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Finitely generated axial algebras

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We investigate generation of (perhaps nonassociative) algebras by idempotents, the main question being, "Under what conditions must an algebra generated by idempotents be finite dimensional?" The monoid algebra generated by two "free" idempotents is infinite dimensional; also Krupnik showed that 3 idempotents can generate arbitrarily large dimensional algebras (and thus infinite dimensional algebras via an ultraproduct argument), so some restriction is needed.

Motivated by group theory and associated schemes, considerable interest has arisen in studying *axes*.

Let $a \in A$ be a semisimple idempotent and $F \ni \lambda, \delta \notin \{0, 1\}$.

Notation 1. *1. For $y \in A$, write L_y for the left multiplication map $z \mapsto yz$ and R_y for the left multiplication map $z \mapsto zy$. We write T when it can be L or R. Thus $T_y(z) \in \{yz, zy\}$.*

2. Write A_η for the left η-eigenspace of A with respect to a. We assume throughout that a is absolutely primitive, in the sense that $A_1 = \mathbb{F}a$. We say that a is a left axis of type (gl) *if $(L_a - \lambda)(L_a - 1)L_a = 0$ and*

$$A = \overbrace{A_0(a) \oplus A_1(a)}^{0\text{-}part} \oplus \overbrace{A_\lambda(a)}^{1\text{-}part},$$

where this is a noncommutative (i.e. two sided) $\mathbb{Z}_2$-grading of A.

A right axis of type (δ) is defined analogously, where R_a satisfies $(R_a-\delta)(R_a-1)R_a = 0$.

An (2-sided) axis of type (λ, δ) is a left axis of type (λ) which is also a right axis of type (δ), satisfying $L_aR_a = R_aL_a$, i.e., $a(ya) = (ay)a$ for all $y \in A$.

3. An axis of Jordan type (λ, δ) is an axis of type (λ, δ) for which $A_{\lambda,0} = A_{0,\delta} = 0$.

Our overriding goal is to develop a computable theory, and it turns out that the axioms for noncommutative axes are enough to develop this theory. The grading of A with respect to a is otherwise known as its *fusion rules*, a special case of general fusion rules. A is a *(noncommutative) axial algebra (resp. of Jordan type)* if it is generated by a set X of axes (resp. of Jordan type), but not necessarily of the same type. If all the axes of X have the same type (λ, δ), we say that A itself has *type* (λ, δ). The *classical case*, is when $A_0^2 = A_0$. The case treated in the literature often is the commutative classical case. It turns out that the axioms for noncommutative axes are enough to develop our theory. We say that A is *finitely generated* if it is generated (as an algebra) by a finite set of axes.

Axes in flexible algebras are always of Jordan type, and we shall see that although they often are commutative, there are counterexamples.

Definition 1. Assume A contains a set X of axes. A *monomial* of *length* n is a product of n axes. We denote by B_X the subspace of A spanned by **all possible monomials** in the elements of X such that in each monomial each element of X appears **at most once**.

Hall, Rehren, and Shpectorov classified all commutative axial algebras $A = A(X)$ for $|X| = 2$, and in each case $A = B(X)$. We recall the very nice main theorem of Gorshkov and Staroletov, which we use as a launching pad:

Suppose that $A = A(X)$ is a commutative axial algebra for $|X| = 3$. Then $A = B_X$.

This leads to the general case:

Main Conjecture.

1. *Suppose that $A = A(X)$ is a finitely generated axial algebra. Then A is finite dimensional.*
2. *When A is commutative, then $A = B_X$.*

Hall, Rehren, and Shpectorov solved (1) for commutative axial algebras of Jordan type $\lambda \neq \frac{1}{2}$, although the proof relies on the classification of simple groups and the given bound of the dimension is rather high. Our objective in this project is to prove that all finitely generated axial algebras are finite dimensional, with a direct proof providing a sharper bound on the dimension.

Our first main result is that Main Conjecture (1) holds even in the noncommutative setting, for axial algebras generated by three axes of Jordan type. Our method is to build an associative algebra from the adjoint algebra of A, which has a strictly larger dimension which nevertheless also is finite dimensional. This also enables us to handle the case when X does not generate A. When A is commutative we recover the Gorshkov-Staroletov Theorem.

For any axes not of Jordan type, there are counterexamples to Main Conjecture (2) even for axial algebras generated by two axes, since $(ab)a$need not be in $B_{\{a,b\}}$. Furthermore, for axial algebras generated by four axes of Jordan type, even in the commutative case, it appears likely that $(ab)((ac)d) \notin B_{\{a,b,c,d\}}$.

In proving these results and more generally attacking Main Conjecture (1), a major tool is the following noncommutative version of a result of Hall, Rehren, and Shpectorov.

Suppose A is generated by a set of axes X. Let $V \subseteq A$ be a subspace containing X such that $xV \subseteq V$ and $Vx \subseteq V$, for all $x \in X$. Then $V = A$.