On a proof of consistency of Arithmetic System

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1. Introduction and Overview

The problem of consistency of the Arithmetic System, is the well-known problem (see for e.g.: Ackermann 1940, Adamowicz 1995, Artemov 2020, Batens 2014, Beklemishev 2010, Bessonov 2018, Boolos 2009, Buldt 2014, Chow 2018, Clote and McAloon 1983, Crossley 2011, Feferman 1960, Ferreira and Ferreira 2013, Gauthier 2013, Gentzen 1936, Gödel 1938, Grzegorczyk 1974, Hájek 1998, Hintikka 1998, Indrzejczak 1998, Khlodovskii 1959, Kotlarski 2004, Krajewski 2003, Kubiński 1963, Lassaigne and de Rougemont 2004, Leivant 1975, Marciszewski 1994, Marek and Mycielski 2001, McCall 204, Meyer 1976, Meyer 2021, Murawski 1987, Murawski 1994, Murawski 1998, Murawski 1999, Murawski 2001, Negri and von Plato 2019, Nelson 2015, Olszewski 2009, Pallares 2004, Perzanowski 2001, von Plato 2009, von Plato 2020, Podnieks 2015, Pogonowski 2011, Pogorzelski and Surma 1969, Pohlers 2009, Poizat 2000, Priest 2003, Rahn etal. 2021, Rasiowa 1973, Rautenberg 2006, Salehi 2012, Smith 2014, Smorynski 1977, Smullyan 1992, Srivastava 2008, Stępień and Stępień 2009, Stępień and Stępień 2010, Stępień and Stępień 2013, Stępień and Stępień 2017, Tait 2005, Tappenden 1995, Tarski 1971, Tennant 2002, Trzęsicki 2010, Tsuji, da Costa and Doria 1998, Urquhart 2008, Visser 2016, Voevodsky 2010, Walsh 2014, Weiermann 2009, Welch 2012, Weyl 2009, Woleński 1989, Wójcicki 1995/96, Wroński 2004, Zach 2006, Zhang and Peace 2014).

The sketch of first version of the proof of consistency of Arithmetic System (done withn this System), was presented in a talk, delivered under the similar title, at *European ASL Summer Meeting* "Logic Colloquium 2009", in Sofia (Bulgaria). The abstract of that talk: T. J. Stępień and Ł. T. Stępień. "On the consistency of Peano's Arithmetic System", *The Bulletin of Symbolic Logic*, 16:132 (2010).

This talk is based on the paper including the full proof of consistency of Arithmetic System (done within this System): T. J. Stępień, Ł. T. Stępień, "On the Consistency of the Arithmetic System", *Journal of Mathematics and System Science*, vol. 7, 43 – 55 (2017), arXiv:1803.11072. I will present now **a sketch** of this proof.

2. Terminology

Let: $\rightarrow$, $\sim$, $\vee$, $\wedge$, $\equiv$ denote the connectives of implication, negation, disjunction, conjunction and equivalence, respectively. $\mathcal{N} = \{1,2,...\}$ denotes the set of all natural numbers.

Next, I denote by $At_0 = \{p_1^1, p_2^1, ..., p_1^2, p_2^2, ..., p_1^k, p_2^k, ...\}$ $(k \in \mathcal{N})$, the set of all propositional variables. Hence, S_0 is the symbol of the set of all well-formed formulas built in the usual manner from propositional variables and by means of logical connectives. $P_0(\phi)$ denotes the set of all propositional variables occuring in ϕ $(\phi \in S_0)$. R_{S_0} is the set of all rules over S_0 (see Pogorzelski, *The Classical Propositional Calculus* 1975). $E(\mathfrak{M})$ is the symbol of the set of all formulas valid in the matrix $\mathfrak{M}$. The $\mathfrak{M}_2$ denotes the classical two-valued matrix.

Z_2 is the set of all formulas valid in the matrix $\mathfrak{M}_2$ (see Pogorzelski, *The Classical Propositional Calculus* 1975).

The symbols: $x_1, x_2, \dots$ denote individual variables, and the symbols: $a_1, a_2, \dots$ denote individual constants. V is the set of all individual variables. C is the set of all individual constants. $P_i^n (i, n \in \mathcal{N} = \{1, 2, \dots\})$ are n-ary predicate letters. The symbols: $f_i^n (i, n \in \mathcal{N})$ are the symbols of n-ary function letters. The symbols: $\bigwedge x_k, \bigvee x_k$ are quantifiers. $\bigwedge x_k$ is the universal quantifier and $\bigvee x_k$ is the existential quantifier. The function letters, applied to the individual variables and individual constants, generate terms. The symbols: $t_1, t_2, \dots$ denote terms. T is the symbol of the set of all terms.

Next: $V \cup C \subseteq T$. The predicate letters, applied to terms, yield simple formulas, i.e. if P_i^k is a predicate letter and $t_1, \dots, t_k$ are terms, then $P_i^k(t_1, \dots, t_k)$ is a simple formula. Smp denotes the set of all simple formulas. Next, At_1 is the set of all atomic formulas, i.e. $At_1 = \{P_i^k(x_{j_1}, \dots, x_{j_k}) : i, k, j_1, \dots, j_k \in \mathcal{N}\}$. At last, the symbol S_1 denotes the set of all well-formed formulas. $FV(\phi)$ is the symbol of the set of all free variables occuring in ϕ, where $\phi \in S_1$. $x_k \in Ff(t_m, \phi)$ expresses that x_k is free for term t_m in ϕ ($\phi \in S_1$). The symbol x_k/t_m denotes the substitution of the term t_m for the individual variable x_k.

$P_1(\phi)$ is the symbol of the set of all predicate letters occuring in ϕ ($\phi \in S_1$). If $FV(\phi) = \{x_1, \dots, x_k\}$, then $\bigwedge \phi = \bigwedge x_1 \dots \bigwedge x_k \phi$. The symbol R_{S_1} denotes the set of all rules over S_1. $\overline{S}_1 = \{\phi \in S_1 : FV(\phi) = \emptyset\}$.

I will use $\Rightarrow, \neg, \mathbb{V}, \&, \Leftrightarrow, \forall, \exists$ as metalogical symbols. Next, $r_0^i (i \in \{0,1\})$ denotes Modus Ponens for propositional calculus and predicate calculus, respectively.

The symbol r_+ denotes the generalization rule. Hence, $R_{0+} = \{r_0^1, r_+\}$.

L_2 is the set of all formulas valid in the classical calculus of quantifiers (see Pogorzelski *The Classical Calculus of Quantifiers*). I write $X \subset Y$ for $X \subseteq Y$ and $Y \neq X$. For any $X \subseteq S_i$, $Cn(R, X)$ is the smallest subset of S_i, containing X and closed under the rules $R \subseteq R_{S_i}$ and $i \in \{0,1\}$. The couple $\langle R, X \rangle$ is called a system, whenever $R \subseteq R_{S_i}$, $X \subseteq S_i$ and $i \in \{0,1\}$ (see: Pogorzelski *The Classical Propositional Calculus*, Pogorzelski *The Classical Calculus of Quantifiers*, and *cf.* Yu. L. Ershov and E. A. Palyutin. *Mathematical Logic*).

Now I repeat some well-known properties of operation of consequence and some well-known definitions (see: R. Murawski. *Recursive Functions and Metamathematics. Problems of Completeness and Decidability, Gödel's Theorems*, Pogorzelski *The Classical Propositional Calculus*, Pogorzelski *The Classical Calculus of Quantifiers*). Let $R \subseteq R_{S_i}$ and $X \subseteq S_i$. Then:

(a_1) $X \subseteq Cn(R, X)$,

(a_2) $X \subseteq Y \Rightarrow Cn(R, X) \subseteq Cn(R, Y)$,

(a_3) $R \subseteq R' \Rightarrow Cn(R, X) \subseteq Cn(R', X)$,

(a_4) $Cn(R, Cn(R, X)) \subseteq Cn(R, X)$,

(a_5) $Cn(R, X) = \bigcup\{Cn(R, Y) : Y \in Fin(X)\}$,

where $Y \in Fin(X)$ denotes that Y is the finite subset of X and $i \in \{0,1\}$.

Now, I recall the definitions of the notions of: consistency in the traditional sense and consistency in the absolute sense (Post's sense).

Definition 1.1. $\langle R, X \rangle \in Cns^T \Leftrightarrow (\neg \exists \alpha \in S_i)[\alpha \in Cn(R, X) \ \& \ {\sim}\alpha \in Cn(R, X)]$, where $i \in \{0,1\}$.

Definition 1.2. $\langle R, X \rangle \in Cns^A \Leftrightarrow Cn(R, X) \neq S_i$, where $i \in \{0,1\}$.

$\langle R, X \rangle \in Cns^T$ denotes that the system $\langle R, X \rangle$ is consistent in the traditional sense, and $\langle R, X \rangle \in Cns^A$ denotes that the system $\langle R, X \rangle$ is consistent in the absolute sense (see: Pogorzelski, *The Classical Propositional Calculus*, Pogorzelski, *The Classical Calculus of Quantifiers*).

2. Basic Theorems

Theorem 2.1. $\langle R_{0+}, L_2 \rangle \in Cns^T$.
Theorem 2.2. $\langle R_{0+}, L_2 \rangle \in Cns^A$.
Theorem 2.3. $(\forall \alpha \in \overline{S}_1)(\forall \beta \in S_1)(\forall X \subseteq S_1)[\beta \in Cn(R_{0+}, A_2 \cup X \cup \{\alpha\}) \Rightarrow (\alpha \to \beta) \in Cn(R_{o+}, A_2 \cup X)]$.
Theorem 2.4. $(\forall \alpha \in \overline{S}_1)(\forall X \subseteq S_1)[Cn(R_{0+}, A_2 \cup X \cup \{\alpha\}) = S_1 \Leftrightarrow {\sim}\alpha \in Cn(R_{0+}, A_2 \cup X)]$.
Theorem 2.5. $(\forall \alpha \in \overline{S}_1)(\forall X \subseteq S_1)[\alpha \notin Cn(R_{0+}, A_2 \cup X) \Leftrightarrow Cn(R_{0+}, A_2 \cup X \cup \{{\sim}\alpha\}) \neq S_1]$,
where A_2 denotes the set of the axioms of the classical calculus of quantifiers (see Pogorzelski *The Classical Calculus of Quantifiers*).

3. Arithmetic Terminology

In the literature, many Authors use many versions of (Peano's) Arithmetic System.

Now: S_A denotes the set of all well-formed formulas of the Arithmetic System. Hence, $Fv(\phi)$ is the symbol of the set of all free variables occuring in ϕ, where $\phi \in S_A$. Next, $x_k \in Ff_A(t_m, \phi)$ expresses that x_k is free for term t_m in ϕ, where $\phi \in S_A$.

$\overline{S}_A = \{\phi \in S_A : Fv(\phi) = \emptyset\}$. The symbol R_{S_A} denotes the set of all rules over S_A. For any $X \subseteq S_A$ and for any $R \subseteq R_{S_A}$, $Cn(R, X)$ is the smallest subset of S_A, containing X and closed under the rules of R. The couple $\langle R, X \rangle$ is called a system, whenever $R \subseteq R_{S_A}$ and $X \subseteq S_A$. Next, $R_{0+}^P = \{r_0^P, r_+^P\} \subseteq R_{S_A}$, where r_0^P and r_+^P are Modus Ponens and generalization rule in the Arithmetic System, respectively.
The symbols: $\psi^1, \psi^2, \psi^3, \psi^4, \psi^5, \psi^6, \psi^7, \psi^8, \psi^9, \psi^{10}, \psi^{11}, \psi^{12}$ denote the specific axioms of the Arithmetic System, where:

ψ^1. $\wedge x_1 \, (x_1 = x_1)$,
ψ^2. $\wedge x_1 \wedge x_2 \, (x_1 = x_2 \to x_2 = x_1)$,
ψ^3. $\wedge x_1 \wedge x_2 \wedge x_3 (x_1 = x_2 \to (x_2 = x_3 \to x_1 = x_3))$,
ψ^4. $\wedge x_1 \wedge x_2 \wedge x_3 \wedge x_4 (x_1 = x_2 \to (x_3 = x_4 \to (x_1 + x_3 = x_2 + x_4)))$,
ψ^5. $\wedge x_1 \wedge x_2 \wedge x_3 \wedge x_4 (x_1 = x_2 \to (x_3 = x_4 \to (x_1 \cdot x_3 = x_2 \cdot x_4)))$,
ψ^6. $\wedge x_1 \wedge x_2 \wedge x_3 \wedge x_4 (x_1 = x_2 \to (x_3 = x_4 \to (x_1 < x_3 \to x_2 < x_4)))$,
ψ^7. $\wedge x_1 \; {\sim}(1 = x_1 + 1)$,
ψ^8. $\wedge x_1 \wedge x_2 \, (x_1 + 1 = x_2 + 1 \to x_1 = x_2)$,
ψ^9. $\wedge x_1 \wedge x_2 \, (x_1 + (x_2 + 1) = (x_1 + x_2) + 1)$,
ψ^{10}. $\wedge x_1 \, (x_1 \cdot 1 = x_1)$,
ψ^{11}. $\wedge x_1 \wedge x_2 \, [x_1 \cdot (x_2 + 1) = (x_1 \cdot x_2) + x_1]$,
ψ^{12}. $\wedge x_1 \wedge x_2 \, [x_1 < x_2 \equiv \vee x_3 \, (x_1 + x_3 = x_2)]$.

Hence, $X_P = \{\psi^1, \psi^2, \psi^3, \psi^4, \psi^5, \psi^6, \psi^7, \psi^8, \psi^9, \psi^{10}, \psi^{11}, \psi^{12}\}$.

Next, the induction schema is the set of the following axioms:

$$\psi^{13}.\ \left(\phi(1) \wedge \bigwedge x_1\big(\phi(x_1) \rightarrow \phi(x_1+1)\big)\right) \rightarrow \bigwedge x_1 \phi(x_1),$$

where $\phi(1), \phi(x), \phi(x+1) \in S_A$.

Hence, the symbol Y_P denotes here the set of all axioms of induction. Thus, the symbols: L_2^r and A_r denote the set of all logical axioms and the set of all specific axioms of the Arithmetic System, respectively, where $A_r = X_P \cup Y_P$ (see A. Grzegorczyk. *An Outline of Mathematical Logic. Fundamental Results and Notions Explained with All Details*, H. Rasiowa. *Introduction to Modern Mathematics*).

Hence, $\langle R_{0+}^P, L_2^r \cup A_r \rangle$ is the Arithmetic System. In: H. Rasiowa. *Introduction to Modern Mathematics*, one can read that the system $\langle R_{0+}^P, L_2^r \cup A_r \rangle$ is a modification of Peano's Arithmetic System.

The symbol L_1^2 denotes the well-known subset of the set L_2^r (see: A. Grzegorczyk. *An Outline of Mathematical Logic. Fundamental Results and Notions Explained with All Details*, Pogorzelski, *The Classical Propositional Calculus*, Pogorzelski, *The Classical Calculus of Quantifiers*, H. Rasiowa. *Introduction to Modern Mathematics*). Namely, $L_1^2 = \{\phi^1, \phi^2, \phi^3, \phi^4, \phi^5, \phi^6, \phi^7, \phi^8, \phi^9, \phi^{10}, \phi^{11}, \phi^{12}\}$, where:

$\phi^1.\ [\alpha \rightarrow (\beta \rightarrow \gamma)] \rightarrow [(\alpha \rightarrow \beta) \rightarrow (\alpha \rightarrow \gamma)],$
$\phi^2.\ (\sim\alpha \rightarrow \alpha) \rightarrow \alpha,$
$\phi^3.\ \sim\alpha \rightarrow (\alpha \rightarrow \beta),$
$\phi^4.\ \alpha \rightarrow (\beta \rightarrow \alpha),$
$\phi^5.\ \alpha \wedge \beta \rightarrow \alpha,$
$\phi^6.\ \alpha \wedge \beta \rightarrow \beta,$
$\phi^7.\ \alpha \rightarrow (\beta \rightarrow \alpha \wedge \beta),$
$\phi^8.\ \alpha \rightarrow \alpha \vee \beta,$
$\phi^9.\ \beta \rightarrow \alpha \vee \beta,$
$\phi^{10}.\ (\alpha \rightarrow \beta) \rightarrow [(\delta \rightarrow \beta) \rightarrow (\alpha \vee \delta \rightarrow \beta)],$
$\phi^{11}.\ \bigwedge x_k \phi \rightarrow \phi\left(\frac{x_k}{t_n}\right)$,if $x_k \in Ff_A(t_n, \phi),$
$\phi^{12}.\ \bigwedge x_k(\phi \rightarrow \psi) \rightarrow (\phi \rightarrow \bigwedge x_k \psi)$, if $x_k \notin Fv(\phi)$

and
$\alpha, \beta, \gamma, \delta, \phi, \psi \in S_A.$

The analogons of Definition 1.1., Definition 1.2., Theorem 2.1., Theorem 2.2., are the following (where $R \subseteq R_{S_A}$ and $X \subseteq S_A$):
Definition 3.1. $\langle R, X \rangle \in Cns_A^T \Leftrightarrow (\neg\exists\alpha \in S_A)[\alpha \in Cn(R, X)\ \ \&\ \sim\alpha \in Cn(R, X)].$
Definition 3.2. $\langle R, X \rangle \in Cns_A^A \Leftrightarrow Cn(R, X) \neq S_A.$
Theorem 3.3. $\langle R_{0+}^P, L_2^r \rangle \in Cns_A^T.$

Theorem 3.4. $\langle R^P_{0+}, L^r_2\rangle \in Cns^A_A$.

As far as the notion of consistency of Arithmetic System is concerned, there in our proof of consistency of Arithmetic System only the definitions: 3.1. and 3.2. are used.

The analogons of Theorem 2.4. and Theorem 2.5., are the following:

Theorem 3.5. $(\forall\alpha \in \overline{S}_A)(\forall X \subseteq S_A)[Cn(R^P_{0+}, L^2_1 \cup X \cup \{\alpha\}) = S_A \Leftrightarrow {\sim}\alpha \in Cn(R^P_{0+}, L^2_1 \cup X)]$.
Theorem 3.6. $(\forall\alpha \in \overline{S}_A)(\forall X \subseteq S_A)[\alpha \notin Cn(R^P_{0+}, L^2_1 \cup X) \Leftrightarrow Cn(R^P_{0+}, L^2_1 \cup X \cup \{{\sim}\alpha\}) \neq S_A]$,

where $L^2_1 \subseteq L^r_2$ (see: Yu. L. Ershov and E. A. Palyutin. *Mathematical Logic*, A. Grzegorczyk. *An Outline of Mathematical Logic. Fundamental Results and Notions Explained with All Details*, R. Murawski. *Recursive Functions and Metamathematics. Problems of Completeness and Decidability, Gödel's Theorems*, W. A. Pogorzelski. *The Classical Calculus of Quantifiers*, H. Rasiowa. *Introduction to Modern Mathematics*, A. Tarski, A. Mostowski and R. M. Robinson. *Undecidable Theories*).

4. The Basic Corollaries and Lemmas

At first, the following formulas are introduced:

(I_1) $O_0 = \psi^7 \equiv {\sim}{\sim}\psi^1$,
(I_2) $u_{27} = {\sim}(1 < 1)$,
(I_3) $O_6 = O_0 \to (\psi^7 \to \psi^1)$,
(I_4) $\alpha^x_2 = \psi^1 \to \psi^7$,
(I_5) $\gamma'_2 = O_0 \to u_{27}$,
(I_6) $\gamma'_0 = (\psi^7 \to \psi^1) \to \psi^{12}$,
(I_7) $\gamma_0 = u_{27} \to \gamma'_0$,
(I_8) $\gamma'_4 = \gamma'_0 \to O_0$.

Next, it is assumed that

(I_9) $(\forall\alpha, \delta \in S_A)[\,_{\alpha}\delta = \alpha \to \delta]$,
(I_{10}) $(\forall X \subseteq S_A)(\forall\alpha \in S_A)[\,_{\alpha}X = (\alpha \to \beta : \beta \in X)]$.

Next, the sets L^1_1 and $_{\psi^7 O_0 u_{27} \sim\psi^1}L^r_2$ are defined, as follows:

(I_{11}) $L^1_1 = L^2_1 \cup \{\psi^1 \to (\psi^7 \to (\psi^{12} \to \omega)) : \omega \in L^r_2 - L^2_1\}$,

(I_{12}) $_{\psi^7 O_0 u_{27} \sim\psi^1}L^r_2 = \{\psi^7 \to (O_0 \to (u_{27} \to ({\sim}\psi^1 \to \beta))) : \beta \in L^r_2 - L^2_1\}$.

Lemma 4.1. $\langle R_{0+}^{P}, L_2^r \cup \{\psi^1, \psi^7, \psi^{12}\}\rangle \notin Cns_A^T \Rightarrow (\forall \alpha \in \overline{S}_A - A^0)\left(\forall \overset{00}{\delta} \in \overline{S}_A - A^0\right)$

$[Cn(R_{0+}^{P}, L_1^1 \cup {}_{\psi^7 O_0 u_{27} \sim \psi^1} L_2^r \cup N\dot{\overline{\Psi}}_3 \cup \left\{\overset{00}{\delta}\right\} \cup \{\alpha \to \psi^7\} \cup \{\xi\}) = S_A \Rightarrow Cn(R_{0+}^{P}, L_1^1 \cup$

${}_{\psi^7 O_0 u_{27} \sim \psi^1} L_2^r \cup N\dot{\overline{\Psi}}_3 \cup \{\alpha \to \psi^7\} \cup \{\xi\}) = S_A]$,

where

$\overset{00}{\delta} = \psi^7 \to \overset{0}{\delta}, \xi = \psi^7 \to (\psi^1 \to \psi^{12})$,

$A^0 = Cn\left(R_{0+}^{P}, L_1^1 \cup {}_{\psi^7 O_0 u_{27} \sim \psi^1} L_2^r \cup N\dot{\overline{\Psi}}_3 \cup \{\xi\}\right)$,

$N\dot{\overline{\Psi}}_3 = \{O_0 \to \gamma_0, \gamma_2', \gamma_4'\}$.

Proof in: T. J. Stępień, Ł. T. Stępień, "On the Consistency of the Arithmetic System", *Journal of Mathematics and System Science*, vol. 7, 43 – 55 (2017), *arXiv:1803.11072*.

Lemma 4.2. $\langle R_{0+}^{P}, L_2^r \cup \{\psi^1, \psi^7, \psi^{12}\}\rangle \notin Cns_A^T \Rightarrow (\exists \alpha' \in \overline{S}_A - A^0)\left(\forall \overset{00}{\delta} \in \overline{S}_A - A^0\right)[Cn(R_{0+}^{P}, L_1^1 \cup$

${}_{\psi^7 O_0 u_{27} \sim \psi^1} L_2^r \cup N\dot{\overline{\Psi}}_3 \cup \left\{\overset{00}{\delta}\right\} \cup \{\alpha' \to \psi^7, \xi\}) = S_A]$,

where

$\overset{00}{\delta} = \psi^7 \to \overset{0}{\delta}, \xi = \psi^7 \to (\psi^1 \to \psi^{12}), A^0 = Cn\left(R_{0+}^{P}, L_1^1 \cup {}_{\psi^7 O_0 u_{27} \sim \psi^1} L_2^r \cup N\dot{\overline{\Psi}}_3 \cup \{\xi\}\right)$,

$N\dot{\overline{\Psi}}_3 = \{O_0 \to \gamma_0, \gamma_2', \gamma_4'\}$.

Proof in: T. J. Stępień, Ł. T. Stępień, "On the Consistency of the Arithmetic System", *Journal of Mathematics and System Science*, vol. 7, 43 – 55 (2017), *arXiv:1803.11072*.

Corollary 4.1. $\langle R_{0+}^{P}, L_2^r \cup A_r\rangle \notin Cns_A^T \Rightarrow (\exists Y_P'' \subseteq Y_P)(\exists X_p'' \subseteq X_P)\{Cn(R_{0+}^{P}, L_2^r \cup Y_p'' \cup X_P'') = S_A$ &
$(\forall Y \subset X_P'' \cup Y_P'')[Cn(R_{0+}^{P}, L_2^r \cup Y) \neq S_A]\}$,

where
$Y_P'' = \{\alpha_1, \dots, \alpha_k\}, X_p'' = \{\alpha_1', \dots, \alpha_n'\}$ and $k, n \in \mathcal{N}$
and $X_P \cup Y_P = A_r$.

Proof in: T. J. Stępień, Ł. T. Stępień, "On the Consistency of the Arithmetic System", *Journal of Mathematics and System Science*, vol. 7, 43 – 55 (2017), *arXiv:1803.11072*.

Corollary 4.2. $\langle R^P_{0+}, L^r_2 \cup A_r\rangle \notin Cns^T_A \Rightarrow (\exists Y''_P \subseteq Y_P)(\exists X''_p \subseteq X_P)[Cn(R^P_{0+}, L^r_2 \cup Y''_P \cup X''_P) = S_A$ & $(\forall Y \subset X''_P \cup Y''_P)[Cn(R^P_{0+}, L^r_2 \cup Y) \neq S_A$ & $(\exists\, Z_P \subseteq \{\psi^2, \psi^3, \psi^4, \psi^5, \psi^6, \psi^8, \psi^9, \psi^{10}, \psi^{11}\} \cup Y_P)$ $[Z_P \subseteq X''_P \cup Y''_P$ & $Z_P \neq \emptyset]]]$,

where
$Y''_P = \{\alpha_1, \dots, \alpha_k\}, X''_P = \{\alpha'_1, \dots, \alpha'_n\}$ and $k, n \in \mathcal{N}$

and $Y_P \cup X_P = A_r$.

Proof in: T. J. Stępień, Ł. T. Stępień, "On the Consistency of the Arithmetic System", *Journal of Mathematics and System Science*, vol. 7, 43 – 55 (2017), *arXiv:1803.11072*.

Using **Corollary 4.2.**, the formulas $\overset{0}{\beta}, \overset{1}{\beta}$ are now defined, as follows:

$(I_{14})\ \overset{0}{\beta} = \alpha_1 \wedge \dots \wedge \alpha_k \wedge \alpha'_1 \wedge \dots \wedge \alpha'_n \wedge \psi^1 \wedge \psi^7 \wedge \psi^{12}$,

$(I_{15})\ \overset{1}{\beta} = \psi^1 \to \left(\psi^7 \to \left(\psi^{12} \to \overset{0}{\beta}\right)\right)$,

where

$\{\alpha_1, \dots, \alpha_k\} = Y''_P$ and $\{\alpha'_1, \dots, \alpha'_n\} = X''_P$ and $k, n \in \mathcal{N}$.

Next, some sets L^1_T and $N\ddot{\overline{\Psi}}_3$ are defined, as follows:

$(I_{16})\ L^1_T = L^2_1 \cup \left\{\overset{0}{\beta} \to \omega : \omega \in L^r_2 - L^2_1\right\}$,

$(I_{17})\ N\ddot{\overline{\Psi}}_3 = \left\{O_0 \to \left(u_{27} \to \overset{1}{\beta}\right), O_0 \to \gamma_0, \gamma'_2, \gamma'_4\right\}$.

Thus,

Corollary 4.3. $\langle R^P_{0+}, L^r_2 \cup A_r\rangle \notin Cns^T_A \Rightarrow Cn\left(R^P_{0+}, L^1_T \cup \left\{\overset{1}{\beta}\right\} \cup \{\psi^1, \psi^7, \psi^{12}\}\right) = S_A$.

Proof (basing on: T. J. Stępień, Ł. T. Stępień, "On the Consistency of the Arithmetic System", *Journal of Mathematics and System Science*, vol. 7, 43 – 55 (2017), *arXiv:1803.11072*):

By using: (I_{14}), (I_{15}), (I_{16}), **Corollary 4.2.** and the definition of the formula $\overset{1}{\beta}$. □

Corollary 4.4. $\langle R^P_{0+}, L^r_2 \cup A_r\rangle \notin Cns^T_A \Rightarrow \left[\sim\overset{0}{\beta} \in Cn(R^P_{0+}, L^1_T)\right]$.

Proof in: T. J. Stępień, Ł. T. Stępień, "On the Consistency of the Arithmetic System", *Journal of Mathematics and System Science*, vol. 7, 43 – 55 (2017), *arXiv:1803.11072*.

Lemma 4.3. $\langle R^P_{0+}, L^r_2 \cup A_r\rangle \notin Cns^T_A \Rightarrow \left(\forall\alpha \in \overline{S}_A - A^1\right)\left(\forall \overset{00}{\delta} \in \overline{S}_A - A^1\right)[Cn(R^P_{0+}, L^1_1 \cup {}_{\psi^7 O_0 u_{27} \sim \psi^1}L^r_2 \cup N\ddot{\overline{\Psi}}_3 \cup \left\{\overset{00}{\delta}\right\} \cup \{\alpha \to \psi^7, \xi\}) = S_A \Rightarrow Cn(R^P_{0+}, L^1_1 \cup {}_{\psi^7 O_0 u_{27} \sim \psi^1}L^r_2 \cup N\ddot{\overline{\Psi}}_3 \cup \{\alpha \to \psi^7, \xi\}) = S_A]$,

where

$\overset{00}{\delta} = \psi^7 \to \overset{0}{\delta}, \xi = \psi^7 \to (\psi^1 \to \psi^{12})$,

$A^1 = Cn\left(R^P_{0+}, L^1_1 \cup {}_{\psi^7 O_0 u_{27} \sim \psi^1}L^r_2 \cup N\ddot{\overline{\Psi}}_3 \cup \{\xi\}\right)$.

Proof. (basing on: T. J. Stępień, Ł. T. Stępień, "On the Consistency of the Arithmetic System", *Journal of Mathematics and System Science*, vol. 7, 43 – 55 (2017), *arXiv:1803.11072*). Let:

(1) $\langle R^P_{0+}, L^r_2 \cup A_r\rangle \notin Cns^T_A$

and

(2) $\neg\left(\forall\alpha \in \overline{S}_A - A^1\right)\left(\forall \overset{00}{\delta} \in \overline{S}_A - A^1\right)[Cn(R^P_{0+}, L^1_1 \cup {}_{\psi^7 O_0 u_{27} \sim \psi^1}L^r_2 \cup N\ddot{\overline{\Psi}}_3 \cup \left\{\overset{00}{\delta}\right\} \cup \{\alpha \to \psi^7, \xi\}) = S_A \Rightarrow$

$Cn(R^P_{0+}, L^1_1 \cup {}_{\psi^7 O_0 u_{27} \sim \psi^1}L^r_2 \cup N\ddot{\overline{\Psi}}_3 \cup \{\alpha \to \psi^7, \xi\}) = S_A]$,

where

(3) $\overset{00}{\delta} = \psi^7 \to \overset{0}{\delta}$,

(4) $\xi = \psi^7 \to (\psi^1 \to \psi^{12})$,

(5) $A^1 = Cn(R^P_{0+}, L^1_1 \cup {}_{\psi^7 O_0 u_{27} \sim \psi^1}L^r_2 \cup N\ddot{\overline{\Psi}}_3 \cup \{\xi\})$.

From (2) – (5), it follows that

(6) $\left(\exists\alpha' \in \overline{S}_A - A^1\right)\left(\exists \overset{00}{\delta}{}' \in \overline{S}_A - A^1\right)[Cn(R^P_{0+}, L^1_1 \cup {}_{\psi^7 O_0 u_{27} \sim \psi^1}L^r_2 \cup N\ddot{\overline{\Psi}}_3 \cup \left\{\overset{00}{\delta}{}'\right\} \cup \{\alpha' \to \psi^7, \xi\}) = S_A \;\&$

$Cn(R^P_{0+}, L^1_1 \cup {}_{\psi^7 O_0 u_{27} \sim \psi^1}L^r_2 \cup N\ddot{\overline{\Psi}}_3 \cup \{\alpha' \to \psi^7, \xi\}) = A \neq S_A]$, where

(7) $\overset{00}{\delta}{}' = \psi^7 \to \overset{0}{\delta}{}'$,

(8) $\xi = \psi^7 \to (\psi^1 \to \psi^{12})$,

(9) $A^1 = Cn(R^P_{0+}, L^1_1 \cup {}_{\psi^7 O_0 u_{27} \sim \psi^1}L^r_2 \cup N\ddot{\overline{\Psi}}_3 \cup \{\xi\})$.

From (6) – (9), I obtain that

$$(10)\ \left(\exists\alpha' \in \overline{S}_A - A^1\right)\left(\exists\overset{00}{\delta}{}' \in \overline{S}_A - A^1\right)[Cn(R^P_{0+}, L^1_1 \cup {}_{\psi^7 O_0 u_{27} \sim \psi^1}L^r_2 \cup N\ddot{\overline{\Psi}}_3 \cup$$

$$\{\alpha' \to \psi^7, \xi\}) = A\ \ \&\ \ A \neq S_A\ \ \&\ \ \overset{00}{\delta}{}' \notin A].$$

From (10) and by **Theorem 3.6.**, I have

$$(11)\ \left(\exists\alpha' \in \overline{S}_A - A^1\right)\left(\exists\overset{00}{\delta}{}' \in \overline{S}_A - A^1\right)[Cn(R^P_{0+}, L^1_1 \cup {}_{\psi^7 O_0 u_{27} \sim \psi^1}L^r_2 \cup N\ddot{\overline{\Psi}}_3 \cup$$

$$\{\alpha' \to \psi^7, \xi\}) = A\ \ \&\ \ Cn(R^P_{0+}, L^1_1 \cup {}_{\psi^7 O_0 u_{27} \sim \psi^1}L^r_2 \cup N\ddot{\overline{\Psi}}_3 \cup \left\{\sim\overset{00}{\delta}{}'\right\} \cup$$

$$\{\alpha' \to \psi^7\} \cup \{\xi\}) = A^*\ \ \&\ \ A^* \neq S_A].$$

From (I_{11}), (I_{14}), (I_{16}), it follows that

$$(12)\ Cn(R^P_{0+}, L^1_T) \subseteq Cn(R^P_{0+}, L^1_1).$$

From (1) – (12), (I_5), (I_6), (I_8), (I_{14}) – (I_{17}), by **Corollary 4.4.**, I have

$$(13)\ \left(\exists\alpha' \in \overline{S}_A - A^1\right)\left(\exists\overset{00}{\delta}{}' \in \overline{S}_A - A^1\right)[Cn(R^P_{0+}, L^1_1 \cup {}_{\psi^7 O_0 u_{27} \sim \psi^1}L^r_2 \cup N\ddot{\overline{\Psi}}_3 \cup$$

$$\left\{\sim\overset{00}{\delta}{}'\right\} \cup \{\alpha' \to \psi^7\} \cup \{\xi\}) = A^*\ \ \& \psi^7, \gamma'_4, \gamma'_2, O_0 \to \gamma_0, \psi^1 \to \psi^{12}, \gamma'_0,$$

$$O_0, u_{27}, \overset{1}{\beta}, \psi^1 \to \overset{0}{\beta}, \sim\overset{0}{\beta}, \sim\psi^1 \in A^*\ \ \&\ \ A^* \neq S_A].$$

Hence, from (I_{12}), it follows that

$$(14)\ \left(\exists\alpha' \in \overline{S}_A - A^1\right)\left(\exists\overset{00}{\delta}{}' \in \overline{S}_A - A^1\right)[Cn(R^P_{0+}, L^1_1 \cup {}_{\psi^7 O_0 u_{27} \sim \psi^1}L^r_2 \cup N\ddot{\overline{\Psi}}_3 \cup$$

$$\left\{\sim\overset{00}{\delta}{}'\right\} \cup \{\alpha' \to \psi^7\} \cup \{\xi\}) = A^*\ \ \&\ \ L^r_2 \subseteq A^*\ \ \&\ \ A^* \neq S_A].$$

From (13), (14), and (I_1), I obtain that

$$(15)\ \left(\exists\alpha' \in \overline{S}_A - A^1\right)\left(\exists\overset{00}{\delta}{}' \in \overline{S}_A - A^1\right)[Cn(R^P_{0+}, L^1_1 \cup {}_{\psi^7 O_0 u_{27} \sim \psi^1}L^r_2 \cup N\ddot{\overline{\Psi}}_3 \cup \left\{\sim\overset{00}{\delta}{}'\right\} \cup$$

$$\{\alpha' \to \psi^7\} \cup \{\xi\}) = A^*\ \ \& \psi^7, \psi^7 \to \psi^1, \sim\psi^1, \psi^1 \in A^*\ \ \&\ \ A^* \neq S_A].$$

From (14) and (15), I have

$$(16)\ \left(\exists\alpha' \in \overline{S}_A - A^1\right)\left(\exists\overset{00}{\delta}{}' \in \overline{S}_A - A^1\right)[Cn(R^P_{0+}, L^1_1 \cup {}_{\psi^7 O_0 u_{27} \sim \psi^1}L^r_2 \cup N\ddot{\overline{\Psi}}_3 \cup \left\{\sim\overset{00}{\delta}{}'\right\} \cup$$

$$\{\alpha' \to \psi^7\} \cup \{\xi\}) = A^*\ \ \& A^* \neq S_A\ \ \&\ \ A^* = S_A].$$

Contradiction. □

Lemma 4.4. $\langle R^P_{0+}, L^r_2 \cup A_r\rangle \notin Cns^T_A \Rightarrow \left(\exists \alpha' \in \overline{S}_A - A^1\right)\left(\forall \overset{00}{\delta} \in \overline{S}_A - A^1\right)[Cn(R^P_{0+}, L^1_1 \cup$

$_{\psi^7 O_0 u_{27} \sim \psi^1} L^r_2 \cup N\ddot{\overline{\Psi}}_3 \cup \left\{\overset{00}{\delta}\right\} \cup \{\alpha' \to \psi^7, \xi\}) = S_A],$

where

$$\overset{00}{\delta} = \psi^7 \to \overset{0}{\delta}, \;\; \xi = \psi^7 \to (\psi^1 \to \psi^{12}),$$

$$A^1 = Cn\left(R^P_{0+}, L^1_1 \cup {}_{\psi^7 O_0 u_{27} \sim \psi^1} L^r_2 \cup N\ddot{\overline{\Psi}}_3 \cup \{\xi\}\right).$$

Proof in: T. J. Stępień, Ł. T. Stępień, "On the Consistency of the Arithmetic System", *Journal of Mathematics and System Science*, vol. 7, 43 – 55 (2017), *arXiv:1803.11072*.

5. The Main Result

Theorem 5.1. $\langle \boldsymbol{R^P_{0+}, L^r_2 \cup A_r}\rangle \in \boldsymbol{Cns^T_A}$.

Proof. (basing on: T. J. Stępień, Ł. T. Stępień, "On the Consistency of the Arithmetic System", *Journal of Mathematics and System Science*, vol. 7, 43 – 55 (2017), *arXiv:1803.11072*). Let:

1) $\langle R^P_{0+}, L^r_2 \cup A_r\rangle \notin Cns^T_A$.

Hence, by **Lemma 4.3.** and **Lemma 4.4.**, it follows that

2) $\left(\exists \alpha' \in \overline{S}_A - A^1\right)[Cn(R^P_{0+}, L^1_1 \cup {}_{\psi^7 O_0 u_{27} \sim \psi^1} L^r_2 \cup N\ddot{\overline{\Psi}}_3 \cup \{\alpha' \to \psi^7\} \cup \{\xi\}) = S_A],$

where

3) $A^1 = Cn(R^P_{0+}, L^1_1 \cup {}_{\psi^7 O_0 u_{27} \sim \psi^1} L^r_2 \cup N\ddot{\overline{\Psi}}_3 \cup \{\xi\}).$

From 2) and 3), by **Theorem 3.5.**, I obtain that

4) $\left(\exists \alpha' \in \overline{S}_A - A^1\right)[\sim(\alpha' \to \psi^7) \in A^1],$

where

5) $A^1 = Cn(R^P_{0+}, L^1_1 \cup {}_{\psi^7 O_0 u_{27} \sim \psi^1} L^r_2 \cup N\ddot{\overline{\Psi}}_3 \cup \{\xi\}).$

From 4) and 5), I have

6) $\left(\exists \alpha' \in \overline{S}_A\right)[\alpha' \notin A^1 \ \& \ \alpha' \in A^1]$,

where

7) $A^1 = Cn(R^P_{0+}, L^1_1 \cup {}_{\psi^7 O_0 u_{27} \sim \psi^1} L^r_2 \cup N\ddot{\overline{\Psi}}_3 \cup \{\xi\})$.

Contradiction □

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