considers pairs of linear polarizers operated in each leg of the apparatus. Each pair has a time switch which interchanges polarizers, the corresponding time being shorter than the time necessary for a light signal to travel from one of the pairs of polarizers to the other (Einstein locality assumption for independence). Below we show that if Bell's inequality is violated, then the total information flow linking a polarizer from one leg to a polarizer from the other leg is strictly positive, that is, in this case the experiment performed at one leg is informationally dependent on the experiment at the other leg.

The Experiment Corresponding to a Pair of Events

Any pair (A, B) of events in a probability space with probabilities $\Pr(A) = \alpha$, $\Pr(B) = \beta$ produces an experiment in Kolmogorov's terminology (cf. Kolmogorov A. N., Foundations of the Theory of Probability, Chelsea Publishing Company, New Yourk 1956), that is, the partition

$$\mathfrak{I} = (A \cap B) \cup (A \cap B^c) \cup (A^c \cap B) \cup (A^c \cap B^c)$$

of the corresponding sample space.
The probabilities of the results of this experiment are

$$\xi_1^{(A,B)} = \Pr(A \cap B), \xi_2^{(A,B)} = \Pr(A \cap B^c),$$

$$\xi_3^{(A,B)} = \Pr(A^c \cap B), \xi_4^{(A,B)} = \Pr(A^c \cap B^c),$$

and the probability distribution

$$(\xi_1, \xi_2, \xi_3, \xi_4) = (\xi_1^{(A,B)}, \xi_2^{(A,B)}, \xi_3^{(A,B)}, \xi_4^{(A,B)})$$

satisfies the linear system

$$\left| \begin{array}{ccccccccc} \xi_1 & + & \xi_2 & & & & & = & \alpha \\ & & & & \xi_3 & + & \xi_4 & = & 1-\alpha \\ \xi_1 & & & + & \xi_3 & & & = & \beta \\ & & \xi_2 & & & + & \xi_4 & = & 1-\beta. \end{array} \right.$$

The solutions depend on one parameter, say $\theta = \xi_1$:

$$\xi_1 = \theta, \xi_2 = \alpha - \theta, \xi_3 = \beta - \theta, \xi_4 = 1 - \alpha - \beta + \theta.$$

The constraint conditions $0 \leq \xi_k \leq 1$, $k = 1, 2, 3, 4$, are equivalent to the property $\theta \in I(\alpha, \beta)$, where $I(\alpha, \beta) = [\ell(\alpha, \beta), r(\alpha, \beta)]$, $\ell(\alpha, \beta) = \max(0, \alpha + \beta - 1), r(\alpha, \beta) = \min(\alpha, \beta)$.

Boltzmann-Shannon entropy of the probability distribution $(\xi_1(\theta), \xi_2(\theta), \xi_3(\theta), \xi_4(\theta))$ is (cf. C. E. Shannon, A Mathematical Theory of Communication, Bell System Technical Journal 27, No 3 (1948), 379-423, No 4 (1948), 623-656.):

$$E_{\alpha,\beta}(\theta) = -\sum_{k=1}^{4} \xi_k(\theta) \ln(\xi_k(\theta)), \theta \in \mathring{I}(\alpha, \beta).$$

The function $E_{\alpha,\beta}(\theta)$ in θ strictly increases on the interval $[\ell(\alpha,\beta), \alpha\beta]$ and strictly decreases on the interval $[\alpha\beta, r(\alpha,\beta)]$, having a global maximum at $\theta = \alpha\beta$. The continuous function $E_{\alpha,\beta}(\theta)$ in $\theta \in I(\alpha,\beta)$ is said to be the *entropy function* and its value at $\theta = \xi_1^{(A,B)}$, is called *entropy of the events A and B*.

Theorem

Let Ω be a sample space with equally likely outcomes and let $(\alpha, \beta) \in (0,1)^2$.
(i) *If $\theta = \alpha\beta$, then the events A and B are independent.*
(ii) *If $\theta = \ell(\alpha, \beta)$, then either $A \subset B^c$ or $B^c \subset A$.*
(iii) *If $\theta = r(\alpha, \beta)$, then either $A \subset B$ or $B \subset A$.*

The above Theorem and the behaviour of the entropy function motivates the use the entropy as a degree of dependence of the events A and B. We "normalize" the entropy function and obtain a continuous function $e_{\alpha,\beta} \colon I(\alpha, \beta) \to [-1, 1]$:

$$e_{\alpha,\beta}(\theta) = \begin{cases} -\frac{E(\alpha\beta) - E(\theta)}{E(\alpha\beta) - E(\ell(\alpha,\beta))} & \text{if } \ell(\alpha, \beta) \leq \theta \leq \alpha\beta \\ \frac{E(\alpha\beta) - E(\theta)}{E(\alpha\beta) - E(r(\alpha,\beta))} & \text{if } \alpha\beta \leq \theta \leq r(\alpha, \beta), \end{cases}$$

The value of the function $e_{\alpha,\beta}$ at $\theta \in I(\alpha, \beta)$, $\theta = \xi_1^{(A,B)}$, is said to be *degree of dependence of the events A and B*.

The function $e_{\alpha,\beta}$ strictly increases on the interval $I(\alpha,\beta)$ from -1 to 1, with $e_{\alpha,\beta}(\alpha\beta) = 0$. The events A and B are said to be *negatively dependent* if $\xi_1^{(A,B)} < \alpha\beta$ and *positively dependent* if $\xi_1^{(A,B)} > \alpha\beta$. When $\xi_1^{(A,B)} = \alpha\beta$ the events A and B are independent (the entropy is maximal).

The experiment $\mathfrak{J}$ is the joint experiment (see Part I, Section 6, Kolmogorov A. N, Foundations of the Theory of Probability, Chelsea Publishing Company, New Yourk 1956) of two simpler binary experiments: $\mathfrak{A} = A \cup A^c$ and $\mathfrak{B} = B \cup B^c$ with $\Pr(A) = \alpha$, $\Pr(B) = \beta$. The *average quantity of information of one of the experiments $\mathfrak{A}$ and $\mathfrak{B}$, relative to the other* is defined by Shannon-Kolmogorov's formula

$$I(\mathfrak{A},\mathfrak{B}) =$$

$$\xi_1 \ln \frac{\xi_1}{\alpha\beta} + \xi_2 \ln \frac{\xi_2}{\alpha(1-\beta)} + \xi_3 \ln \frac{\xi_3}{(1-\alpha)\beta} + \xi_4 \ln \frac{\xi_4}{(1-\alpha)(1-\beta)},$$

see Gelfand I. M., Kolmogorov A. N., Yaglom A. M, Amount of Information and Entropy for Continuous Distributions. Mathematics and Its Applications, Selected Works of A. N. Kolmogorov, III: Information Theory and the Theory of Algorithms, 33–56, Springer Science+Business Media Dordrecht 1993. In this particular case we have $I(\mathfrak{A},\mathfrak{B})(\theta)=E_{\alpha,\beta}(\alpha\beta)-E_{\alpha,\beta}(\theta)$. Since $E_{\alpha,\beta}(\alpha\beta)-E_{\alpha,\beta}(\ell(\alpha,\beta))=\max_{\ell(\alpha,\beta)\leq\tau\leq\alpha\beta} I(\mathfrak{A},\mathfrak{B})(\tau)$, $E_{\alpha,\beta}(\alpha\beta)-E_{\alpha,\beta}(r(\alpha,\beta))=\max_{\alpha\beta\leq\tau\leq r(\alpha,\beta)} I(\mathfrak{A},\mathfrak{B})(\tau)$, we can write

$$e_{\alpha,\beta}(\theta)=\begin{cases}-\frac{I(\mathfrak{A},\mathfrak{B})(\theta)}{\max_{\ell(\alpha,\beta)\leq\tau\leq\alpha\beta} I(\mathfrak{A},\mathfrak{B})(\tau)} & \text{if } \ell(\alpha,\beta)\leq\theta\leq\alpha\beta\\ \frac{I(\mathfrak{A},\mathfrak{B})(\theta)}{\max_{\alpha\beta\leq\tau\leq r(\alpha,\beta)} I(\mathfrak{A},\mathfrak{B})(\tau)} & \text{if } \alpha\beta\leq\theta\leq r(\alpha,\beta).\end{cases}$$

In particular, the degree of dependence of two events does not depend on the choice of unit of information.

$\mathcal{H}$: 2-dimensional unitary space with inner product $\langle x|y\rangle$ which is linear in the second slot and anti-linear in the first slot;
$\mathbb{I} = \mathbb{I}_{\mathcal{H}}$: the identity linear operator on $\mathcal{H}$;
$\mathcal{H}^{\otimes 2} = \mathcal{H} \otimes \mathcal{H}$: the unitary tensor square with inner product $\langle x_1 \otimes x_2|y_1 \otimes y_2\rangle = \langle x_1|y_1\rangle\langle x_2|y_2\rangle$;
$\mathcal{U}^{(2)}$: the unit sphere in $\mathcal{H}^{\otimes 2}$;
Spec(A): the real spectre of a self-adjoined linear operator A on $\mathcal{H}$ with trace zero, having the form $\text{Spec}(A) = \{\lambda_1^{(A)}, \lambda_2^{(A)}\}$, $\lambda_1^{(A)} + \lambda_2^{(A)} = 0$;
$u^{(A)} = \{u_1^{(A)}, u_2^{(A)}\}$: the orthonormal frame for $\mathcal{H}$, formed by the corresponding eigenvectors of A;
$\mathcal{H}_i^{(A)}$: the eigenspaces $\mathbb{C}u_i^{(A)}$ of A, $i = 1, 2$;

We fix an orthonormal frame $h = \{h_1, h_2\}$ for $\mathcal{H}$ and identify the self-adjoined operators with their matrices with respect to h. For any $\mu \in [0, \pi]$ we denote by A_μ the self-adjoined operator

$$\left(\begin{array}{cc} \cos\mu & \sin\mu \\ \sin\mu & -\cos\mu \end{array} \right).$$

We have $\lambda_1^{(A_\mu)} = 1, \lambda_2^{(A_\mu)} = -1$, and

$$u_1^{(A_\mu)} = (\cos\frac{\mu}{2})h_1 + (\sin\frac{\mu}{2})h_2, u_2^{(A_\mu)} = (-\sin\frac{\mu}{2})h_1 + (\cos\frac{\mu}{2})h_2.$$

For any $\nu \in [0, \pi]$ we set $B_\nu = A_\nu$.
Note that $\{h_1 \otimes h_1, h_1 \otimes h_2, h_2 \otimes h_1, h_2 \otimes h_2\}$ and $u^{(A_\mu)} \otimes u^{(B_\nu)} = \{u_1^{(A_\mu)} \otimes u_1^{(B_\nu)}, u_1^{(A_\mu)} \otimes u_2^{(B_\nu)}, u_2^{(A_\mu)} \otimes u_1^{(B_\nu)}, u_2^{(A_\mu)} \otimes u_2^{(B_\nu)}\}$ are orthonormal frames for $\mathcal{H}^{\otimes 2}$.

Let us set $\mathcal{A}_\mu = A_\mu \otimes \mathbb{I}$, $\mathcal{B}_\nu = \mathbb{I} \otimes B_\nu$. It is a straightforward check that the last two linear operators on $\mathcal{H}^{\otimes 2}$ are also self-adjoined with $\lambda_1^{(\mathcal{A}_\mu)} = \lambda_1^{(\mathcal{B}_\nu)} = 1$, $\lambda_2^{(\mathcal{A}_\mu)} = \lambda_2^{(\mathcal{B}_\nu)} = -1$, the $\lambda_i^{(\mathcal{A}_\mu)}$-eigenspace $\mathcal{H}_i^{(\mathcal{A}_\mu)} = \mathcal{H}_i^{(A_\mu)} \otimes \mathcal{H}$ has orthonormal frame $\{u_i^{(A_\mu)} \otimes u_1^{(B_\nu)}, u_i^{(A_\mu)} \otimes u_2^{(B_\nu)}\}$, and the $\lambda_i^{(\mathcal{B}_\nu)}$-eigenspace $\mathcal{H}_i^{(\mathcal{B}_\nu)} = \mathcal{H} \otimes \mathcal{H}_i^{(B_\nu)}$ has orthonormal frame $\{u_1^{(A_\mu)} \otimes u_i^{(B_\nu)}, u_2^{(A_\mu)} \otimes u_i^{(B_\nu)}\}$, $i = 1, 2$. Since $u^{(A_\mu)} \otimes u^{(B_\nu)}$ is an orthonornal frame of $\mathcal{H}^{\otimes 2}$ consisting of eigenvectors of both $\mathcal{A}_\mu$ and $\mathcal{B}_\nu$, then the last two operators commute.

Let $\psi \in \mathcal{U}^{(2)}$ and let $S(\psi; \mathcal{A}_\mu, \mathcal{B}_\nu)$ be the sample space with set of outcomes $u^{(A_\mu)} \otimes u^{(B_\nu)} = \{u_1^{(A_\mu)} \otimes u_1^{(B_\nu)}, u_1^{(A_\mu)} \otimes u_2^{(B_\nu)}, u_2^{(A_\mu)} \otimes u_1^{(B_\nu)}, u_2^{(A_\mu)} \otimes u_2^{(B_\nu)}\}$ and probability assignment $\{p_{11}, p_{12}, p_{21}, p_{22}\}$ with $p_{ij} = |\langle u_i^{(A_\mu)} \otimes u_j^{(B_\nu)} | \psi\rangle|^2$, $i, j = 1, 2$.

With an abuse of the language, we consider the observable $\mathcal{A}_\mu$ as a random variable $\mathcal{A}_\mu\colon u^{(A_\mu)}\otimes u^{(B_\nu)}\to\mathbb{R}$,
$\mathcal{A}_\mu(u_1^{(A_\mu)}\otimes u_j^{(B_\nu)})=\lambda_1^{(\mathcal{A}_\mu)}$, $\mathcal{A}_\mu(u_2^{(A_\mu)}\otimes u_j^{(B_\nu)})=\lambda_2^{(\mathcal{A}_\mu)}$, $j=1,2$, on the sample space $S(\psi;\mathcal{A}_\mu,\mathcal{B}_\nu)$ with probability distribution
$p_{\mathcal{A}_\mu}(\lambda_i^{(A)})=|\langle u_i^{(A_\mu)}\otimes u_1^{(B_\nu)}|\psi\rangle|^2+|\langle u_i^{(A_\mu)}\otimes u_2^{(B_\nu)}|\psi\rangle|^2$, $i=1,2$, and $p_{\mathcal{A}_\mu}(\lambda)=0$ for $\lambda\notin\operatorname{Spec}(\mathcal{A}_\mu)$.
We also consider the observable $\mathcal{B}_\nu$ as a random variable
$\mathcal{B}_\nu\colon u^{(A_\mu)}\otimes u^{(B_\nu)}\to\mathbb{R}$, $\mathcal{B}_\nu(u_j^{(A_\mu)}\otimes u_1^{(B_\nu)})=\lambda_1^{(\mathcal{B}_\nu)}$,
$\mathcal{B}_\nu(u_j^{(A_\mu)}\otimes u_2^{(B_\nu)})=\lambda_2^{(\mathcal{B}_\nu)}$, $j=1,2$, on the sample space $S(\psi;\mathcal{A}_\mu,\mathcal{B}_\nu)$ with probability distribution
$p_{\mathcal{B}_\nu}(\lambda_i^{(A)})=|\langle u_1^{(A_\mu)}\otimes u_i^{(B_\nu)}|\psi\rangle|^2+|\langle u_2^{(A_\mu)}\otimes u_i^{(B_\nu)}|\psi\rangle|^2$, $i=1,2$, and $p_{\mathcal{B}_\nu}(\lambda)=0$ for $\lambda\notin\operatorname{Spec}(\mathcal{B}_\nu)$.
We identify the event $\{u_i^{(A_\mu)}\otimes u_1^{(B_\nu)},u_i^{(A_\mu)}\otimes u_2^{(B_\nu)}\}$ with the "event" $\mathcal{A}_\mu=\lambda_i^{(\mathcal{A}_\mu)}$ and the event $\{u_1^{(A_\mu)}\otimes u_i^{(B_\nu)},u_2^{(A_\mu)}\otimes u_i^{(B_\nu)}\}$ with the "event" $\mathcal{B}_\nu=\lambda_i^{(\mathcal{B}_\nu)}$.

We also identify the intersection $(\mathcal{A}_\mu = \lambda_i^{(\mathcal{A}_\mu)}) \cap (\mathcal{B}_\nu = \lambda_j^{(\mathcal{B}_\nu)})$ with the event $\{u_i^{(A_\mu)} \otimes u_j^{(B_\nu)}\}$, $i, j = 1, 2$, in the sample space $S(\psi; \mathcal{A}_\mu, \mathcal{B}_\nu)$.

In particular, let us set $\psi = \frac{1}{\sqrt{2}}(h_1 \otimes h_2 - h_2 \otimes h_1)$. Taking into account the form of the eigenvectors of the matrices A_μ and B_ν, we obtain

$$\text{pr}(\mathcal{A}_\mu = \lambda_i^{(\mathcal{A}_\mu)}) = \text{pr}(\mathcal{B}_\nu = \lambda_j^{(\mathcal{B}_\nu)}) = \frac{1}{2}, i, j = 1, 2,$$

$$\text{pr}((\mathcal{A}_\mu = \lambda_1^{(\mathcal{A}_\mu)}) \cap (\mathcal{B}_\nu = \lambda_1^{(\mathcal{B}_\nu)})) = \frac{1}{2} \sin^2\left(\frac{\mu - \nu}{2}\right),$$

$$\text{pr}((\mathcal{A}_\mu = \lambda_1^{(\mathcal{A}_\mu)}) \cap (\mathcal{B}_\nu = \lambda_2^{(\mathcal{B}_\nu)})) = \frac{1}{2} \cos^2\left(\frac{\mu - \nu}{2}\right),$$

$$\text{pr}((\mathcal{A}_\mu = \lambda_2^{(\mathcal{A}_\mu)}) \cap (\mathcal{B}_\nu = \lambda_1^{(\mathcal{B}_\nu)})) = \frac{1}{2} \cos^2\left(\frac{\mu - \nu}{2}\right),$$

$$\text{pr}((\mathcal{A}_\mu = \lambda_2^{(\mathcal{A}_\mu)}) \cap (\mathcal{B}_\nu = \lambda_2^{(\mathcal{B}_\nu)})) = \frac{1}{2} \sin^2\left(\frac{\mu - \nu}{2}\right).$$

The random variable $\mathcal{A}_\mu\mathcal{B}_\nu$ has probability distribution

$$p_{\mathcal{A}_\mu\mathcal{B}_\nu}(1) = \sin^2\left(\frac{\mu-\nu}{2}\right), p_{\mathcal{A}_\mu\mathcal{B}_\nu}(-1) = \cos^2\left(\frac{\mu-\nu}{2}\right),$$

and $p_{\mathcal{A}_\mu\mathcal{B}_\nu}(\lambda) = 0$ for $\lambda \neq \pm 1$. The expected value of this random variable is $\mathcal{E}(\mathcal{A}_\mu\mathcal{B}_\nu) = -\cos(\mu-\nu)$. Let us set $A = (\mathcal{A}_\mu = \lambda_1^{(\mathcal{A}_\mu)})$, $B = (\mathcal{B}_\nu = \lambda_1^{(\mathcal{B}_\nu)})$, so $A^c = (\mathcal{A}_\mu = \lambda_2^{(\mathcal{A}_\mu)})$, $B^c = (\mathcal{B}_\nu = \lambda_2^{(\mathcal{B}_\nu)})$. $\alpha = \mathrm{pr}(A) = \frac{1}{2}$, $\beta = \mathrm{pr}(B) = \frac{1}{2}$. The pair (A, B) of events in the sample space $S(\psi; \mathcal{A}, \mathcal{B})$ with $\alpha = \beta = \frac{1}{2}$ produces an experiment $\mathfrak{J} = (A\cap B) \cup (A \cap B^c) \cup (A^c \cap B) \cup (A^c \cap B^c)$ and the probabilities of its results in this case are

$$\theta = \xi_1 = \mathrm{pr}(A\cap B) = \frac{1}{2}\sin^2\left(\frac{\mu-\nu}{2}\right), \xi_2 = \mathrm{pr}(A\cap B^c) = \frac{1}{2}\cos^2\left(\frac{\mu-\nu}{2}\right)$$

$$\xi_3 = \mathrm{pr}(A^c\cap B) = \frac{1}{2}\cos^2\left(\frac{\mu-\nu}{2}\right), \xi_4 = \mathrm{pr}(A^c\cap B^c) = \frac{1}{2}\sin^2\left(\frac{\mu-\nu}{2}\right). \quad (2)$$

The entropy of the probability distribution $(\xi_1, \xi_2, \xi_3, \xi_4)$ is $E(\theta) = -\sum_{k=1}^{4} \xi_k(\theta) \ln(\xi_k(\theta)) = -2\theta \ln \theta - 2(\frac{1}{2} - \theta) \ln(\frac{1}{2} - \theta)$ and the function $E(\theta)$ can be extended as continuous on the interval $[0, \frac{1}{2}]$. In particular, $\max_{\theta \in [0, \frac{1}{2}]} E(\theta) = E(\frac{1}{4}) = 2 \ln 2$. Since $\min_{\theta \in [0, \frac{1}{4}]} E(\theta) = E(0) = \ln 2 = E(\frac{1}{2}) = \min_{\theta \in [\frac{1}{4}, \frac{1}{2}]} E(\theta)$, we obtain $\min_{\theta \in [0, \frac{1}{2}]} E(\theta) = \ln 2$. Taking into account the values of extrema of entropy function, we obtain

$$e(\theta) = \begin{cases} -2 + \frac{E(\theta)}{\ln 2} & \text{if } 0 \leq \theta \leq \frac{1}{4} \\ 2 - \frac{E(\theta)}{\ln 2} & \text{if } \frac{1}{4} \leq \theta \leq \frac{1}{2} \end{cases}$$

for degree of dependence function $e = e_{\frac{1}{2}, \frac{1}{2}}$. It turns out that the events A and B are *independent* exactly when the entropy is maximal (equal to $2 \ln 2$), that is, when $e(\theta) = 0$. In cases $e(\theta) = -1$ and $e(\theta) = 1$ the entropy is minimal and equal to $\ln 2$.

In the context of the bipartite quantum system that describes Aspect's optical version of EPR experiment (see Aspect A., Dalibard J., Roger G., Experimental Test of Bell's Inequalities Using Time-Varying Analysers, Physical Review Letters, 49 No 25 (1982), 1804-1807), we consider the pairs of linear polarizers operated in each leg of the apparatus as pairs of self-adjoined linear operators A_{μ_i} and $B_{\nu_j} = A_{\nu_j}$, where $\mu_i, \nu_j \in [0, \pi]$, $i, j = 1, 2$, are the angles of the polarizers. Note that each pair has a time switch which interchanges polarizers, the corresponding time being shorter than the time necessary for a light signal to travel from one of the pairs of polarizers to the other (Einstein locality assumption for independence). Each pair of operators A_{μ_1}, A_{μ_2} and B_{ν_1}, B_{ν_2} acts on the state space of the corresponding quantum subsystem (a unitary plane). By tensoring with the unit operator on the other plane, we obtain the above two pairs of self-adjoined linear operators $\mathcal{A}_{\mu_1}$, $\mathcal{A}_{\mu_2}$ and $\mathcal{B}_{\nu_1}$, $\mathcal{B}_{\nu_2}$ with spectre $\{1, -1\}$ on the state space $\mathcal{H}^{\otimes 2}$ of the whole quantum system.

Moreover, for each $i, j = 1, 2$, the operators $\mathcal{A}_{\mu_i}$ and $\mathcal{B}_{\nu_j}$ commute because the state space of the whole quantum system has an orthonormal frame consisting of eigenvectors of both operators. The experiment $\mathfrak{J}$ is the joint experiment of two simple binary trials: $\mathfrak{A}_\mu = A \cup A^c$ and $\mathfrak{B}_\nu = B \cup B^c$ with $\mathrm{pr}(A) = \mathrm{pr}(B) = \frac{1}{2}$. The average quantity of information of one of the experiments $\mathfrak{A}_\mu$ and $\mathfrak{B}_\nu$, relative to the other is defined in this particular case by the formula $I(\mathfrak{A}_\mu, \mathfrak{B}_\nu)(\theta) =$ $\xi_1(\theta) \ln 4\xi_1(\theta) + \xi_2(\theta) \ln 4\xi_2(\theta) + \xi_3(\theta) \ln 4\xi_3(\theta) + \xi_4(\theta) \ln 4\xi_4(\theta)$. The above notation is correct since the interchanges of A and A^c or B and B^c causes permutations of ξ_i's. Thus, we obtain $I(\mathfrak{A}_\mu, \mathfrak{B}_\nu)(\theta) = \max_{\theta \in [0, \frac{1}{2}]} E(\theta) - E(\theta)$. Now, the definition of the degree function $e(\theta)$ yields immediately $I(\mathfrak{A}_\mu, \mathfrak{B}_\nu)(\theta) = |e(\theta)| \ln 2$ for $\theta \in [0, \frac{1}{2}]$. Translating into the language of information theory, we have $e(\theta) = -1$ or $e(\theta) = 1$ if and only if $I(\mathfrak{A}_\mu, \mathfrak{B}_\nu)(\theta) = \max_{0 \leq \tau \leq \frac{1}{2}} I(\mathfrak{A}_\mu, \mathfrak{B}_\nu)(\tau) = \ln 2$.

Finally, we have $e(\theta) = 0$ if and only if $I(\mathfrak{A}_\mu, \mathfrak{B}_\nu)(\theta) = 0$ and under this condition the experiments $\mathfrak{A}_\mu$ and $\mathfrak{B}_\nu$ are said to be *informationally independent*.

Let us set $I^{(s)}(\mathcal{A}_\mu, \mathcal{B}_\nu)(\theta) = e(\theta) \ln 2$ for $\theta \in [0, \frac{1}{2}]$ and call this quantity *average quantity of signed information of one of the events* $\mathcal{A}_\mu = \lambda_1^{(\mathcal{A}_\mu)}$ *and* $\mathcal{B}_\nu = \lambda_1^{(\mathcal{B}_\nu)}$*, relative to the other*. Then $I(\mathfrak{A}_\mu, \mathfrak{B}_\nu) = |I^{(s)}(\mathcal{A}_\mu, \mathcal{B}_\nu)|$ and since the function e is invertible, we obtain $\theta = e^{-1}(\frac{1}{\ln 2} I^{(s)}(\mathcal{A}_\mu, \mathcal{B}_\nu))$. In particular, the value of the signed information flow $I^{(s)}(\mathcal{A}_\mu, \mathcal{B}_\nu)$ reproduces the probability distribution (2) predicted by quantum mechanics.

For any $\mu_1, \mu_2, \nu_1, \nu_2 \in [0, \pi]$ we consider the self-adjoined operators A_{μ_i}, B_{ν_j}, $i, j = 1, 2$, and extend notation: $\theta_{ij} = \frac{1}{2}\sin^2\left(\frac{\mu_i - \nu_j}{2}\right)$, $\theta_{ij} \in [0, \frac{1}{2}]$, $\mathfrak{A}_{\mu_i}$, $\mathfrak{B}_{\nu_j}$, $I(\mathfrak{A}_{\mu_i}, \mathfrak{B}_{\nu_j}) = |e(\theta_{ij})| \ln 2$, $i, j = 1, 2$. The sum $I(\mathfrak{A}, \mathfrak{B}) = \sum_{i,j=1}^{2} I(\mathfrak{A}_{\mu_i}, \mathfrak{B}_{\nu_j})$ is said to be the *average quantity of information of one of the pairs of experiments* $\mathfrak{A} = \{\mathfrak{A}_{\mu_1}, \mathfrak{A}_{\mu_2}\}$ *and* $\mathfrak{B} = \{\mathfrak{B}_{\nu_1}, \mathfrak{B}_{\nu_2}\}$ *relative to the other*, or, *total information flow*, or, *total information noise*. The sum $I^{(s)}(\mathfrak{A}, \mathfrak{B}) = \sum_{i,j=1}^{2} I^{(s)}(\mathcal{A}_{\mu}, \mathcal{B}_{\nu})$ is said to be the *average quantity of signed information of one of the pairs of experiments* $\mathfrak{A} = \{\mathfrak{A}_{\mu_1}, \mathfrak{A}_{\mu_2}\}$ *and* $\mathfrak{B} = \{\mathfrak{B}_{\nu_1}, \mathfrak{B}_{\nu_2}\}$ *relative to the other*, or, *total signed information flow*, or, *total signed information noise*.

Thus, we obtain the functions

$$I(\mathfrak{A},\mathfrak{B})\colon [0,\pi]^4 \to \mathbb{R}, (\mu_1,\mu_2,\nu_1,\nu_2) \mapsto (\ln 2) \sum_{i,j=1}^{2} |e(\theta_{ij})|,$$

and

$$I^{(s)}(\mathfrak{A},\mathfrak{B})\colon [0,\pi]^4 \to \mathbb{R}, (\mu_1,\mu_2,\nu_1,\nu_2) \mapsto (\ln 2) \sum_{i,j=1}^{2} e(\theta_{ij}),$$

which represents the intensity of information flow (respectively, signed information flow) between the pairs of experiments $\mathfrak{A}$ and $\mathfrak{B}$. It turns out that their ranges coincide with the intervals $[0, 4\ln 2]$ and $[-4\ln 2, 4\ln 2]$), respectively.

The equality $|\mathcal{A}_{\mu_1}\mathcal{B}_{\nu_1} + \mathcal{A}_{\mu_1}\mathcal{B}_{\nu_2} + \mathcal{A}_{\mu_2}\mathcal{B}_{\nu_1} - \mathcal{A}_{\mu_2}\mathcal{B}_{\nu_2}| = 2$ yields (with an abuse of the probability theory) Bell's inequality

$$|\mathcal{E}(\mathcal{A}_{\mu_1}\mathcal{B}_{\nu_1}) + \mathcal{E}(\mathcal{A}_{\mu_1}\mathcal{B}_{\nu_2}) + \mathcal{E}(\mathcal{A}_{\mu_2}\mathcal{B}_{\nu_1}) - \mathcal{E}(\mathcal{A}_{\mu_2}\mathcal{B}_{\nu_2}| \leq 2,$$

that is, $|b(\mu_1, \mu_2, \nu_1, \nu_2)| \leq 2$, where $b(\mu_1, \mu_2, \nu_1, \nu_2) = \cos(\mu_1 - \nu_1) + \cos(\mu_1 - \nu_2) + \cos(\mu_2 - \nu_1) - \cos(\mu_2 - \nu_2)$. J. S. Bell in Bell J., On the Einstein Podolski Rosen Paradox, Physics, 1 (1964) 195-200, proves that if there exist "...additional variables which restore to the (quantum) theory causality and locality", then the above inequality is satisfied. The equality $I(\mathfrak{A}, \mathfrak{B}) = 0$ is equivalent to $|\mu_i - \nu_j| = \frac{\pi}{2}$, $i, j = 1, 2$, and this yields $b = 0$. Therefore if Bell's inequality is violated, then the total information flow $I(\mathfrak{A}, \mathfrak{B})$ is strictly positive, that is, the experiments $\mathfrak{A}$ and $\mathfrak{B}$ are informationally dependent.

Examples

Note that the results of all calculations below are rounded up to the 7-th digit.
1) (Aspect's experiment) $\mu_1 = \frac{\pi}{8}, \mu_2 = \frac{3\pi}{8}$, $\nu_1 = \frac{\pi}{4}$, $\nu_2 = 0$. Then we obtain $b(\frac{\pi}{8}, \frac{3\pi}{8}, \frac{\pi}{4}, 0) = 2.3889551$, $I(\mathfrak{A}, \mathfrak{B}) = 0.1615415$ and $I^{(s)}(\mathfrak{A}, \mathfrak{B}) = -0.2330551$.
2) $\mu_1 = \pi, \mu_2 = \frac{2\pi}{3}$, $\nu_1 = 0$, $\nu_2 = \frac{\pi}{3}$. Then we have $b(\pi, \frac{2\pi}{3}, 0, \frac{\pi}{3}) = -2.5$, $I(\mathfrak{A}, \mathfrak{B}) = 1.0855833$ and $I^{(s)}(\mathfrak{A}, \mathfrak{B}) = 1.1887219$.
3) $\mu_1 = \frac{\pi}{2}, \mu_2 = 0$, $\nu_1 = \frac{\pi}{4}$, $\nu_2 = \frac{3\pi}{4}$. Then $b(\frac{\pi}{2}, 0, \frac{\pi}{4}, \frac{3\pi}{4}) = 2\sqrt{2}$, $I(\mathfrak{A}, \mathfrak{B}) = 0.9053727$ and $I^{(s)}(\mathfrak{A}, \mathfrak{B}) = -0.22827767$.

Numerical Examples

Examples

4) $\mu_1 = \pi, \mu_2 = 0,\ \nu_1 = 0,\ \nu_2 = \pi$. Then we have
$b(\pi, 0, 0, -\pi) = 2,\ I(\mathfrak{A}, \mathfrak{B}) = 4 \ln 2 = 2.7725887 = \max I(\mathfrak{A}, \mathfrak{B})$
and $I^{(s)}(\mathfrak{A}, \mathfrak{B}) = 0$.
5) $\mu_1 = \frac{5\pi}{6}, \mu_2 = \frac{2\pi}{3},\ \nu_1 = \frac{\pi}{3},\ \nu_2 = \frac{\pi}{2}$. In this case we have
$b(\frac{5\pi}{6}, \frac{2\pi}{3}, \frac{\pi}{3}, \frac{\pi}{2}) = 1 - \frac{\sqrt{3}}{2},\ I(\mathfrak{A}, \mathfrak{B}) = 0.7089624$ and
$I^{(s)}(\mathfrak{A}, \mathfrak{B}) = -0.267929$.

Examples

6) The link
http://www.math.bas.bg/algebra/valentiniliev/
contains a Java experimental implementation "dependencemeasurements2" depending on five parameters: an non-negative integer n and four real numbers $\mu_1, \mu_2, \nu_1, \nu_2$ from the closed interval $[0, \pi]$. In case $n = 0$ one inputs μ_i, ν_j, $i = 1, 2$, manually in the form $r\pi$, where $0 \leq r \leq 1$ is a rational fraction, and then one obtains b, $I(\mathfrak{A}, \mathfrak{B})$, and $I^{(s)}(\mathfrak{A}, \mathfrak{B})$. In case $n \geq 1$ the parameters μ_i, ν_j are randomly chosen and the n iterations have outputs $b^{(k)}$, $I(\mathfrak{A}, \mathfrak{B})^{(k)}$, and $I^{(s)}(\mathfrak{A}, \mathfrak{B})^{(k)}$, $1 \leq k \leq n$.

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"While we have thus shown that the wave function does not provide a complete description of the physical reality, we left open the question of weather or not such a description exists. We believe, however, that such a theory is possible".
EPR paper

"It is operationally impossible to separate reality and information."
Anton Zeilinger