(2, 3)-GENERATION OF THE GROUPS

$PSL_6(q)$

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Common features

- A group $G$ is called $(2, 3)$-generated if $G = \langle x, y \rangle$ for some elements $x$ and $y$ of orders 2 and 3, respectively.
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The theorem of Liebeck-Shalev and Lübeck-Malle states that all finite simple groups, except the symplectic groups $PSp_4(2^m)$, $PSp_4(3^m)$, the Suzuki groups $Sz(2^m)$ ($m$ odd), and finitely many other groups, are $(2, 3)$-generated (see [11])

For the $PSL_n(q)$, 
$(2, 3)$-generation has been proved in the cases $n = 2, q \neq 9$ [8], $n = 3, q \neq 4$ [4], [1], $n = 4, q \neq 2$ [12], [13], [9], $n = 5$, any $q$ [14], $n \geq 5$, odd $q \neq 9$ [2],[3], and $n \geq 13$, any $q$ [10].
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Main theorem

Theorem

The group $\text{PSL}_6(q)$ is $(2, 3)$-generated for any $q$. 
Preliminaries

- $G = \text{SL}_6(q)$, $\overline{G} = G/Z(G) = \text{PSL}_6(q)$, where $q = p^m$ and $p$ is a prime. Set $d = (6, q - 1)$, also $Q = q^5 - 1$ if $q \neq 3, 7$ and $Q = (q^5 - 1)/2$ if $q = 3$ or 7.

- The group $G$ acts naturally on a six-dimensional vector space $V$ over the field $F = GF(q)$ and $\overline{G}$ acts on the corresponding projective space $P(V)$. 
Lemma 1

Let $\overline{M}$ be a maximal subgroup of the group $\overline{G}$. Then either $\overline{M}$ is reducible on the space $P(V)$ or $\overline{M}$ has no element of order $Q/(d, Q)$.
The maximal subgroups of $\text{PSL}_6(q)$ are determined (up to conjugacy) in [5]. In particular, this implies that one of the following holds:

(i) $\bar{M}$ belongs to the family $C_1$ of reducible subgroups of $\bar{G}$;

(ii) $\bar{M}$ is a member of one of the remaining families $C_2, C_3, C_4, C_5, C_8$ of (irreducible) geometric subgroups of $\bar{G}$;

(iii) $\bar{M} \cong \text{PSL}_3(q)$ if $q$ is odd or $\bar{M} \cong \text{PSL}_2(11), A_7, M_{12}, \text{PSL}_3(4).\mathbb{Z}_2, \text{PSU}_4(3)$, or $\text{PSU}_4(3).\mathbb{Z}_2$ for specific values of $p$ and $q$. 
Case 1: $q \neq 2, 4$

Let $\omega \in GF(q^5)^*$, $|\omega| = Q$

$$f(t) = \prod_{i=0}^{4} (t - \omega^{q^i}) = t^5 - \alpha t^4 + \beta t^3 - \gamma t^2 + \delta t - \varepsilon.$$  

Then $f(t) \in F[t]$ and the polynomial $f(t)$ is irreducible over $F$. 
Case 1

**Generators**

\[ x = \begin{pmatrix} -1 & 0 & 0 & \gamma \varepsilon^{-1} & 0 & \gamma \\ 0 & -1 & 0 & \beta \varepsilon^{-1} & 0 & \beta \\ 0 & 0 & 0 & \alpha \varepsilon^{-1} & -1 & \delta \\ 0 & 0 & -1 & \delta \varepsilon^{-1} & 0 & \alpha \\ 0 & 0 & 0 & \varepsilon^{-1} & 0 & 0 \end{pmatrix}, \ x \in G, |x| = 2, \]

\[ y = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}, \ y \in G, |y| = 3. \]
Case 1

\[ z = xy = \begin{pmatrix} 0 & 0 & -1 & 0 & \gamma & \gamma \varepsilon^{-1} \\ -1 & 0 & 0 & 0 & \beta & \beta \varepsilon^{-1} \\ 0 & 0 & 0 & -1 & \delta & \alpha \varepsilon^{-1} \\ 0 & 0 & 0 & 0 & \varepsilon & 0 \\ 0 & -1 & 0 & 0 & \alpha & \delta \varepsilon^{-1} \\ 0 & 0 & 0 & 0 & 0 & \varepsilon^{-1} \end{pmatrix}. \]

The characteristic polynomial of \( z \) is

\[ f_z(t) = (t - \varepsilon^{-1}) f(t) \]

and the characteristic roots \( \varepsilon^{-1}, \omega, \omega q, \omega q^2, \omega q^3, \omega q^4 \) of \( z \) are pairwise distinct. Then, in \( GL_6(q^5) \), \( z \) is conjugate to

\[ \text{diag}(\varepsilon^{-1}, \omega, \omega q, \omega q^2, \omega q^3, \omega q^4) \]

and hence \( z \) is an element of \( G \) of order \( Q \).
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Lemma 2

Let $H = \langle x, y \rangle$, $H \leq G$. The group $H$ acts irreducibly on the space $V$. 
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Let now $q = 2$ or $4$

The element $y \in G$, $|y|=3$ is the same like in Case 1 and

$$x = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & \eta & 0 & \eta^2 \\ 0 & 0 & 0 & \eta & 1 & \eta^2 \\ 0 & 0 & 0 & 0 & 0 & \eta \\ 0 & 0 & 1 & \eta & 0 & \eta^2 \\ 0 & 0 & 0 & \eta^2 & 0 & 0 \end{pmatrix}, \quad x \in G, |x| = 2.$$

Here $\langle \eta \rangle = F^*$. 
\[ z = xy = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & \eta^2 & \eta \\ 0 & 0 & 0 & 1 & \eta^2 & \eta \\ 0 & 0 & 0 & 0 & \eta & 0 \\ 0 & 1 & 0 & 0 & \eta^2 & \eta \\ 0 & 0 & 0 & 0 & 0 & \eta^2 \end{pmatrix} \]

The characteristic polynomial of \( z \) is \( f_z(t) = (t + \eta^2)g(t) \), where
\[
g(t) = t^5 + \eta^2 t^4 + \eta^2 t^3 + \eta^2 t^2 + (\eta^2 + \eta) t + \eta.
\]
It follows that both for \( q = 2 \) and \( q = 4 \) the element \( z \) has order \( q^5 - 1 = Q \).
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Now, in $\overline{G}$, the elements $\overline{x}$, $\overline{y}$, and $\overline{z}$ have orders 2, 3, and $Q/(d, Q)$ in Case 1 ($Q/d$ - Case 2), respectively. So the group $\overline{H} = \langle \overline{x}, \overline{y} \rangle$ has an element of order $Q/(d, Q)$ (or $Q/d$) and $\overline{H}$ is irreducible on $P(V)$ as $H$ is irreducible on $V$ by Lemma 2. Lemma 1 implies that $\overline{H}$ cannot be contained in any maximal subgroup of $\overline{G}$. Thus $\overline{H} = \overline{G}$ and $\overline{G} = \langle \overline{x}, \overline{y} \rangle$ is a $(2, 3)$-generated group.
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