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New examples of fundamental algebras

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Preliminaries

We fix a field K of characteristic 0 and consider the free associative algebra (the algebra of polynomials in noncommuting variables)

$$K\langle X\rangle = K\langle x_1, x_2, \ldots\rangle.$$

The polynomial $f(x_1, \ldots, x_n) \in K\langle X\rangle$ is a polynomial identity for an algebra R if

$$f(r_1, \ldots, r_n) = 0 \text{ for all } r_1, \ldots, r_n \in R.$$

The algebra is called PI-algebra if it satisfies a nontrivial polynomial identity.

The set $T(R)$ of all polynomial identities of R is a T-ideal, i.e. it is an ideal of $K\langle X\rangle$ which is invariant under all endomorphisms of $K\langle X\rangle$. The factor algebra

$$F(R) = K\langle X\rangle / T(R)$$

is called a relatively free algebra (of countable rank) in the variety of associative algebras $\text{var}(R)$ generated by R.

Let

$$P_n = \mathrm{span}\{x_{\sigma(1)} \cdots x_{\sigma(n)} \mid \sigma \in S_n\}$$

be the multilinear component of degree n in $K\langle X\rangle$. (Here S_n is the symmetric group of degree n.) If R is a PI-algebra then $T(R)$ is generated as a T-ideal by its multilinear elements $\bigcup_{n\geq 1}(P_n \cap T(R))$.

From quantitative point of view it is more convenient to study the factor spaces

$$P_n(R) = P_n/(P_n \cap T(R)) \text{ instead of } P_n \cap T(R), n = 1, 2, \ldots.$$

The symmetric group S_n acts from the left on P_n by

$$\sigma : x_{i_1} \cdots x_{i_n} \to x_{\sigma(i_1)} \cdots x_{\sigma(i_n)}, \quad x_{i_1} \cdots x_{i_n} \in P_n, \sigma \in S_n,$$

and as an S_n-module $P_n \cong KS_n$ (the group algebra of S_n as a left S_n-module). Then $P_n(R)$ is an S_n-factor module of P_n.

The irreducible representations of S_n are described with partitions of n

$$\lambda = (\lambda_1, \ldots, \lambda_k) \vdash n, \lambda_1 \geq \cdots \geq \lambda_k \geq 0, \lambda_1 + \cdots + \lambda_k = n,$$

with S_n-character χ_λ. Then

$$\chi_n(R) = \chi_{S_n}(P_n(R)) = \sum_{\lambda \vdash n} m_\lambda(R)\chi_\lambda$$

is the n-th cocharacter of R where $m_\lambda(R) \in \mathbb{N}_0$ is the multiplicity of χ_λ in $\chi_n(R)$.

Kemer developed the structure theory of T-ideals in the spirit of the theory of ideals of finitely generated commutative algebras. He used it to solve the Specht problem for the finite basis property of the polynomial identities of associative algebras. One of the main results in the theory of Kemer is that every finitely generated PI-algebra is PI-equivalent (i.e. has the same polynomial identities) to a finite dimensional algebra. See his book for the account.

A.R. Kemer, Ideals of Identities of Associative Algebras, Translations of Math. Monographs 87, AMS, Providence, RI, 1991.

A special place in the theory of Kemer has a class of finite dimensional algebras called by Procesi fundamental.

C. Procesi, The geometry of polynomial identities (Russian translation), Izv. Ross. Akad. Nauk, Ser. Mat. 80 (2016), No. 5, 103-152. English original: Izv. Math. 80 (2016), No. 5, 910-953.

Let R be a finite dimensional algebra. By the Wedderburn-Malcev theorem

$$R = \overline{R} + J, \text{ where } \overline{R} = R_1 \oplus \cdots \oplus R_m$$

is the semisimple component of R presented as a direct sum of its simple components R_j and J is the Jacobson radical. Then the (t,s)-index $\mathrm{Ind}_{t,s}(R)$ of R is equal to (t_R, s_R) where $t_R = \dim(\overline{R})$ and the radical J is nilpotent of index $s_R + 1$, i.e. $J^{s_R} \neq 0$ and $J^{s_R+1} = 0$.

Consider the free product $\overline{R} * K\langle X_s\rangle$, where $K\langle X_s\rangle$ is the free unitary associative algebra of rank $s = s_R$, and factorize $\overline{R} * K\langle X_s\rangle$ first modulo the ideal generated by all polynomial identities of R and then modulo the ideal of elements of degree $\geq s$ in the variables X_s. Let $R^{(0)}$ be the obtained factor algebra. Let also

$$R^{(j)} = (R_1 \oplus R_{j-1} \oplus R_{j+1} \oplus R_m) + J, \quad j = 1, \ldots, m.$$

The finite dimensional algebra R is called fundamental, if it is not PI-equivalent to the direct sum of the algebras $R^{(0)}, R^{(1)}, \ldots, R^{(m)}$ defined above.

Typical examples of fundamental algebras are the block triangular matrices.
A. Giambruno, M. Zaicev, Minimal varieties of algebras of exponential growth, Adv. Math. 174 (2003), 310-323.

Fundamental algebras are the building blocks used to generate any variety of finite basic rank (i.e. a variety generated by a finitely generated algebra. Every finitely generated PI-algebra is PI-equivalent to a finite direct sum of fundamental algebras.

A characterization of fundamental algebras in terms of the cocharacter sequence was given by Giambruno, Polcino Milies and Zaicev.

A. Giambruno, C. Polcino Milies, M. Zaicev, A characterization of fundamental algebras through S_n-characters, J. Algebra 541 (2020), 51-60.

Theorem. *Let R be a finite dimensional algebra of (t,s)-index (t_R, s_R) and let*

$$\chi_n(R) = \sum_{\lambda \vdash n} m_\lambda \chi_\lambda, \quad n = 1, 2, \ldots,$$

be the cocharacter sequence of R. The algebra R is fundamental if and only if for all sufficiently large n there is a partition $\lambda = (\lambda_1, \ldots, \lambda_n) \vdash n$ such that $\lambda_{t_R+1} + \cdots + \lambda_n = s_R$ and $m_\lambda \neq 0$.

In other words, the algebra R is fundamental if for all sufficiently large n there is an irreducible S_n-character χ_λ which participate with a nonzero multiplicity in the n-th cocharacter $\chi_n(R)$ of R and such that the corresponding Young diagram has s_R boxes below the t_R-th row.

Using their characterisation of fundamental algebras Giambruno, Polcino Milies and Zaicev showed that *the Grassmann (or exterior) algebra E_d of a d-dimensional vector space is fundamental if and only if the underlying vector space is of even dimension.*

Sketch of the proof. The Grassmann algebra E_d is generated by the set $\{e_1, \dots, e_d\}$ and has defining relations

$$e_ie_j + e_je_i = 0, \quad i, j = 1, \dots, d.$$

It is well known that $T(E_{2p}) = T(E_{2p+1})$
O.M. Di Vincenzo, A note on the identities of the Grassmann algebras, Boll. Un. Mat. Ital. (7) A-5 (1991), No. 3, 307-315.
and

$$\chi_n(E_{2p}) = \chi_{(n)} + \chi_{(n-1,1)} + \cdots + \chi_{(n-2p,1^{2p})}, \quad n = 1, 2, \dots$$

(and $\chi_{(n-k,1^k)}$ does not participate as a summand in $\chi_n(E_{2p})$ if $n \leq k$).

The semisimple component is one-dimensional (spanned by 1), $J = J(E_{2p})$ is generated by $e_1, \dots, e_{2p}$ and $J^{2p} \neq 0$, $J^{2p+1} = 0$. Hence $(s,t) = (2p,1)$. Since the Young diagram of the partition $(n-2p, 1^{2p})$ has $2p$ boxes below the first row, we obtain that E_{2p} is fundamental.
The polynomial identities of E_{2p+1} are the same as the polynomial identities of E_{2p}. But $J^{2p+1}(E_{2p+1}) \neq 0$ and the n-th cocharacter of E_{2p+1} does not contain irreducible summands with Young diagrams with $2d+1$ boxes below the first row. This shows that E_{2p+1} is not fundamental.

Tensor product of Grassmann algebras

Our first main result is the following.

Theorem. *The tensor product of finite dimensional Grassmann algebras* $E_{d_1} \otimes \cdots \otimes E_{d_k}$ *is fundamental if and only if all underlying* d_i*-dimensional vector spaces are of even dimension,* $i = 1, \ldots, k$.

Idea of the proof. We use the same arguments as in the case of E_d. The semisimple component of the tensor product $E_{d_1} \otimes \cdots \otimes E_{d_k}$ is one-dimensional and the radical is nilpotent of class $d_1 + \cdots + d_k + 1$. When all d_i are even for sufficiently large n there is a irredusible character χ_λ which participates in $\chi_n(E_{d_1} \otimes \cdots \otimes E_{d_k})$ such that partition $\lambda = (\lambda_1, \dots, \lambda_m)$ satisfies $\lambda_2 + \cdots + \lambda_m = d_1 + \cdots + d_k$ and there are no characters in $\chi(E_{d_1} \otimes \cdots \otimes E_{d_k})$ with $\lambda_2 + \cdots + \lambda_m = d_1 + \cdots + d_k + 1$. If some d_i is odd, then the polynomial identities of $E_{d_1} \otimes \cdots \otimes E_{d_k}$ are the same as those of $\chi(E_{d_1} \otimes \cdots \otimes E_{d_i - 1} \otimes \cdots \otimes E_{d_k})$ and there are no irreducible characters χ_λ in $\chi_n(E_{d_1} \otimes \cdots \otimes E_{d_k})$ with $\lambda_2 + \cdots + \lambda_m = d_1 + \cdots + d_k$. This implies that $E_{d_1} \otimes \cdots \otimes E_{d_k}$ is not fundamental.

In the proof we use also some old results:

- By Leron and Vapne that *If* A_1, A_2, B_1, B_2 *are algebras with the property* $T(A_1) = T(A_2)$ *and* $T(B_1) = T(B_2)$ *then* $T(A_1 \otimes B_1) = T(A_2) \otimes B_2)$.
  U. Leron, A. Vapne, Polynomial identities of related rings, Isr. J. Math. 8, 127-137.
- By the speaker for the relations between the cocharacters of the polynomial identities of a unitary algebra with the cocharacters of its proper polynomial identities.
  V. Drensky, Codimensions of T-ideals and Hilbert series of relatively free algebras, J. Algebra 91 (1984), 1-17.

Triangular products

Let R_1 and R_2 be two algebras and let M be an (R_1, R_2)-bimodule. Then the algebra of upper triangular block matrices

$$R = \begin{pmatrix} R_1 & M \\ 0 & R_2 \end{pmatrix}$$

has the property $T(R) \supseteq T(R_1)T(R_2)$. As a corollary of a theorem of Lewin in the form stated by Giambruno and Zaicev (Corollary 1.8.2 in their book) if R_1 and R_2 are relatively free algebras of infinite rank and M is a free (R_1, R_2)-bimodule with infinite set of free generators, then $T(R) = T(R_1)T(R_2)$.

J. Lewin, A matrix representation for associative algebras. I, II, Trans. Am. Math. Soc. 188 (1974), 293-308, 309-317.

A. Giambruno, M. Zaicev, Polynomial Identities and Asymptotic Methods, Mathematical Surveys and Monographs, vol. 122, AMS, Providence, RI, 2005.

Our second main result is the following.

Theorem. *If* R_1 *and* R_2 *are fundamental algebras, then there is a fundamental algebra* R *such that* $T(R) = T(R_1)T(R_2)$. *The algebra* R *is constructed as a triangular product in terms of* R_1 *and* R_2.

The proof is transparent but uses many facts from the theory of PI-algebras established in the last 50 years:

- The theorem of Lewin about the polynomial identities of the triangular product of relatively free algebras.
- The theorem of Berele and Regev about the cocharacters of products of T-ideals.
  A. Berele, A.Regev, Codimensions of products and of intersections of verbally prime T-ideals, Isr. J. Math. 103 (1998), 17-28.
- A generic construction in the spirit of the construction of generic matrices. Starting with a PI-algebra R we embed the relatively free algebra $F(R)$ in the tensor product $K[X] \otimes R$.
- The theorem of Kemer that the relatively free algebra $F_d(R)$ can be embeded into a matrix algebra over the extension of the base field.
- Other well known facts for varieties of algebras.