Algebra and Logic Seminar
Institute of Mathematics and Informatics
Bulgarian Academy of Sciences, Sofia

Derivations of skew Ore polynomial semirings

Dimitrinka Vladeva
University of Forestry, Sofia, Bulgaria
d_vladeva@abv.bg

December 18, 2020

This project follows

D. Vladeva (2020) **Derivations of skew Ore polynomial semirings**, Communications in Algebra, 48, No 11, 4718-4731, DOI: 10.1080/00927872.2020.1769641.

Note that if in this paper we choose the derivation δ from the the Ore commutation rule to be a zero derivation we obtain most of the results in

D. Vladeva (2020) **Derivations of polynomial semirings**, International Journal of Algebra and Computation, 2020, Vol.30, No.1, 1 - 12, DOI: 10.1142/S0218196719500620.

We show that multiplying each polynomial by x on left is a derivation and construct idempotent semiring consisting of derivations of a skew polynomial semiring. Generalized hereditary derivations are derivations acting only over the coefficients of the polynomial. We construct an S-derivation in the classical sense of Jacobson and show that an arbitrary δ - derivation can be represented by a generalized hereditary derivation and an S-derivation.

1. Introduction

A semiring is a 4-tuple $(S, +, ., 0)$ such that $(S, +, 0)$ is a commutative monoid, $(S, .)$ is a semigroup and distributive laws are fulfilled. Semiring for which the operation of addition is idempotent is called an additively idempotent. The interest in additively idempotent semirings arises from fields of optimization theory, the theory of discrete-event dynamical systems, automata theory, formal language theory, theoretical computer science and theoretical physics.

In 2001 G. Thierrin first considers derivation in semiring S as in the rings – a map $\delta : S \to S$ such that $\delta(a + b) = \delta(a) + \delta(b)$ and $\delta(ab) = \delta(a)b + a\delta(b)$.
From the Ore commutation rule $xa = ax + b$ for polynomial rings follows that $b = \delta(a)$, where δ is a derivation in the ground ring. But for polynomial semirings the same rule is considered if we suppose that δ is a derivation. Firstly we show constructions of two types of derivations of the ground additively idempotent semiring.

Let S be an arbitrary (not necessarily commutative) additively idempotent semiring and $C(S)$ be the center of S. The element $s \in S$ is called left (right) Ore element if for all $a \in S$ there is $b \in S$ such that
$sa = as + b \ \ (as = sa + b)$ or equivalently $sa \geq as \ (as \geq sa)$.

Proposition 1. *Let s be a left (right) Ore element of the semiring S. The map $\delta_s^\ell : S \to S$ ($\delta_s^r : S \to S$) such that $\delta_s^\ell(a) = sa$ ($\delta_s^r(a) = as$) for any $a \in S$ is a derivation of S.*

Now we consider $S[x]$ to be a noncomutative additively idempotent semiring of polynomials $a_0 + a_1 x + \cdots + a_m x^m$, $a_i \in S$ for $i = 0, \ldots, m$ such that

$$xa = ax + \delta(a), \tag{1}$$

where $a, \delta(a) \in S$ and δ is a derivation of S. For example $\delta = \delta_s^\ell$, where s is a left Ore element of S or $\delta = \delta_s^r$, where s is a right Ore element of S.

Proposition 2. *Let $\delta : S \to S$ be the derivation of S, used in (1). Let $\delta_{\mathrm{her}} : S[x] \to S[x]$ be the map such that*

$$\delta_{\mathrm{her}}(P(x)) = \delta(a_0) + \delta(a_1)x + \cdots + \delta(a_m)x^m,$$

where $P(x) = a_0 + a_1x + \cdots + a_mx^m \in S[x]$. Then δ_{her} is a derivation of $S[x]$.

For an arbitrary $P(x) \in S[x]$ we denote

$$d_x(P(x)) = x\,P(x). \tag{2}$$

From (2) and the proof of the Proposition 2. it follows

$$d_x(P(x)) = \delta_{\mathrm{her}}(P(x)) + P(x)x. \tag{3}$$

From (3) we have

Proposition 3. *The map $d_x : S[x] \to S[x]$ such that $d_x(P(x)) = xP(x)$ for any polynomial $P(x) \in S[x]$ is a derivation in $S[x]$.*

Lemma 1. *The map $d_x^k : S[x] \to S[x]$ such that $d_x^k(P(x)) = x^k P(x)$ for any polynomial $P(x) \in S[x]$, where $k \in \mathbb{N}$ is a derivation in $S[x]$.*

Let $C(S)$ be the center of the semiring S, 0 the zero of S and $Co_\delta(S)$ the subsemiring of S, consisting the constants under the derivation δ that is the elements $s \in S$ such that $\delta(s) = 0$. In the general case $\Gamma(S) = C(S) \cap Co_\delta(S)$ is a nontrivial subsemiring of S.

Lemma 2. *The map $d_{\alpha x^k} : S[x] \to S[x]$, where $\alpha \in \Gamma(S)$, $k \in \mathbb{N}$ such that $d_{\alpha x^k}(P(x)) = \alpha x^k P(x)$ for an arbitrary polynomial $P(x) \in S[x]$ is a derivation in $S[x]$.*

Let us denote

$$\Gamma(S)[d_x] = \left\{ \alpha_0 \mathrm{i} + \cdots + \alpha_k d_x^k \;\middle|\; \alpha_i \in \Gamma(S), i = 0, \ldots, k,\ k \in \mathbb{N} \right\}.$$

Note that the identity map i is a derivation in any idempotent semiring.

Theorem 1. *Each element of the commutative additively idempotent semiring $\Gamma(S)[d_x]$ is a derivation of $S[x]$.*

Let $P(x) = a_0 + a_1x + \cdots + a_mx^m$ be an arbitrary polynomial. By combinatorial reasoning we can express the coefficient of the polynomial $d_x^k(P(x))$ by a_i, $i = 0, \ldots, m$, $k \in \mathbb{N}$: $d_x^k(P(x)) =$

$$= \sum_{i=0}^{m} \left(a_ix^{i+k} + \delta(a_i)x^{i+k-1} + \delta^2(a_i)x^{i+k-2} + \cdots + \delta^k(a_i)x^i \right).$$

This result shows that all of these coefficients are sums of $\delta^j(a_i)$, where $i = 1, \ldots, m$, $j = 1, \ldots, k$. Hence the coefficients of the polynomial $d(P(x))$, where d is an arbitrary derivation from $\Gamma(S)[d_x]$ are linear combinations of $\delta^j(a_i)$, $i = 0, \ldots, m$ and $j \in \mathbb{Z}$, $j \geq 0$, with coefficients from the semiring $\Gamma(S)$.
The ideal I of the semiring S is called a δ-ideal of S if $\delta(I) \subseteq I$, where δ is the derivation considered in (1).
Corollary 1. *Let I be a a δ-ideal of S. Then $IS[x]$ is a d-ideal of $S[x]$ for any derivation $d \in \Gamma(S)[d_x]$.*

3. Representation of derivations

The derivations considered in the previous section has an important property.
Let $a \in S$. From (1) we have $d_x(a) = \delta(a) + ax$ and $d_x^2(a) = x(\delta(a) + ax) = \delta^2(a) + \delta(a)x + ax^2$. By induction it follows that

$d_x^k(a) = \delta^k(a) + \cdots + \delta(a)x^{k-1} + ax^k$ for an arbitrary positive integer k. Thus for a derivation

$$d = \alpha_0 + \alpha_1 d_x + \cdots + \alpha_k d_x^k \in \Gamma(S)[d_x],$$

it follows that $d(a) =$

$$= \alpha_0 a + \cdots + \alpha_k \delta^k(a) + (\alpha_1 a + \cdots + \alpha_k \delta^{k-1}(a))x + \cdots + \alpha_k a x^k.$$

The polynomial $d(a)$ so obtained has the following property: $d(\delta(a)) = \delta_{\mathrm{her}}(d(a))$.

A derivation $d : S[x] \to S[x]$ such that $d(\delta(a)) = \delta_{\mathrm{her}}(d(a))$ for any $a \in S$ is called a δ - derivation.

Note that an arbitrary S - derivation in sense of Jacobson is evidently a δ - derivation. So, there are sufficiently many δ - derivations.
From the definition, it follows that if d_1 and d_2 are δ - derivations then $d_1 + d_2$ also is a δ - derivation.
Our aim is to find a representation of all δ - derivations in the semiring $S[x]$. We define a map $\partial : S[x] \to S[x]$ such that $\partial(P(x)) = a_1 + a_2x + \cdots + a_mx^{m-1}$ for each polynomial $P(x) = a_0 + a_1x + a_2x^2 + \cdots + a_mx^m \in S[x]$.
Proposition 4. *For an arbitrary polynomial $P(x) \in S[x]$, $\alpha \in S$, $k, \ell, m \in \mathbb{N}$, it follows:*
(a) $\partial(\alpha P(x)) = \alpha\partial(P(x))$;
(b) $\partial(\delta^k_{\mathrm{her}}(P(x)) = \delta^k_{\mathrm{her}}(\partial(P(x))$;
(c) $\partial(P(x)x^\ell) = P(x)x^{\ell-1}$;
(d) $\partial(P(x))x^m + P(x)x^{m-1} = P(x)x^{m-1}$.

Proposition 5. *The map* $\partial : S[x] \to S[x]$ *such that* $\partial(P(x)) = a_1 + a_2x + \cdots + a_mx^{m-1}$ *for each polynomial* $P(x) = a_0 + a_1x + a_2x^2 + \cdots + a_mx^m \in S[x]$ *is a derivation in* $S[x]$.

Let $d : S[x] \to S[x]$ be an arbitrary δ - derivation. Let $P(x) \in S[x]$, $P(x) = a_0 + a_1x + \cdots + a_mx^m$, where $a_i \in S$ for $0 \le i \le m$. Note that $d(a_i)$ for each $0 \le i \le m$ are in general arbitrary polynomials. Now we define a map

$$\Delta_d : S[x] \to S[x],$$

such that

$$\Delta_d(P(x)) = d(a_0) + d(a_1)x + \cdots + d(a_m)x^m. \tag{4}$$

This map Δ_d satisfies

Proposition 6. *For each* δ *- derivation* $d : S[x] \to S[x]$ *the map* Δ_d*, defined by (4) is a derivation.*

The derivation Δ_d is called the generalized hereditary derivation, generated by the δ - derivation d.

Let Δ_{d_1} and Δ_{d_2} be generalized hereditary derivations. For an arbitrary polynomial $P(x) = a_0 + a_1 x + \cdots + a_m x^m$, where $a_i \in S$ for $0 \leq i \leq m$ we obtain

$$\Delta_{d_1+d_2}(P(x)) = (d_1+d_2)(a_0)+(d_1+d_2)(a_1)x+\cdots+(d_1+d_2)(a_m)x^m$$

$$= \Delta_{d_1}(P(x)) + \Delta_{d_2}(P(x)).$$

Hence $\Delta_{d_1} + \Delta_{d_2} = \Delta_{d_1+d_2}$. Thus the set of generalized hereditary derivations is an additive semigroup. We denote this semigroup by $\mathcal{D}_{gh}$.

The last theorem is the main result of the paper. We prove that an arbitrary δ - derivation can be represented by a generalized hereditary derivation and the derivation ∂.

Theorem 2. *Let S be an additively idempotent semiring. Assume that for the multiplication in the polynomial semiring $S[x]$ the equality (1) holds. Then for an arbitrary δ - derivation $D : S[x] \to S[x]$ and polynomial $P(x) \in S[x]$ there exists a generalized hereditary derivation $\Delta_D \in \mathcal{D}_{gh}$ such that*

$$D(P(x)) = \Delta_D(P(x)) + \partial(P(x))\, D(x).$$

Thank you!