On the Lvov-Kaplansky conjecture

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Introduction

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Definition

Given a polynomial $f(x_1, \dots, x_m) \in F\langle X\rangle$ and an F-algebra A, we define the image of f on A as

$$Im(f) = \{f(a_1, \dots, a_m); a_1, \dots, a_m \in A\}$$

Some examples

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4. the image of $f(x, y, z) = xyz - zyx$ on $M_n(F)$ is equal to $M_n(F)$ [**D. Khurana**, **T. Lam** (2012)]

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2. $Im(f)$ is closed under conjugation by invertible elements of A;
3. the linear span of $Im(f)$ is a Lie ideal of A (an ideal of the Lie algebra $A^{(-)}$).

Taking $A = M_n(F)$ where $char(F) \neq 2$ or $n \neq 2$, we have the following

I. Herstein (1955)

The Lie ideals of $M_n(F)$ are $\{0\}, F, sl_n(F)$ and $M_n(F)$.

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Corolary

If f is multilinear, then $span(Im(f))$ on $M_n(F)$ is $\{0\}, F, sl_n(F)$ or $M_n(F)$.

M. Bresar, I. Klep, *Values of noncommutative polynomials, Lie skew-ideals and tracial Nullstellensätze*, Math. Res. Lett. **16** (2009), 605-626

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Lvov-Kaplansky conjecture

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Equivalently,

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The image of a multilinear polynomial over F on $M_n(F)$ is $\{0\}, F, sl_n(F)$ or $M_n(F)$.

Some positive solutions

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Let $f(x, y) = \alpha xy + \beta yx$.
If $\lambda = \alpha + \beta \neq 0$, then $A = f(\lambda^{-1}A, I_n)$. Hence $Im(f) = M_n(F)$.
If $\lambda = 0$, then $f(x, y) = \alpha[x, y]$. Hence $Im(f) \in \{\{0\}, sl_n(F)\}$.

K. Dykema, I. Klep (2016)

If n is even or $n < 17$, then the image of a degree three multilinear polynomial on $M_n(\mathbb{C})$ is $\{0\}, sl_n(\mathbb{C})$ or $M_n(\mathbb{C})$.

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A. Kanel-Belov, S. Malev, L. Rowen (2012)

Let F be a quadratically closed field and let $f \in F\langle X \rangle$ be a multilinear polynomial. Then $Im(f)$ on $M_2(F)$ is $\{0\}, F, sl_2(F)$ or $M_2(F)$.

The strictly upper triangular matrices case

Denote by $J = J(UT_n)$ the algebra of $n \times n$ strictly upper triangular matrices.

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Note that $Im(f) \subset J^m$.

Moreover J satisfies the identity $x_1 \cdots x_n = 0$ and therefore we may assume $m < n$ modulo $Id(J)$.

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Our main goal is to prove that modulo $Id(J)$ we also have $Im(f) \supset J^m$. That is, we want to prove the following theorem

P. Fagundes (2019)

Let $f(x_1, \ldots, x_m) \in F\langle X \rangle$ be a multilinear polynomial where F is any field. Then $Im(f)$ on J is equal to J^m iff $f \notin Id(J)$.

Sketch of the proof:

Given a matrix $A \in J^m$, write $A = \sum_{i=m+1}^{n} A_i$, where A_i is the i-th diagonal of A. We will show that there exist $B_i, B_2, \ldots, B_m \in J$ such that $A_i = f(B_i, B_2, \ldots, B_m)$.

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It will follow that

$$A = \sum_{i=m+1}^{n} A_i = \sum_{i=m+1}^{n} f(B_i, B_2, \ldots, B_m) = f(\sum_{i=m+1}^{n} B_i, B_2, \ldots, B_m)$$

Rewrite f as $\sum_{j=1}^{m} f_j$ where f_j is the sum of all monomials of f which $j-$th variable is equal to x_1.

Taking $x_1^{(m+1)} = \sum_{k=1}^{n-1} y_k^{(m+1)} \boldsymbol{e_{k,k+1}}, x_j = \sum_{k=1}^{n-1} y_k^{(j)} \boldsymbol{e_{k,k+1}}$ where the y's are commutative variables, we compute $f(x_1^{(m+1)}, x_2, \ldots, x_m)$

$$f(x_1^{(m+1)}, x_2, \dots, x_m) = \sum_{k=1}^{n-m} \left(y_k^{(m+1)} \sum_{\sigma \in S_m^{(1)}} \alpha_\sigma y_{k+1}^{(\sigma(2))} \cdots y_{k+m-1}^{(\sigma(m))} \right.$$

$$\left. + y_{k+1}^{(m+1)} \delta_2^{(m+1)}(x_2, \dots, x_m) + \cdots + y_{k+m-1}^{(m+1)} \delta_m^{(m+1)}(x_2, \dots, x_m) \right) \boldsymbol{e_{k,k+m}},$$

where $y_{k+j-1}^{(m+1)} \delta_j^{(m+1)}(x_2, \dots, x_m)$ stands for the $(k, k+m)$ entry of the matrix $f_j(x_1^{(m+1)}, x_2, \dots, x_m)$ and $S_m^{(1)} = \{\sigma \in S_m ; \sigma(1) = 1\}$.

In order to find $B_{m+1}, B_2, \ldots, B_m$, we are looking for a solution of the following system

$$y_k^{(m+1)} \sum_{\sigma \in S_m^{(1)}} \alpha_\sigma y_{k+1}^{(\sigma(2))} \cdots y_{k+m-1}^{(\sigma(m))} + y_{k+1}^{(m+1)} \delta_2^{(m+1)}(x_2, \ldots, x_m)$$

$$+ \cdots + y_{k+m-1}^{(m+1)} \delta_m^{(m+1)}(x_2, \ldots, x_m) = a_k^{(m+1)}, \text{ for } k = 1, \ldots, n-m,$$

where $A_{m+1} = \sum_{k=1}^{n-m} a_k^{(m+1)} \boldsymbol{e_{k,k+m}}$

Claim: there exist evaluations of the variables $y^{(2)}, \ldots, y^{(m)}$ by elements of F such that

$$\sum_{\sigma \in S_m^{(1)}} \alpha_\sigma y_{k+1}^{(\sigma(2))} \cdots y_{k+m-1}^{(\sigma(m))}$$

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is nonzero for all k.

Then one can find a solution of the previous system recursively. In others words, we are able to realize the first nonzero diagonal of J^m as an evaluation of f on some matrices in J.

In order to define B_i we take $x_2, \ldots, x_m$ as before and

$$x_1^{(i)} = \sum_{k=1}^{n-i+m} y_k^{(i)} \boldsymbol{e_{k,k+i-m}}, i = m+2, \ldots, n$$

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$$x_1^{(i)} = \sum_{k=1}^{n-i+m} y_k^{(i)} \mathbf{e_{k,k+i-m}}, i = m+2, \dots, n$$

and then

$$f(x_1^{(i)}, x_2, \dots, x_m) = \sum_{k=1}^{n-i+1} \left(y_k^{(i)} \sum_{\sigma \in S_m^{(1)}} \alpha_\sigma y_{k+i-m}^{(\sigma(2))} \cdots y_{k+i-2}^{(\sigma(m))} \right.$$

$$\left. + y_{k+1}^{(i)} \delta_2^{(i)}(x_2, \dots, x_m) + \cdots + y_{k+m-1}^{(i)} \delta_m^{(i)}(x_2, \dots, x_m) \right) \mathbf{e_{k,k+i-1}}.$$

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Therefore each $A_i = f(B_i, B_2, \ldots, B_m)$ as we would like to prove.

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We conclude $f(J) = J^m$.

P. Fagundes (2019)

Let $f(x_1, \ldots, x_m) \in F\langle X \rangle$ where F is any field. Then $Im(f)$ on J^k is equal to J^{mk} iff $f \notin Id(J^k)$.

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Thank you for your attention

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