The Periodic Hamming Space and D-hyperoctahedral Groups

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Denote by H_n the *Hamming space* of dimension n.
This space consists of all n-tuples

$$(a_1, \ldots, a_n), \quad a_i \in \{0, 1\}, 1 \le i \le n.$$

The distance d_{H_n} between two such n-tuples is equal to the number of coordinates where they differ. The isometry group $Isom H_n$ of the metric space H_n is isomorphic to the wreath product $W_n = Z_2 \wr S_n$.

The group W_n consists of all pairs $[\sigma, f]$, where $\sigma \in S_n$, $f \in Z_2^{\underline{n}}$, $\underline{n} = \{1, \ldots, n\}$. Denote $f(i) = a_i$, $(1 \leq i \leq n)$. Each pair $[\sigma, f]$ corresponds to a unique sequence $[\sigma; a_1, \ldots, a_n]$. Then the group operation in $Z_2 \wr S_n$ is determined by the equality

$$[\sigma; a_1, \ldots, a_n][\eta; b_1, \ldots, b_n] = [\sigma\eta; a_1 + b_{1^\sigma}, \ldots, a_n + b_{n^\sigma}],$$

where $+$ denotes the addition in Z_2.

The inverse of the element $[\sigma; a_1, \ldots, a_n]$ is the element

$$[\sigma^{-1}; a_{1^{\sigma^{-1}}}, \ldots, a_{n^{\sigma^{-1}}}].$$

Transformation $u = [\sigma; a_1, \ldots, a_n]$ acts on the vector $\bar{t} = (t_1, \ldots, t_n) \in Z_2^n$ according to the rule:

$$t^u = (t_{1^\sigma} + a_1, \ldots, t_{n^\sigma} + a_n).$$

Let $\{0,1\}^{\mathbb{N}}$ be the set of all infinite sequences of elements of the set $\{0,1\}$, i.e. the set of all infinite binary sequences.
Equip $\{0,1\}^{\mathbb{N}}$ with a pseudo-metric $\hat{d}_B$, defined for arbitrary sequences $\mathrm{x} = (x_1, x_2, \ldots)$ and $\mathrm{y} = (y_1, y_2, \ldots)$ from $\{0,1\}^{\mathbb{N}}$ by the equality

$$\hat{d}_B(x,y) = \limsup_{n\to\infty} \frac{1}{n} d_{H_n}((x_1,\ldots,x_n),(y_1,\ldots,y_n)).$$

The pseudo-metric $\hat{d}_B$ defines an equivalence $\sim_{\hat{d}_B}$ on $\{0,1\}^{\mathbb{N}}$, i.e.

$$\mathrm{x} \sim_{\hat{d}_B} \mathrm{y}$$

if and only if

$$\hat{d}_B(\mathrm{x},\mathrm{y}) = 0.$$

Denote by $X_B = \{0,1\}^{\mathbb{N}} / \sim_{\hat{d}_B}$ the quotient set by this equivalence. The function $\hat{d}_B$ induces the metric d_B on X_B.

The metric space (X_B, d_B) is called the Besicovitch space or the Besicovitch-Hamming space
(see F. Blanchard , E. Formenti, P. Kurka, *Cellular Automata in Cantor, Besicovitch and Weil Topological Spaces*, Complex Systems, V. 11, 1997, pp. 107–123,
or A. M. Vershik, *The Pascal automorphism has a continuous spectrum*, Funct. Anal. Appl., V. 45, 2011, pp 173–186).

The metric space (X_B, d_B) is complete, nonseparable, and not locally compact.
We consider a continuum family of compact separable subspaces of the Besicovitch space, naturally parameterized by supernatural numbers.

Let $\mathbb{P}$ be the set of all primes. A *supernatural number* (or Steinitz number) is an infinite formal product of the form

$$\prod_{p \in \mathbb{P}} p^{k_p}$$

where $k_p \in \mathbb{N} \cup \{0, \infty\}$. Denote by $\mathbb{SN}$ the set of all supernatural numbers. The elements of the set $\mathbb{SN} \setminus \mathbb{N}$ are called *infinite supernatural* numbers.

An infinite sequence a $= (a_1, a_2, \ldots)$, $a_i \in B$ is said to be *periodic* if there exists a natural number k such that the equality $a_i = a_{i+k}$ holds for all $i \in \mathbb{N}$. In this case the number k is called a *period* of the sequence a.

A periodic sequence a is called *u-periodic* for some supernatural number u if its minimal period divides u.

Let u be some infinite supernatural number. Denote by $\mathcal{H}(u)$ the subspace of the Besicovitc space (X_B, d_B) consisting of all u-periodic sequences over the set $\{0, 1\}$. We call the metric space $\mathcal{H}(u)$ the *u-periodic Hamming space*.

Proposition

Let u, v be supernatural numbers. Then the spaces $\mathcal{H}(u)$ and $\mathcal{H}(v)$ are isometric iff $u = v$.

Proposition

[P. J. Cameron, S. Tarzi] The completions $\mathcal{H}$ of u-periodic Hamming spaces are independent of choice of u.

Problem

[A. M., Vershik]
Let $(x_1, x_2, \ldots)$ be from $\{0, 1\}^{\mathbb{N}}$ and let x be the equivalence class defined by the sequence $(x_1, x_2, \ldots)$. Is there an algorithm to determine whether a class x belongs to the space $\mathcal{H}$?

A sequence of positive integers $\tau = (m_1, m_2, \ldots)$ is called *divisible* if $m_i | m_{i+1}$ for all $i \in \mathbb{N}$.
Let $\tau = (m_1, m_2, \ldots)$ be an increasing divisible sequence. Denote by $(s_1, s_2, \ldots)$ the sequence of ratios of the sequence τ, i.e.

$$s_1 = m_1, \qquad s_{i+1} = \frac{m_{i+1}}{m_i}, \ i \geq 1.$$

The supernatural number

$$s_1 \cdot s_2 \cdot s_3 \ldots$$

is called the *characteristic of the sequence* τ and denoted by $char(\tau)$.

Assume that T_τ is a spherically homogeneous rooted tree with spherical index $[s_1; s_2; \ldots]$. We consider the boundary ∂T_τ of the tree T_τ, i.e. the set of all infinite simple paths starting at the root.

Define a distance ρ on the set ∂T_τ as

$$\rho_\tau(\gamma_1, \gamma_2) = \begin{cases} \frac{1}{k+1}, & \text{if } \gamma_1 \neq \gamma_2 \\ 0, & \text{if } \gamma_1 = \gamma_2 \end{cases},$$

where k is the length of the common beginning of rooted paths γ_1 and γ_2.

The set of all rooted paths from ∂T_τ passing through a vertex v is denoted by

$$C_v = \{\gamma \in \partial T_\tau \mid v \in \gamma\}$$

and called the *cylindrical set* C_v corresponding to v.

Define the Bernoulli measure μ on the Borel σ-algebra of ∂T_τ by the rule:

$$\mu(C_v) = \frac{1}{n_v},$$

where n_v is the number of vertices of T_τ on the level containing the vertex v.

Denote by $(Homeo\partial T_\tau \cap Aut(\partial T_\tau, \mu))$ the group of all homeomorphisms of boundary ∂T_τ that preserve the measure μ. The group $(Homeo\partial T_\tau \cap Aut(\partial T_\tau, \mu))$ acts on $C(\partial T_\tau, Z_2)$ by generalized translations: given $g \in (Homeo\partial T_\tau \cap Aut(\partial T_\tau, \mu))$ and $h \in C(\partial T_\tau, Z_2)$, put

$$h^g(x) = h(x^g), x \in \partial T_\tau.$$

This action is an automorphism of $C(\partial T_\tau, Z_2)$. Consequently, we can consider the semidirect product $C(\partial T_\tau, Z_2) \rtimes (Homeo\partial T_\tau \cap Aut(\partial T_\tau, \mu))$.

Theorem (B. Ol., V. Sushchansky, 2013)

Let u be a supernatural number and let $\tau = (m_1, m_2, \ldots)$ be a strictly increasing divisible sequence of positive integers with $char(\tau) = u$. The isometry group $Isom\mathcal{H}(u)$ of the u-periodic Hamming space $\mathcal{H}(u)$ is isomorphic as a transformation group to the semidirect product

$$C(\partial T_\tau, Z_2) \rtimes (Homeo\partial T_\tau \cap Aut(\partial T_\tau, \mu)),$$

where T_τ is the spherically homogeneous rooted tree and μ is the Bernoulli measure on the σ-algebra of clopen sets of ∂T_τ.

Theorem (B. Ol., V. Sushchansky, 2013)

The isometry group $Isom\mathcal{H}$ *of the completion of the space* $H(u)$ *of u-periodic Hamming space is isomorphic as a transformation group to the the semidirect product*

$$Fun_{\mu}(\partial T_{\tau}, Z_2) \rtimes Aut(\partial T_{\tau}, \mu),$$

where T_{τ} *is the spherically homogeneous rooted tree and* μ *is the Bernoulli measure on the* σ*-algebra of clopen sets of* ∂T_{τ}*.*

Problem

[P. J. Cameron, S. Tarzi] What is the structure of the isometry group of the periodic Hamming space over some finite alphabet? What is the structure of the isometry group of its completion?

Define an embedding of the permutation group $(W_{m_i}, Z_2^{m_i})$ into the permutation group $(W_{m_{i+1}}, Z_2^{m_{i+1}})$ by a pair of maps

$$h_i : W_{m_i} \to W_{m_{i+1}}, \qquad \delta_i : Z_2^{m_i} \to Z_2^{m_{i+1}},$$

such that for each $i \in \mathbb{N}$ we have:

1. $h_i([\sigma; a_1, \dots, a_{m_i}]) = [\theta^{s_{i+1}}\sigma; (\underbrace{a_1, \dots, a_{m_i}, \dots, a_1, \dots, a_{m_i}}_{m_i \cdot s_{i+1}})]$,
2. $\delta_i(t_1, \dots, t_{m_i}) = (\underbrace{t_1, \dots, t_{m_i}, t_1, \dots, t_{m_i}, \dots, t_1, \dots, t_{m_i}}_{m_i \cdot s_{i+1}})$,

where $\sigma \in S_{m_i}$, $(a_1, \dots, a_{m_i}), (t_1, \dots, t_{m_i}) \in Z_2^{m_i}$ and

$$\theta^{s_{i+1}}\sigma = \left(\begin{array}{ccc|c|ccc} 1 & \dots & m_i & \dots & (s_{i+1}-1)m_i + 1 & \dots & s_{i+1}m_i \\ 1^\sigma & \dots & m_i^\sigma & \dots & (s_{i+1}-1)m_i + 1^\sigma & \dots & (s_{i+1}-1)m_i + m_i^\sigma \end{array} \right)$$

The increasing divisible sequence $\tau = (m_1, m_2, \ldots)$ determines the direct spectrum

$$\langle (W_{m_i}, Z_2^{m_i}), F_i \rangle_{i \in \mathbb{N}}. \tag{1}$$

of hyperoctahedral groups $(W_{m_i}, Z_2^{m_i})$.
We call the direct limit of directed system (1) the *D-hyperoctahedral group* corresponding to the sequence τ and denote it by $W(\tau)$.

Equip the group of homeomorphisms $Homeo\partial T_\tau$ and the group $C(\partial T_\tau, Z_2)$ with the metrics

$$\sigma_\tau(f,g) = \max_{x \in \partial T_\tau} \rho_\tau(x^g, x^f), \qquad \text{for all } f,g \in Homeo\partial T_\tau,$$

$$\hat{\sigma}_\tau(h,t) = \begin{cases} 1, & \text{if } h \neq t \\ 0, & \text{if } h = t \end{cases}, \qquad \text{for all } h,t \in C(\partial T_\tau, Z_2).$$

Theorem (B.Ol., V. Sushchansky, 2013)

The isometry group $\mathcal{H}(u)$ *of the u-periodic Hamming space* $\mathcal{H}(u)$ *is the closure of D-hyperoctahedral group* $W(\tau)$, *char*$\tau = u$, *regarded as a subgroup of* $C(\partial T_\tau, Z_2) \rtimes Homeo\partial T_\tau$ *in the Tychonoff product of topologies induced by the metrics* σ_τ *and* $\hat{\sigma}_\tau$.

Theorem (B.Ol., V. Sushchansky, 2014)

Let τ_1, τ_2 be increasing divisible sequences. The groups $W(\tau_1)$ and $W(\tau_2)$ are isomorphic iff $char \tau_1 = char \tau_2$.

Let $B(u)$ be the subgroup of $W(u)$ consisting of elements of the form

$$[e, a_1, a_2, \ldots], \quad a_i \in Z_2.$$

We denote by $B_0(u)$ the subgroup of sequences $[e; a_1, a_2, \ldots] \in B(u)$ such that for certain number n, $n \mid u$, the equality

$$a_1 + a_2 + \ldots a_n = 0(mod 2)$$

holds.

Denote by C the subgroup of $B(u)$ containing only sequences $[e, 0, 0, \ldots]$ and $[e, 1, 1, \ldots]$. Define subgroups of the group $W(u)$ by the rule:

$$U = S(r) \cdot \mathcal{B}_0(r), \quad V = gp(W'(r), [(1, 2); 1, 0, \ldots]), \quad H = A(r) \cdot \mathcal{B}(r),$$

where $gp(X)$ is the subgroup generated by the set X.

Theorem

Let u be an infinite supernatural number.
1) If $2^{\infty} \nmid r$, then the lattice of normal subgroups of the group $W(r)$ has the form depicted on Fig. 1 in case $2 \mid r$ and the form depicted on Fig. 2 in case $2 \nmid r$.
2) If $2^{\infty} \nmid r$ then the lattice of normal subgroups of the group $W(r)$ has the form depicted on Fig. 3.

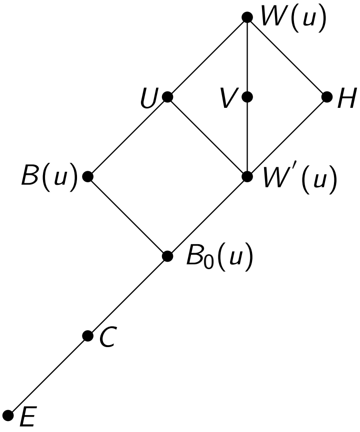

Figure:

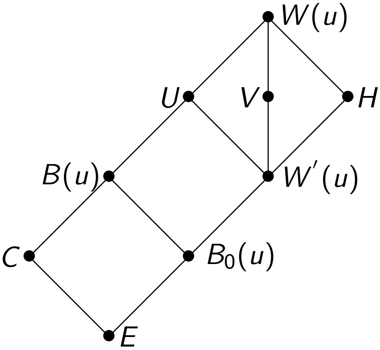

Figure:

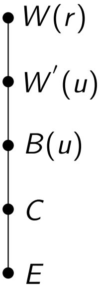

Figure:

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