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Dedicated to Vesselin Drensky on the occasion of his 70th birthday

Margarete Wolf, symmetric polynomials in noncommuting variables and noncommutative invariant theory

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Dedicated to Vesselin Drensky on the occasion of his 70th birthday - our teacher, colleague and friend!

Satetlite Talk to the Webinar
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Every mathematics student knows the Fundamental theorem of symmetric polynomials

Every symmetric polynomial can be expressed in a unique way as a polynomial of the elementary symmetric polynomials.

More precisely: We fix a field K, a set of d variables $X_d = \{x_1, \ldots, x_d\}$ and consider the polynomial algebra $K[X_d] = K[x_1, \ldots, x_d]$. We define an action of the symmetric group S_d on $K[X_d]$ by

$$\sigma : f(x_1, \ldots, x_d) \to f(x_{\sigma(1)}, \ldots, x_{\sigma(d)}), \quad \sigma \in S_d, f \in K[X_d].$$

Then:

(1) *The algebra of symmetric polynomials*

$$K[X_d]^{S_d} = \{f(X_d) \in K[X_d] \mid \sigma(f) = f \text{ for all } \sigma \in S_d\}$$

is generated by

$$e_1 = x_1 + \cdots + x_d = \sum_{i=1}^{d} x_i,$$

$$e_2 = x_1x_2 + x_1x_3 + \cdots + x_{d-1}x_d = \sum_{i<j} x_ix_j,$$

$$\cdots$$

$$e_d = x_1 \cdots x_d;$$

(2) *If* $f \in K[X_d]^{S_d}$*, then there exists a unique polynomial* $p \in K[y_1, \dots, y_d]$ *such that* $f = p(e_1, \dots, e_d)$*. In other words, the elementary symmetric polynomials are algebraically independent.*

Even more well known are the Vieté formulas

If the algebraic equation

$$f(x) = a_0x^d + a_1x^{d-1} + \cdots + a_{d-1}x + a_d = 0$$

has roots $x_1, \ldots, x_d$ *(in some extension of the base field* K*), then*

$$a_i = \frac{(-1)^i}{a_0} e_i(x_1, \ldots, x_d), \quad i = 1, \ldots, d.$$

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    \+ radice 2.

Apotome lateris Polygoni sexaginta quatuor laterum.

Radix binomia 2

\+ Radice binomia { 2
  \+ Radice binomia { 2
    \+ radice binomia { 2
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Et eo continuo progressu.

Sit autem Diameter 1. Circulus 1 N.

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Viète's formula, as printed in Viète's
Variorum de rebus mathematicis responsorum, liber VIII (1593)

The history of symmetric functions of roots of equations

Not so many people know the history of these theorems. Details can be found in
H.G. Funkhouser, A short account of the history of symmetric functions of roots of equations, *Amer. Math. Monthly* 37 (1930), 357-365.

From the paper by Funkhouser

It has been my particular experience to find that new interest and vitality come to a subject when there is linked up with it a knowledge of its origin and the vicissitudes of its progress, and an acquaintance with those renowned and worthy scholars of previous generations who have provided such a noble heritage for the student of today. One approaches his work with a somewhat different conception of it when he realizes that he is sampling a stream that has been flowing steadily for centuries and will continue to flow in like manner after he is gone.

Invariant theory studies the following generalization:

Let $\mathrm{char}(K) = 0$. The group $GL_d(K)$ of $d \times d$ invertible matrices acts canonically from the left on the vector space with basis $X_d = \{x_1, \ldots, x_d\}$. This action is extended diagonally on $K[X_d]$ by the rule

$$g\big(f(x_1, \ldots, x_d)\big) = f\big(g(x_1), \ldots, g(x_d)\big), \quad g \in GL_d(K), f \in K[X_d].$$

Definition

If G is a subgroup of $GL_d(K)$, then **the algebra of G-invariants** is

$$K[X_d]^G = \{f \in K[X_d] \mid g(f) = f \text{ for all } g \in G\}.$$

Remark

Usually one considers another action of $GL_d(K)$ and assumes that $GL_d(K)$ acts on a d-dimensional vector space V_d with basis $\{v_1, \dots, v_d\}$. Then one defines the polynomial functions

$$x_i : V_d \to K, \quad i = 1, \dots, d,$$

by

$$x_i(\xi_1 v_1 + \cdots + \xi_d v_d) = \xi_i, \quad \xi_1, \dots, \xi_d \in K.$$

If $f(X_d) \in K[X_d]$ and $g \in GL_d(K)$, then

$$g(f) : v \to f\big(g^{-1}(v)\big), \quad v \in V_d.$$

Remark

Both ways do not differ essentially. The group $GL_d(K)$ and its opposite $GL_d(K)^{\mathsf{op}}$ (acting on V_d from the right) are isomorphic by

$$GL_d(K) \ni g \to (g^t)^{-1} \in GL_d(K)^{\mathsf{op}},$$

where g^t is the transpose of g, and then the "classical" action of $GL_d(K)$ on the polynomials considered as functions is the same as our "diagonal" action induced by the canonical action from the left of $GL_d(K)$ on the vector space with basis X_d.
For our generalizations it is more convenient to consider our action of $GL_d(K)$.

Problem: Describe $K[X_d]^G$.

(1) *Is the algebra $K[X_d]^G$ finitely generated for all subgroups G of $GL_d(K)$?* This is the main motivation for the 14-th problem of Hilbert from the International Congress of Mathematicians in Paris in 1900.

Answers. G – finite – YES (Emmy Noether);

G-reductive (in some sense "nice") – YES (Although not stated in this generality, the (nonconstructive) proof is contained in the work of Hilbert from 1890–1893);

In the general case – NO (the counterexample of Nagata in the 1950s).

The origins of invariant theory

- can be found in the work of Lagrange in the 1770s and Gauss (in his *Disquititiones Arithmeticae* from 1801) who studied the representation of integers by quadratic binary forms and used the discriminant to distinguish nonequivalent forms.
- But the real invariant theory began in the 1840s with works by George Boole in England and by Otto Hesse in Germany.
- Later, the further development of the theory continued in the work of a pleiad of distinguished mathematicians, among them Cayley, Sylvester, Clebsch, Gordan (known as "König der Invariantentheorie"), and Hilbert.

Problem: Describe $K[X_d]^G$.

(2) If $K[X_d]^G$ is generated by $f_1, \dots, f_m$, then it is a homomorphic image of $K[Y_m]$ ($\pi : K[Y_m] \to K[X_d]^G$ is defined by $\pi(y_j) = f_j$). **Find generators of the ideal $\ker(\pi)$.**

Answers. Explicit sets of generators for different groups G.

Hilbert's Basissatz. *Every ideal of $K[Y_m]$ is finitely generated.* (Nonconstructive proof.)

As other nonconstructive proofs of Hilbert, the proof of Hilbert's Basissatz was accepted sceptically by many of his contemporary mathematicians. Gordan commented the proof as

"Das ist nicht Mathematik, das ist Theologie."

It is not clear if Gordan really said this since the earliest reference to it is 25 years after the events and after his death, and nor is it clear whether the quote was intended as criticism, or praise, or a subtle joke. Gordan himself encouraged Hilbert and used Hilbert's results and methods, and the widespread story that he opposed Hilbert's work on Invariant Theory is a myth.

Chevalley-Shephard-Todd

For G finite $K[X_d]^G \cong K[Y_d]$ if and only if $G < GL_d(K)$ is generated by pseudo-reflections (matrices of finite multiplicative order with all eigenvalues except one equal to 1).

How many are the invariants?

$K[X_d]^G$ is a graded vector space. If

$$R = K[X_d] = R_0 \oplus R_1 \oplus R_2 \oplus \cdots$$

is the natural grading of the polynomial algebra (R_n is the vector space of the homogeneous polynomials of degree n), then

$$K[X_d]^G = R_0^G \oplus R_1^G \oplus R_2^G \oplus \cdots .$$

The **Hilbert** (or **Poincaré**) **series** of the algebra $K[X_d]^G$ is the formal power series

$$H(K[X_d]^G, z) = \sum_{n\geq 0} \dim(R_n^G) z^n.$$

Molien formula

If G is finite, then

$$H(K[X_d]^G, z) = \frac{1}{|G|} \sum_{g \in G} \frac{1}{\det(1 - gz)}.$$

Molien-Weyl integral formula

G – infinite: Replace the sum with an integral!

Hilbert–Serre theorem

If G is reductive, then $H(K[X_d]^G, z)$ is a rational function.

Noncommutative generalizations

Problem. Replace the polynomial algebra $K[X_d]$ with another noncommutative algebra which shares many of the properties of $K[X_d]$.

The most natural candidate is the free associative algebra $K\langle X_d\rangle$ (or the algebra of polynomials in d noncommuting variables). This algebra has the same **universal property** as $K[X_d]$:

If R is a commutative algebra, then every mapping $X_d \to R$ can be extended in a unique way to a homomorphism $K[X_d] \to R$.

If R is an associative algebra, then every mapping $X_d \to R$ can be extended in a unique way to a homomorphism $K\langle X_d\rangle \to R$.

Problem

Describe the symmetric polynomials in $K\langle X_d\rangle$.

Answer

M.C. Wolf, Symmetric functions of non-commutative elements, *Duke Math. J.* 2 (1936), No. 4, 626-637.

Next step

Develop noncommutative invariant theory and study $K\langle X_d\rangle^G$.

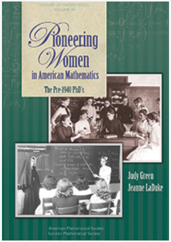
Go further

Study $F(X_d)^G$, where $F(X_d)$ is an algebra with universal property similar to those of $K[X_d]$ and $K\langle X_d\rangle$ (the free Lie algebra $L(X_d)$, the free nonassociative algebra $K\{X_d\}$, the relatively free algebra $F_d(\mathfrak{V})$ of a variety of algebras $\mathfrak{V}$).

Who is Margarete Wolf?

- More than 14 percent of the PhD's in mathematics awarded in the United States during the first four decades of the twentieth century went to women, a proportion not achieved again until the 1980s.
- But there are not so many sisters who defended their Ph.D. Theses in that period. Maybe Louise Adelaide Wolf (Oct. 20, 1898–Nov. 14, 1962) and Margarete Caroline Wolf (Hopkins after her marriage) (Nov. 3, 1911–April 3, 1998) are the only examples in algebra.

Later we follow the book
Judy Green and Jeanne LaDuke, Pioneering Women in American Mathematics: *The Pre-1940 PhD's, History of Mathematics, vol. 34, Providence, RI, American Mathematical Society*, 2008.



Supplementary material containing extended biographies and bibliographical information is available from the companion website for the book:
http://www.ams.org/publications/authors/books/postpub/hmath-34-PioneeringWomen.pdf

The sisters were born in Milwaukee, Wisconsin, in the family of Caroline (1875–1973, born in Germany as Kupperian and immigrated to the United States in 1892) and John Theodore Wolf (1872–1953, conductor on the street railroad and later a truck gardener).

In 1928-1929 Margarete Wolf entered the Milwaukee Extension Division of the University of Wisconsin and then went to Madison to continue her study at the University.

The acting chair Mark Hoyt Ingraham who later became the advisor of the Ph.D. Thesis of Margarete Wolf wrote in his recommendation for a scholarship:
"She has now been at the University for five semesters and in all her courses has received an "A" – this in spite of the fact that she has uniformly elected as hard a program as she could; has at times taken correspondence courses during vacations for the fun of it; has entered classes where she was warned that she did not have the prerequisite and then proceeded to lead the class."

In 1934–35 Margarete Wolf was a research assistant at the University of Wisconsin–Madison.

- Margarete Wolf was a Ph.D. student of Mark Hoyt Ingraham (1896–1982).
- Her advisor received his undergraduate degree from Cornell University, graduating early to serve with the American Expeditionary Forces in Europe during World War I.
- After his military service, he received his M.A. in mathematics from the University of Wisconsin in 1922 and his Ph.D. from the University of Chicago in 1924.
- He was appointed as professor of mathematics at the University of Wisconsin–Madison in 1927 and remained there as a faculty member until his retirement in 1966.
- In the 1930s, Ingraham established the first academic computing center in the United States on the University of Wisconsin campus.

In 1935 Margarete Wolf defended her Ph.D. Thesis

Symmetric Functions of Matrices.

Later the main results of the thesis were published in
M.C. Wolf, Symmetric functions of non-commutative elements, *Duke Math. J.* 2 (1936), 626-637.

Margarete Wolf published three more papers:

M.H. Ingraham, M.C. Wolf, Relative linear sets and similarity of matrices whose elements belong to a division algebra, *Trans. Amer. Math. Soc.* 42 (1937),16-31.

M.C. Wolf, Transformation of bases for relative linear sets, *Bull. Amer. Math. Soc.* 44 (1938), 71-718.

M. H. Ingraham, M.C. Wolf, Convergence of a sequence of linear transformations, *Amer. J. Math.* 60 (1938), 107-19.

- Louise Adelaide, the elder sister of Margarete Wolf finished the Milwaukee–Downer College (a women's college in Milwaukee, Wisconsin) in 1916.
- The next years she had several positions working in a dental office and in a public library in Milwaukee, teaching two years in a district school in Nevada, and working two years in Florida.
- In 1928 Louise Adelaide joined her younger sister first at the Milwaukee Extension Division of the University of Wisconsin and then at the Madison campus of the University of Wisconsin.

Louise Wolf was a senior mathematics major with a straight "A" record in math when she was appointed as an assistant. Usually the University would not appoint a senior to such a position but in the request to the dean it was written *"Miss Wolf is an exceedingly capable woman, over 30 years of age, who has had experience in teaching and whose university education has been delayed by the fact that she is helping educate a younger sister"*.

In 1935 she defended her Ph.D. Thesis
Similarity of Matrices in Which the Elements are Real Quaternions
as a student again of Mark Ingraham. The results of the thesis were published in
L.A. Wolf, Similarity of matrices in which the elements are real quaternions, *Bull. Am. Math. Soc.* 42 (1936), 737-743.

Ingraham, as the department chairman, wrote to E.J. Moulton at Northwestern on August 8, 1935, *"By the way, we have just gone through the examinations and completed signing the theses of the two Wolf sisters – Margarete and Louise. They are among the best women doctors I have known. If you know of any positions for corking good people, both mathematicians and teachers, who are at the same time women, please give me a tip"*.

Louise Wolf immediately took a position with the University of Wisconsin's Extension Division later becoming a part of the University of Wisconsin–Milwaukee. She worked there until her retirement in 1961.

- In 1938 Margarete Wolf moved to Wayne University in Detroit, where she was an instructor until 1941.
- In August 1941 she married Edward John Hopkins (1908–1985), an electronics engineer with the US Navy in Brooklyn, where couple made their home.
- They had two children: Edward John (who earned a bachelor's degree in chemistry, a master's degree in science education, and a doctorate in meteorology) and Margaret Louise (who received a bachelor's degree in physics and a master's degree in meteorology).

- When she went to New York, Margarete Wolf Hopkins taught in the evening and extension sessions of Hunter College during the 1942–43 and 1943–44 academic years.
- She returned to her mathematics career in 1958, when she joined the faculty at St. Joseph's College for Women in Brooklyn where she was consecutively an assistant professor, an associate professor, and professor and chairman of the mathematics department.
- After the death of her husband in 1985 Margarete Hopkins moved to Madison and lived with her daughter's family. She and Elizabeth Hirschfelder attended the UW-Madison Mathematics PhD Centennial Conference held in Madison in May 1997; they were then eighty-five and ninety-five, respectively.

Back to mathematics – the main results of Margarete Wolf

Let K be any field and $X_d = \{x_1, \ldots, x_d\}$, $d > 1$. The free associative algebra $K\langle X_d\rangle = K\langle x_1, \ldots, x_d\rangle$ has a basis consisting of the set $\langle X_d\rangle$ of all monomials $x_{i_1} \cdots x_{i_n}$ in the noncommutative variables X_d. The symmetric group S_d acts on $K\langle X_d\rangle$ as in the commutative case:

$$\sigma : f(x_1, \ldots, x_d) \to f(x_{\sigma(1)}, \ldots, x_{\sigma(d)}), \quad \sigma \in S_d, f \in K\langle X_d\rangle,$$

and f is symmetric if $\sigma(f) = f$.

If $f(x_1,\ldots,x_d) \in K\langle X_d\rangle$, then we shall use the standard notation

$$\sum f = \sum_{\sigma \in S_d} \sigma(f)$$

for the corresponding symmetric polynomial.

The symmetric group S_d acts on the set of monomials $\langle X_d\rangle$ and splits it in orbits. If we choose one monomial u from each orbit, then $K\langle X_d\rangle^{S_d}$ has a basis consisting of all such $\sum u$.

Now we shall present some of the main results in the paper by Margarete Wolf on symmetric polynomials.

> This paper shows that as the number of elements and the degree are increased, an infinite sequence of symmetric polynomials, consisting of a finite set of one or more for each degree can be chosen so that every symmetric polynomial may be expressed uniquely in terms of the polynomials of this sequence and the coefficients of the original polynomial, with coefficients which are integral. This sequence may be chosen in more than one way but the number for each degree is unique.

From the introduction of the paper

Translation in our language

Theorem. *The algebra of symmetric polynomials $K\langle X_d\rangle^{S_d}$, $d>1$, is a free associative algebra. It has a homogeneous system of free generators $\{f_j \mid j\in J\}$ such that for any $n\geq 1$ there is at least one generator of degree n. The number of homogeneous polynomials of degree n is the same in every homogeneous free generating system. If $f\in K\langle X_d\rangle^{S_d}$ has the presentation*

$$f=\sum_{p=(p_1,\ldots,p_m)}\alpha_p f_{p_1}\cdots f_{p_m},\quad \alpha_p\in K,$$

then the coefficients α_p are linear combination with integer coefficients of the coefficients of $f(x_1,\ldots,x_d)$.

Sketch of a modification of the original proof.

We order the monomials $u \in \langle X_d \rangle$ first by degree and then lexicographically assuming that $x_1 > \cdots > x_d$. Then the leading monomial $u = x_{i_1} \cdots x_{i_n}$ of $\sum v$, $v \in \langle X_d \rangle$, $\deg(v) = n > 0$, has the following properties:

(i) $i_1 = 1$;

(ii) If $u = x_{i_1} \cdots x_{i_k} x_{i_{k+1}} w$, $v \in \langle X_d \rangle$, $v = x_{i_1} \cdots x_{i_k}$ depends essentially on all $x_1, \ldots, x_p$ and $i_{k+1} \neq 1, \ldots, p$, then $i_{k+1} = p + 1$.

(iii) Every monomial $u \in \langle X_d \rangle$ satisfying (i) and (ii) is the leading monomial of $\sum u$.

The leading monomials u of the symmetric polynomials $\sum v$, $v \in \langle X_d \rangle$ for small degree n are the following:

$n = 1$: x_1;

$n = 2$: $x_1x_1, \quad x_1x_2$;

$n = 3$: $x_1x_1x_1, \quad x_1x_1x_2, \quad x_1x_2x_1, \quad x_1x_2x_2, \quad x_1x_2x_3$;

$n = 4$:

$$\begin{array}{lllll} x_1x_1x_1x_1, & x_1x_1x_1x_2, & x_1x_1x_2x_1, & x_1x_1x_2x_2, & x_1x_2x_1x_3 \\ x_1x_2x_1x_1, & x_1x_2x_1x_2, & x_1x_2x_1x_3, & x_1x_2x_2x_1, & x_1x_2x_2x_2 \\ x_1x_2x_2x_3, & x_1x_2x_3x_1, & x_1x_2x_3x_2, & x_1x_2x_3x_3, & x_1x_2x_3x_4 \end{array}$$

Table of the Number of Simple Symmetric Polynomials for the Degrees 1, 2, · · ·, 8

Degree

Number of distinct elements in a term	1	2	3	4	5	6	7	8
1	1	1	1	1	1	1	1	1
2		1	3	7	15	31	63	127
3			1	6	25	90	301	966
4				1	10	65	350	1701
5					1	15	140	1050
6						1	21	266
7							1	28
8								1
Total	1	2	5	15	52	203	877	4140

We shall denote the leading monomial of $f \in K\langle X_d\rangle$, $f \neq 0$, by $\overline{f}$. Obviously for the leading monomial $\overline{f_1 f_2}$ of the product of two nonzero polynomials f_1 and f_2 in $K\langle X_d\rangle$ is equal to the product of their leading monomials $\overline{f_1}$ and $\overline{f_2}$.

We shall apply induction on the leading monomials in the basis of the vector space $K\langle X_d\rangle^{S_d}$ constructed in the previous slides. The basis of the induction is $\sum x_1$ and we add it to the generating set of the algebra $K\langle X_d\rangle^{S_d}$ which we shall construct.

If the leading monomial of a symmetric polynomial f is of the form $\overline{f} = vx_1$, $\deg(v) > 0$, then v is the leading monomial of $\sum v$ and $\overline{f} = \left(\overline{\sum v}\right)\left(\overline{\sum x_1}\right)$. Hence the leading monomial $\overline{h}$ of the symmetric polynomial $h = \sum v \sum x_1$ is smaller than $\overline{f}$ and by the inductive assumption h can be expressed as a polynomial of the already constructed polynomials in the generating set of $K\langle X_d\rangle^{S_d}$.

Similarly, if the leading monomial of a symmetric polynomial f is of the form $\overline{f} = v_1x_1x_2v_2$, $\deg(v_1) > 0$, then v_1 is the leading monomial of $\sum v_1$ and $x_1x_2v_2$ is the leading monomial of $\sum x_1x_2v_2$. Hence $\overline{f} = \left(\overline{\sum v_1}\right)\left(\overline{\sum x_1x_2v_2}\right)$ and again we apply the inductive assumption.

It is easy to see that the leading monomial u of a symmetric polynomial f, $\deg(u) > 1$, cannot be presented as a product of two leading monomials of symmetric polynomials of lower degree if u is neither of the form $u = vx_1$ nor of the form $u = v_1x_1x_2v_2$, $\deg(v_1) > 0$. Then we add the symmetric polynomial $\sum u$ to the generating system of $K\langle X_d\rangle^{S_d}$.

The polynomials of the constructed generating system of $K\langle X_d\rangle^{S_d}$ are free generators of $K\langle X_d\rangle^{S_d}$ because the leading monomial of every $f \in K\langle X_d\rangle^{S_d}$ can be presented in a unique way as a product of the leading monomials of the system.

$H_1 = \Sigma x_1,$

$H_2 = \Sigma x_1 x_2,$

$H_3^{(1)} = \Sigma x_1 x_2^2, \qquad H_3^{(2)} = \Sigma x_1 x_2 x_3,$

$H_4^{(1)} = \Sigma x_1 x_2 x_1 x_3, \qquad H_4^{(2)} = \Sigma x_1 x_2^3, \qquad H_4^{(3)} = \Sigma x_1 x_2^2 x_3,$

$H_4^{(4)} = \Sigma x_1 x_2 x_3 x_2, \qquad H_4^{(5)} = \Sigma x_1 x_2 x_3^2, \qquad H_4^{(6)} = \Sigma x_1 x_2 x_3 x_4.$

TABLE FOR THE NUMBER OF $E_m^{(j)}$ FOR $m = 1, 2, \cdots, 6$

Degree

Number of distinct elements in a term

	1	2	3	4	5	6
1	1					
2		1	1	1	1	1
3			1	4	12	33
4				1	8	44
5					1	13
6						1
Total	1	1	2	6	22	92

Symmetric polynomials in two variables

THEOREM. *The simple symmetric polynomials of degree m which involve two elements x_1, x_2 require for representation as polynomials in a fundamental set of order m one and only one fundamental polynomial of every degree* $1, 2, \cdots, m$.

First proof. It follows immediately from the proof of the general case that $K\langle X_2\rangle^{S_2}$ *is freely generated by the symmetric polynomials*

$$\sum x_1 x_2^{n-1} = x_1 x_2^{n-1} + x_2 x_1^{n-1}, \quad n \geq 1.$$

Second proof. Let $\mathrm{char}(K) \neq 2$. We change linearly the free generators of $K\langle X_2\rangle$ by

$$y_1 = \frac{1}{2}(x_1 + x_2), y_2 = \frac{1}{2}(x_1 - x_2).$$

Then $\sigma = (12) \in S_2$ acts on X_2 and $Y_2 = \{y_1, y_2\}$ by

$$x_1 \to x_2, x_2 \to x_1 \text{ and } y_1 \to y_1, y_2 \to -y_2.$$

Hence $K\langle Y_2\rangle^{S_2} = K\langle X_2\rangle^{S_2}$ is spanned by the monomials $u \in \langle Y_2\rangle$ which are of even degree in y_2. Such monomials are written as

$$u = y_1^{n_0}(y_2 y_1^{n_1} y_2)\cdots(y_2 y_1^{n_k} y_2) y_1^{n_{k+1}}$$

and $K\langle Y_2\rangle^{S_2}$ is freely generated by

$$y_1, \quad y_2 y_1^n y_2, \quad n \geq 0.$$

The original proof of Margarete Wolf

Assume that all simple symmetric polynomials through degree $m-1$ can be expressed as polynomials in a set of fundamental polynomials $F_1, F_2, \cdots, F_{m-1}$, where there is one and only one F_k of each weight $k \leqq m-1$. There is a totality of 2^{m-1} simple symmetric polynomials of degree m involving two letters, since m positions can be filled by two letters in 2^m ways, but those two permutations belong to one simple symmetric polynomial which differ only in the interchange of x_1 and x_2. From the above assumption the total number of products of the F_k of weight m is $2^{m-1}-1$ because the total number of compositions[1] of m from $1, 2, \cdots, m$ is 2^{m-1}, and if m is excluded, the number is $2^{m-1}-1$. The representation of the simple symmetric polynomials of degree m, the $S_m^{(j)}(x_n)$,

[1] Macmahon, *Combinatorial Analysis*, vol. I, p. 151.

as polynomials in the F_k must be unique; hence the F_k must be so chosen that a polynomial $P_1(F_1, F_2, \cdots, F_m)$ can equal another polynomial

$$P_2(F_1, F_2, \cdots, F_m),$$

if and only if the coefficients of like terms are equal. Consider the products of the F_k of weight m as $2^{m-1} - 1$ equations in the 2^{m-1} unknowns $S_m^{(j)}(x_n)$. It has been proved that every $S_m^{(j)}(x_n)$ is present in at least one product, namely, $(\Sigma x_1)^m$, and it has been proved that a product of simple symmetric polynomials, and therefore a product of the F_k, is a sum of simple symmetric polynomials with coefficients positive unity. The rank of the matrix of the coefficients of the $S_m^{(j)}(x_n)$ is equal to $2^{m-1} - 1$, which is the number of equations. That is, the rank is equal to the number of products of the F_k which are of weight m, because there is no polynomial relationship between the products. Since there are 2^{m-1} unknown $S_m^{(j)}(x_n)$, it is necessary that one $S_m^{(j)}(x_n)$ be assigned arbitrarily so that one can solve for the other $S_m^{(j)}(x_n)$ in terms of the products of the F_k, $k = 1, 2, \cdots, m-1$, of weight m and the fixed $S_m^{(j)}(x_n)$. Let this $S_m^{(j)}(x_n) = F_m$.

Translation of the original proof

By the previous theorem we already know that $K\langle X_2\rangle^{S_2}$ is a free associative algebra which has a homogeneous free generating system. Let the $K\langle X_2\rangle^{(n)}$ be the vector space of the homogeneous polynomials of degree n in $K\langle X_2\rangle$. We shall prove the theorem by induction. Let $n \geq 1$. Assume that all symmetric polynomials of degree $\leq n-1$ can be expressed as polynomials in a set of symmetric polynomials $f_1, \ldots, f_{n-1}$, $\deg(f_k) = k$, $k = 1, \ldots, n-1$.

The vector space $K\langle X_2\rangle^{(n)}$ of the homogeneous polynomials of degree n is equal to 2^n because there are 2^n monomials $x_{i_1} \cdots x_{i_n}$, $i_k = 1, 2$. Every monomial $u(x_1, x_2) \in K\langle X_2\rangle^{(n)}$ is different from the monomial $u(x_2, x_1)$ and the 2^{n-1} polynomials $u(x_1, x_2) + u_2(x_2, x_1)$ form a basis of the vector space of symmetric polynomials of degree n. Hence $\dim(K\langle X_2\rangle^{(n)})^{S_2} = 2^{n-1}$.

Let $f_{k_1} f_{k_2} \cdots f_{k_p} = \sum \alpha_j x_{j_1} \cdots x_{j_n}$ be a product of degree n. There is a 1-1 correspondence between such products and the $(p-1)$-tuples $(k_1 + 1, k_1 + k_2 + 1, \ldots, k_1 + \cdots + k_{p-1} + 1)$. The $((p-1)$-tuple indicates the positions in the monomials $x_{j_1} \cdots x_{j_n}$ where the monomials in $f_2, \ldots, f_p$ start, respectively.
For example, the product

$$f_2 f_4 f_1 = \left(\sum \alpha x_{i_a} x_{a_2}\right) \left(\sum \beta x_{b_1} x_{b_2} x_{b_3} x_{b_4}\right) \left(\sum \gamma x_c\right)$$

corresponds to $(3, 7)$.

There are $\binom{n-1}{p-1}$ possibilities to choose $f_{k_1}\cdots f_{k_p}$. Hence all possibilities are

$$\sum_{p=2}^{n}\binom{n-1}{p-1}=2^{n-1}-1.$$

The products $f_{k_1}\cdots f_{k_p}$ of degree n are linearly independent and span a vector subspace of codimension 1 of $(K\langle X_2\rangle^{(n)})^{S_2}$. Hence we need one more symmetric polynomial f_n of degree n to express all homogeneous symmetric polynomials of degree n.

What happened with noncommutative symmetric polynomials after Margarete Wolf?

Symmetric functions in commuting variables are studied from different points of view. The same have happened in the noncommutative case. In her paper Margarete Wolf studied the algebraic properties of $K\langle X_d\rangle^{S_d}$.

The next result in this direction appeared more than 30 years later in **G.M. Bergman, P.M. Cohn**, Symmetric elements in free powers of rings, *J. Lond. Math. Soc., II. Ser. 1 (1969), 525-534* where the authors generalized the main result of Wolf.

There is an enourmous literature devoted to different aspects in the theory. We shall mention few papers and one book only.

- **I.M. Gelfand, D. Krob, A. Lascoux, B. Leclerc, V.S. Retakh, J.-Y. Thibon**, Noncommutative symmetric functions, *Adv. Math.* 112 (1995), No. 2, 218-348.
- **S. Fomin and C. Greene**, Noncommutative Schur functions and their applications, *Discrete Math.* 193 (1998), 179-200.
- **M.H. Rosas, B.E. Sagan**, Symmetric functions in noncommuting variables, *Trans. Am. Math. Soc.* 358 (2006), No. 1, 215-232.

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- **D.S. Kaliuzhnyi-Verbovetskyi, V. Vinnikov**, Foundations of Free Noncommutative Function Theory, Mathematical Surveys and Monographs, vol. 199, Providence, RI, American Mathematical Society, 2014.

Noncommutative invariant theory

In what follows we assume that the field K is of charactersitic 0. As in the commutative case we assume that the general linear group $GL_d(K)$ acts on the vector space with basis X_d and extend this action diagonally on $K\langle X_d\rangle$ by the rule

$$g(f(x_1,\ldots,x_d))=f(g(x_1),\ldots,g(x_d)),\quad g\in GL_d(K),f\in K\langle X_d\rangle.$$

If G is a subgroup of $GL_d(K)$, then **the algebra of G-invariants** is

$$K\langle X_d\rangle^G=\{f\in K\langle X_d\rangle\mid g(f)=f\text{ for all }g\in G\}.$$

Similarity and differences between commutative and noncommutative invariant theory

The first natural question to answer is:

Which results in commutative invariant theory hold also in the noncommutative case?

Which results are not true?

The problem for finite generation

The group $G \subset GL_d(K)$ acts on the vector space with basis X_d by **scalar multiplication** if G consists of scalar matrices.

If G is finite and acts by scalar multiplication, then G is cyclic. If $|G| = q$ then $K\langle X_d\rangle^G$ is generated by all monomials of degree q. The number of such monomials is equal to d^q and hence the algebra $K\langle X_d\rangle^G$ is isomorphic to the free algebra $K\langle Y_{d^q}\rangle$.

It has turned out that the analogue of the theorem of **Emmy Noether** for the finite generation of $K[X_d]^G$ for finite groups G holds for $K\langle X_d\rangle^G$ in this very special case only.

Theorem. (Koryukin, Dicks and Formanek, Kharchenko)

Let G be a finite subgroup of $GL_d(K)$. Then $K\langle X_d\rangle^G$ is finitely generated if and only if G acts on the vector space with basis X_d by scalar multiplication.

W. Dicks, E. Formanek, Poincaré series and a problem of S. Montgomery, *Lin. Multilin. Algebra* 12 (1982), 21-30.
V.K. Kharchenko, Noncommutative invariants of finite groups and Noetherian varieties, *J. Pure Appl. Algebra* 31 (1984), 83-90.

Finite generation with additional action

Theorem. (Koryukin)

Let the symmetric group S_n of degree n, $n = 1, 2, \ldots$, act from the right on the homogeneous elements of degree n in $K\langle X_d\rangle$ by the rule

$$(x_{i_1} \cdots x_{i_n})^{\sigma^{-1}} = x_{i_{\sigma(1)}} \cdots x_{i_{\sigma(n)}}, \quad \sigma \in S_n.$$

We equip the algebra $K\langle X_d\rangle$ with this additional action. If $\mathrm{char}(K) = 0$ and G is a finite subgroup of $GL_d(K)$, then the algebra $K\langle X_d\rangle^G$ with this additional action is finitely generated.

A.N. Koryukin, Noncommutative invariants of reductive groups (Russian), *Algebra i Logika* 23 (1984), No. 4, 419-429. Translation: Algebra Logic 23 (1984), 290-296.

What happens with the Chevalley-Shephard-Todd theorem

Theorem. (Lane, Kharchenko)

Let G be a finite subgroup of $GL_d(K)$. Then the algebra of noncommutative G-invariants $K\langle X_d\rangle^G$ is free.

D.R. Lane, Free Algebras of Rank Two and Their Automorphisms, *Ph.D. Thesis, Bedford College, London*, 1976.

V.K. Kharchenko, Algebra of invariants of free algebras (Russian), *Algebra i Logika* 17 (1978), 478-487. Translation: Algebra and Logic 17 (1978), 316-321.

Galois theory for free algebras

Theorem. (Kharchenko)

The map $H \longrightarrow K\langle X_d\rangle^H$ *gives a 1-1 correspondence between the subgroups of the finite group* $G \subset GL_d(K)$ *and the free subalgebras of* $K\langle X_d\rangle$ *containing* $K\langle X_d\rangle^G$*.*

V.K. Kharchenko, Algebra of invariants of free algebras (Russian), *Algebra i Logika* 17 (1978), 478-487. Translation: Algebra and Logic 17 (1978), 316-321.

The analogue of the Molien formula

Theorem. (Dicks and Formanek)

If $G \subset GL_d(K)$ is a finite group, then

$$H(K\langle X_d\rangle^G, z) = \frac{1}{|G|}\sum_{g\in G}\frac{1}{1-\mathrm{tr}(g)z}.$$

W. Dicks, E. Formanek, Poincaré series and a problem of S. Montgomery, *Lin. Multilin. Algebra* 12 (1982), 21-30.

How many are the free generators of a free algebra?

Lemma.

The free associative algebra $K\langle X_d\rangle$ *is* $\mathbb{N}_0^d$*-graded with respect to the grading counting the degree of each variable in the monomials. Then the corresponding Hilbert series is*

$$H(K\langle X_d\rangle, z_1, \ldots, z_d) = \frac{1}{1-(z_1+\cdots+z_d)}.$$

Corollary.

Let $Y = Y_1 \cup Y_2 \cup \cdots$ *be a set of variables and let the elements of* Y_n *are assumed to be of degree* n*. This induces a grading on* $K\langle Y\rangle$*. The Hilbert series of* $K\langle Y\rangle$ *is*

$$H(K\langle Y\rangle, z) = \frac{1}{1 - g(z)},$$

where

$$g(z) = \sum_{n\geq 1} |Y_n| z^n$$

is the generating function of the graded set Y*.*

Corollary.

If G is a finite group, then the generating function of the free generators of $K\langle X_d\rangle^G$ is

$$g(z) = 1 - \frac{1}{H(K\langle X_d\rangle^G, z)}.$$

One more proof for the number of free generators of $K\langle X_2\rangle^{S_2}$

Let $\text{char}(K) = 0$. We embed $S_2 = \{\text{id}, (12)\}$ into $GL_2(K)$:

$$\text{id} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad (12) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

and obtain for the Hilbert series of $K\langle X_2\rangle^{S_2}$

$$H(K\langle X_2\rangle^{S_2}, z) = \frac{1}{2}\left(\frac{1}{1 - \text{tr}(\text{id})z} + \frac{1}{1 - \text{tr}(12)z}\right)$$

$$= \frac{1}{2}\left(\frac{1}{1-2z} + 1\right) = \frac{1-z}{1-2z}.$$

Hence the generating function of the free generators of $K\langle X_2\rangle^{S_2}$ is

$$g(z) = 1 - \frac{1}{H(K\langle X_2\rangle^{S_2}, z)} = 1 - \frac{1-2z}{1-z} = \frac{z}{1-z} = z + z^2 + \cdots .$$

Hence the free algebra $K\langle X_2\rangle^{S_2}$ has one generator of degree n for every $n \geq 1$.

Is there an analogue of the Hilbert–Serre theorem?

By the Dicks–Formanek formula $H(K\langle X_d\rangle^G, z)$ is a rational function for G finite. But in the general case of G reductive the Hilbert series $H(K\langle X_d\rangle^G, z)$ may be not rational. The simplest example is

$$H(K\langle X_2\rangle^{SL_2(K)}, z) = \frac{1-\sqrt{1-4z^2}}{2z^2}.$$

G. Almkvist, W. Dicks, E. Formanek, Hilbert series of fixed free algebras and noncommutative classical invariant theory, *J. Algebra* 93 (1985), 189-214.

Remark. The algebra $K\langle X_2\rangle^{SL_2(K)}$ coincides with the subalgebra of $K\langle X_2\rangle$ consisting of all $f(x_1, x_2)$ with the property $f(x_1+x_2, x_2) = f(x_1, x_1+x_2) = f(x_1, x_2)$.

Further reading for invariant theory for other noncommutative algebras

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THANK YOU FOR ATTENTION!