Regularity of algebras of $\mathrm{O}(n)$-invariants using Hilbert series

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Definitions and notations

- Let G be a reductive complex linear algebraic group (e.g. $G = \mathrm{GL}(n)$, $\mathrm{SL}(n)$, $\mathrm{O}(n)$, $\mathrm{SO}(n)$, $\mathrm{Sp}(2k)$). A finite-dimensional representation W of G is called *coregular* if the algebra of invariants $\mathbb{C}[W]^G$ is regular, i.e. isomophic to a polynomial algebra.
- The irreducible coregular representations of connected simple complex algebraic groups were classified by Kac, Popov and Vinberg. They also described the degrees of the generators in a minimal generating set.
- The reducible coregular representations of connected simple complex algebraic groups were classified by Schwarz.

Question: What can we say about $\mathbb{C}[W]^{\mathrm{O}(n)}$?

Definitions and notations

Definition

Let $A = \bigoplus_{i \geq 0} A^i$ *be a finitely generated graded algebra over* $\mathbb{C}$ *such that* $A^0 = \mathbb{C}$ *or* $A^0 = 0$. *The Hilbert series of* A *is the formal power series*

$$H(A, t) = \sum_{i \geq 0} (\dim A^i) t^i.$$

Definition

A finitely generated graded algebra A *is called Cohen-Macaulay if there exist homogeneous elements* $u_1, \dots, u_k \in A$ *with the two properties:*

(i) *The elements* u_i *are algebraically independent, i.e., the algebra they generate is a polynomial ring in the* u_i.

(ii) *The ring* A *is a free module over* $\mathbb{C}[u_1, \dots, u_k]$.

Definitions and notations

- By Hochster-Roberts theorem, for any reductive linear algebraic group G the algebra of invariants $\mathbb{C}[W]^G$ is Cohen-Macaulay. Therefore, its Hilbert series has the form

$$H(\mathbb{C}[W]^G, t) = \frac{p(t)}{\prod_i (1 - t^{h_i})}, \quad \text{where } p(t) = \sum_j t^{l_j}.$$

- If $p(t) \neq 1$ then $\mathbb{C}[W]^G$ is not polynomial, hence W is not coregular.

Definitions and notations

- Let $\lambda = (\lambda_1 \geq \cdots \geq \lambda_n) \in (\mathbb{N}_0)^n$ be a non-negative integer partition. By V_λ we denote the irreducible $\mathrm{GL}(n)$-module with highest weight λ.

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Let W be a polynomial $\mathrm{GL}(n)$-module and let $S(W) = \oplus_{i \geq 0} S^i W$ denote the symmetric algebra of W. For $G = \mathrm{O}(n)$, $\mathrm{SO}(n)$, $\mathrm{Sp}(2k)$ we have that

$$\mathbb{C}[W]^G \cong S(W)^G.$$

Question: When is $S(W)^{\mathrm{O}(n)}$ a polynomial algebra, i.e. when is W a coregular $\mathrm{O}(n)$-representation?

Coregular $\mathrm{O}(2)$-representations

Theorem

Let W be a polynomial $\mathrm{GL}(2)$-module. If the algebra $S(W)^{\mathrm{O}(2)}$ is polynomial, then up to an $\mathrm{O}(2)$-isomorphism W is one of the following:

(1) V, S^2V, S^3V, S^4V, Λ^2V, $V_{(3,1)}$;

(2) $V \oplus V$, $V \oplus S^2V$, $S^2V \oplus S^2V$, $V \oplus \Lambda^2V$, $S^2V \oplus \Lambda^2V$.

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In the proof we use the following:

- Every subrepresentation of a coregular representation must be coregular.
- Consider the irreducible $\mathrm{GL}(2)$-module $V_{(k,l)}$.
  - (i) If at least one of k and l is even, then $V_{(k,l)} \cong S^{k-l}V$ as $\mathrm{O}(2)$-modules;
  - (ii) If both k and l are odd, then $V_{(k,l)} \cong V_{(k-l+1,1)}$ as $\mathrm{O}(2)$-modules.
- Branching rules for S^kV and $V_{(k,1)}$ as $\mathrm{O}(2)$-modules.

Coregular $\mathrm{O}(2)$-representations

Table : Hilbert series for $n = 2$

W	$H(S(W)^{\mathrm{O}(2)}, t)$	$H(S(W)^{\mathrm{SO}(2)}, t)$
$V = \mathbb{C}^2$	$\frac{1}{1-t^2}$	$\frac{1}{1-t^2}$
$S^2 V$	$\frac{1}{(1-t)(1-t^2)}$	$\frac{1}{(1-t)(1-t^2)}$
$S^3 V$	$\frac{1}{(1-t^2)^2(1-t^4)}$	$\frac{1+t^4}{(1-t^2)^2(1-t^4)}$
$S^4 V$	$\frac{1}{(1-t)(1-t^2)^2(1-t^3)}$	$\frac{1+t^3}{(1-t)(1-t^2)^2(1-t^3)}$
$S^5 V$	$\frac{1+t^2+3t^4+4t^6+5t^8+4t^{10}+3t^{12}+t^{14}+t^{16}}{(1-t^8)(1-t^6)(1-t^4)(1-t^2)^2}$	$\frac{1+t^2+6t^4+9t^6+12t^8+9t^{10}+6t^{12}+t^{14}+t^{16}}{(1-t^8)(1-t^6)(1-t^4)(1-t^2)^2}$
$S^6 V$	$\frac{1+t^2+t^3+2t^4+t^5+2t^6+t^7+t^8+t^{10}}{(1-t)(1-t^2)^2(1-t^3)(1-t^4)(1-t^5)}$	$\frac{1+t^2+3t^3+4t^4+4t^5+4t^6+3t^7+t^8+t^{10}}{(1-t)(1-t^2)^2(1-t^3)(1-t^4)(1-t^5)}$
$\Lambda^2 V$	$\frac{1}{1-t^2}$	$\frac{1}{1-t}$
$V_{(3,1)}$	$\frac{1}{(1-t^2)^2}$	$\frac{1}{(1-t)(1-t^2)}$
$V_{(5,1)}$	$\frac{1+t^4}{(1-t^2)^3(1-t^3)}$	$\frac{1+t^3}{(1-t)(1-t^2)^2(1-t^3)}$

Table : Hilbert series for $n = 2$

W	$H(S(W)^{\mathrm{O}(2)}, t)$	$H(S(W)^{\mathrm{SO}(2)}, t)$
$V \oplus V$	$\frac{1}{(1-t^2)^3}$	$\frac{1+t^2}{(1-t^2)^3}$
$V \oplus S^2V$	$\frac{1}{(1-t)(1-t^2)^2(1-t^3)}$	$\frac{1+t^3}{(1-t)(1-t^2)^2(1-t^3)}$
$S^2V \oplus S^2V$	$\frac{1}{(1-t)^2(1-t^2)^3}$	$\frac{1+t^2}{(1-t)^2(1-t^2)^3}$
$V \oplus \Lambda^2V$	$\frac{1}{(1-t^2)^2}$	$\frac{1}{(1-t)(1-t^2)}$
$S^2V \oplus \Lambda^2V$	$\frac{1}{(1-t)(1-t^2)^2}$	$\frac{1}{(1-t)^2(1-t^2)}$
$\Lambda^2V \oplus \Lambda^2V$	$\frac{1+t^2}{(1-t^2)^2}$	$\frac{1}{(1-t)^2}$
$V \oplus V \oplus V$	$\frac{1+t^2+t^4}{(1-t^2)^5}$	$\frac{1+4t^2+t^4}{(1-t^2)^5}$
$V \oplus V \oplus S^2V$	$\frac{(1+t+t^2+t^3+t^4)(1+t^3)}{(1-t^2)^4(1-t^3)^2}$	$\frac{1+2t^2+4t^3+2t^4+t^6}{(1-t)(1-t^2)^3(1-t^3)^2}$
$V \oplus S^3V$	$\frac{1+t^2+3t^4+t^6+t^8}{(1-t^2)^3(1-t^4)^2}$	$\frac{1+2t^2+8t^4+2t^6+t^8}{(1-t^2)^3(1-t^4)^2}$
$V \oplus S^4V$	$\frac{(1+t^4)(1+t+t^2+t^3+t^4+t^5+t^6)}{(1-t^2)^3(1-t^3)^2(1-t^5)}$	$\frac{1+t^2+4t^4+t^5+t^7}{(1-t^2)^3(1-t^3)(1-t^5)(1-t^6)}$

Comparing algebras of $\mathrm{O}(n)$-invariants and algebras of $\mathrm{SO}(n)$-invariants

Let $V = \mathbb{C}^n$ denote the standard $\mathrm{GL}(n)$-module and let $W = S^2V$ be the second symmetric power of V. Then

$$H(S(S^2V)^{\mathrm{O}(n)}, t) = H(S(S^2V)^{\mathrm{SO}(n)}, t) = \prod_{i=1}^{n} \frac{1}{1-t^i}.$$

Corollary

For all n we have

$$S(S^2V)^{\mathrm{O}(n)} = S(S^2V)^{\mathrm{SO}(n)}.$$

In particular, $S(S^2V)^{\mathrm{O}(n)}$ is a polynomial algebra.

Remark: The above corollary is trivially true for $n = 2k+1$ since $\mathrm{O}(2k+1) = \mathrm{SO}(2k+1) \times \{\mathrm{Id}, -\mathrm{Id}\}$.

Comparing algebras of $\mathrm{O}(n)$-invariants and algebras of $\mathrm{SO}(n)$-invariants

Next, let us take $W = \Lambda^2 V$, the second exterior power of V. Then

$$H(S(\Lambda^2 V)^{\mathrm{O}(n)}, t) = \prod_{i=1}^{\lfloor \frac{n}{2} \rfloor} \frac{1}{1 - t^{2i}},$$

$$H(S(\Lambda^2 V)^{\mathrm{SO}(n)}, t) = \begin{cases} \frac{1}{1 - t^k} \prod_{i=1}^{k-1} \frac{1}{1 - t^{2i}}, & \text{if } n = 2k, \\ \prod_{i=1}^{k} \frac{1}{1 - t^{2i}}, & \text{if } n = 2k + 1. \end{cases}$$

For $n = 2k + 1$ we have again $S(\Lambda^2 V)^{\mathrm{O}(2k+1)} = S(\Lambda^2 V)^{\mathrm{SO}(2k+1)}$.

Comparing algebras of $\mathrm{O}(n)$-invariants and algebras of $\mathrm{SO}(n)$-invariants

Lemma

If $\dim S^l(W)^{\mathrm{SO}(n)} = 1$ *and if* f *is a generator of* $S^l(W)^{\mathrm{SO}(n)}$ *then* f *or* f^2 *is an* $\mathrm{O}(n)$*-invariant.*

Corollary

Let $n = 2k$. *There exists a generating set* $\{f_{2i}\}_{i=1}^{k-1} \cup \{g\}$ *of* $S(\Lambda^2 V)^{\mathrm{SO}(2k)}$ *(where* $\deg f_j = j$ *and* $\deg g = k$*) such that* $\{f_{2i}\}_{i=1}^{k-1} \cup \{g^2\}$ *is a generating set for* $S(\Lambda^2 V)^{\mathrm{O}(2k)}$. *In particular,* $S(\Lambda^2 V)^{\mathrm{O}(2k)}$ *is a polynomial algebra.*

Comparing algebras of $\mathrm{O}(n)$-invariants and algebras of $\mathrm{SO}(n)$-invariants

Let $W = V \oplus \Lambda^2 V$. Then, when $n = 2k$ we obtain

$$H(S(W)^{\mathrm{O}(2k)}, t) = \prod_{i \leq k} \frac{1}{(1 - t^{2i})^2},$$

$$H(S(W)^{\mathrm{SO}(2k)}, t) = \frac{1}{(1 - t^{2k})(1 - t^k)} \prod_{i \leq k-1} \frac{1}{(1 - t^{2i})^2}.$$

For $n = 2k + 1$ we have

$$H(S(W)^{\mathrm{O}(2k+1)}, t) = \frac{1}{1 - t^{n+1}} \prod_{i \leq k} \frac{1}{(1 - t^{2i})^2},$$

$$H(S(W)^{\mathrm{SO}(2k+1)}, t) = \frac{1}{1 - t^{k+1}} \prod_{i \leq k} \frac{1}{(1 - t^{2i})^2}.$$

Comparing algebras of $\mathrm{O}(n)$-invariants and algebras of $\mathrm{SO}(n)$-invariants

The degrees of the generators of $S(V \oplus \Lambda^2 V)^{\mathrm{SO}(n)}$ (using the fact that $S(V \oplus \Lambda^2 V) = S(V) \otimes S(\Lambda^2 V)$) are as follows (see Schwarz):

- For $S(W)^{\mathrm{SO}(2k)}$ we have generators in degrees

  $$(0,2),(0,4),\ldots,(0,2k-2);(2,0),(2,2),(2,4),\ldots,(2,2k-2);(0,k).$$

- For $S(W)^{\mathrm{SO}(2k+1)}$ we have generators in degrees

  $$(0,2),(0,4),\ldots,(0,2k);(2,0),(2,2),(2,4),\ldots,(2,2k-2);(1,k).$$

Comparing algebras of $\mathrm{O}(n)$-invariants and algebras of $\mathrm{SO}(n)$-invariants

Corollary

Let $n = 2k + 1$. Let $\{f_{2i}, g_{2i}\}_{i=1}^{k} \cup \{h\}$ be a generating set for $S(V \oplus \Lambda^2 V)^{\mathrm{SO}(2k+1)}$ (where $\deg f_j = (0, j)$, $\deg g_j = (2, j-2)$, $\deg h = (1, k)$). Then $\{f_{2i}, g_{2i}\}_{i=1}^{k} \cup \{h^2\}$ is a generating set for $S(V \oplus \Lambda^2 V)^{\mathrm{O}(2k+1)}$. Hence, $S(V \oplus \Lambda^2 V)^{\mathrm{O}(2k+1)}$ is a polynomial algebra.

Corollary

Let $n = 2k$. There exists a generating set $\{f_{2i}\}_{i=1}^{k-1} \cup \{g_{2i}\}_{i=1}^{k} \cup \{h\}$ of $S(V \oplus \Lambda^2 V)^{\mathrm{SO}(2k)}$ (where $\deg f_j = (0, j)$, $\deg g_j = (2, j-2)$, $\deg h = (0, k)$) such that $\{f_{2i}\}_{i=1}^{k-1} \cup \{g_{2i}\}_{i=1}^{k} \cup \{h^2\}$ is a generating set for $S(V \oplus \Lambda^2 V)^{\mathrm{O}(2k)}$. Hence, $S(V \oplus \Lambda^2 V)^{\mathrm{O}(2k)}$ is a polynomial algebra.

The end

Thank you for your attention!