International Conference
"Trends in Combinatorial Ring Theory"
September 20-24, 2021, Sofia, Bulgaria

Unpublished results of A.I. Shirshov

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September 24, 2021

Introduction

 

Anatoly Illarionovich Shirshov

In 1964 A.I. Malcev wrote a recommendation for the election of A.I. Shirshov for corresponding member of the Soviet Academy of Sciences. There he described the main results of Shirshov. He especially emphasized the works on Lie algebras and the paper on special Jordan algebras. Malcev also compared the results of Shirshov with results of Jacobson and Witt.

There are still important results of Shirshov which are not published in papers or monographs. Many of them are in his Ph.D. Thesis (in Russian) defended in 1953 at the Moscow State University and which can be download in http://math.nsc.ru/LBRT/a1/ssylki.html

The theorem on subalgebras of free Lie algebras

Every subalgebra of the free Lie algebra is free.

The theorem is included in the Ph.D. Thesis and is published in Mat. Sbornik, 1953. Now it is known as the theorem of Shirshov-Witt. Here Shirshov uses the Hall basis of the free Lie algebra and the *Shirshov elimination*. (Compare this with the Lazard elimination, 1960.)

Discovery of an "abstract" series of bases in a free Lie algebra over a field or a commutative algebra

This is included in the Ph.D. Thesis and is published in:
A.I. Shirshov, Bases of free Lie algebra, Algebra and Logic, 1962.

Each specific basis of this "Hall – Shirshov" series is based on some (partial) ordering of non-associative monomials on the generators. The result covers the Hall basis (1950) based on "Deg-Lex" ordering of the monomials. It covers also the Shirshov basis from his Ph.D. Thesis based on "Comm-Lex" ordering of the monomials in terms of their commutative bases.
Example: The commutative content of the monomial $((a_2a_3)(a_1a_3))$ is the word $a_3a_3a_2a_1$.

This Shirshov basis of "Comm-Lex" order has not yet been published!

Shirshov uses this new basis to prove a sufficient condition for the validity of the Poincaré-Birkhoff-Witt Theorem for a Lie algebra over a commutative algebra (Dissertation 1953).

The proof is not published!

The statement of this sufficient condition is given in the paper A.I. Shirshov, Representation of Lie rings in associative rings, Usp. Mat. Nauk, 1953.

An example of a Lie algebra over a commutative algebra that cannot be embedded into an associative algebra is given. This work quickly received a response abroad. In 1958, the work of Pierre Cartier (Paris) appeared with a citation and a further continuation of the work of A.I. Shirshov (with another example). More examples of such Lie algebras were proposed by Paul Cohn (London) in 1963. The examples of Shirshov and Cartier were over commutative algebras over the field GF(2), Cohn's examples were over all fields GF(p). So far, there is no example over a commutative algebra of characteristic zero.

It is important to note that the "Noncomm-Lex" order of monomials in terms of their non-commutative bases is also suitable for the Hall-Shirshov Series.
An example is the non-commutative content of the monomial $((a_2a_3)(a_1a_3))$ is the word $a_2a_3a_1a_3$.
In this case, we get the Lyndon-Shirshov basis, and with it (omitting the brackets) the Lyndon associative words!

Without a doubt, Shirshov investigated this order as well and he was sure that it is also suitable for his "abstract" basis. (The proof is the same as in the "Comm-Lex" case.)
It is a big and rare piece of luck to find an order which gives rise to a new "abstract" basis and Shirshov could not miss one more specific basis of his scheme.

Even now, only 4 particular bases of free Lie algebra from the series "abstract" bases are known. In addition to those mentioned above, we shall mention a basis corresponding to the "Polynilpotent $(n_1, n_2, \ldots, n_k)$-order" of monomials, where $k > 0$, $n_i > 1$ (see L.B., A basis for free polynilpotent Lie algebras, Algebra i Logika 2 (1963), No. 4, 1320).
The initial pieces of the basis give a basis of the free polynilpotent Lie algebra $L(X)/(\cdots((L(X)^{n_1})^{n_2})\cdots)^{n_k}$. In particular, for $n_i = 2$ we obtain a basis for the free solvable Lie algebra.

So already in 1953 Shirshov probably knew the Lyndon – Shirshov basis (1958) and the Lyndon words (1954)!

The paper A.I. Shirshov, On free Lie algebras, Mat. Sbornik 1958

In this paper it was necessary to use the Noncomm-Lex basis to prove the theorem on the embedding of a countably generated Lie algebra over a field into a 2-generated one.

Shirshov could not use his "abstract" basis theorem because it was not published. As a result, he used a new definition of the Lyndon-Shirshov basis in terms of the scheme of Hall, but starting from his knowledge of "regular words" (the Lyndon words).

The paper in Mat. Sb. (1958) became another milestone work. Here is a list of discoveries in the paper:

- The discovery of the "factorization property of regular (Lyndon) words".
- Using this property, Shirshov defined a "special arrangement of parentheses on a regular word with a fixed regular subword".
- It was established that any Lie algebra with not more than countable set of generators can be embedded into a 2-generated Lie algebra.
- Finally, Shirshov discovered "Gauss" elimination of the leading word in the Lie polynomial f with the help of another polynomial g.

The theory of Gröbner-Shirshov bases

All the above, together with the systematic use of the inclusion of the free Lie algebra Lie(X) over a field k into the free associative algebra $k\langle X\rangle$, served as the beginning of the discovery of the theory of Gröbner-Shirshov bases for Lie algebras and associative algebras in the work
A.I. Shirshov, Some algorithmic problems for Lie algebras, SibMatZh, 1962.
It remained for Shirshov only to add the definition of the operation of composition of two associative polynomials (s-polynomial in the theory of Buchberger, 1965-1970). Then Shirshov discovered the "composition" for Lie polynomials with a "special" arrangement of parentheses in their associative composition.

THANK YOU VERY MUCH FOR YOUR ATTENTION!