Hilbert series and invariant theory of symplectic and orthogonal groups

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Definitions and notations

Joint work with Vesselin Drensky

The ground field is $\mathbb{C}$.

Topic of classical invariant theory: Let G be a group and let V be a finite dimensional representation of G. By $\mathbb{C}[V]$ we denote the algebra of polynomial functions on V. The goal of classical invariant theory is to describe the subalgebra of G-invariant polynomial functions

$$\mathbb{C}[V]^G = \{f \in \mathbb{C}[V] : g \cdot f = f \text{ for all } g \in G\}$$

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Questions:

1. Is $\mathbb{C}[V]^G$ a finitely generated algebra over $\mathbb{C}$?
2. Is $\mathbb{C}[V]^G$ a polynomial algebra?
3. Can we describe $\mathbb{C}[V]^G$ in terms of generators and relations?
4. Can we give some upper bounds on the number of generators in a minimal generating set?

Definitions and notations

Definition

Let $A = \bigoplus_{i \geq 0} A^i$ *be a finitely generated graded algebra over* $\mathbb{C}$ *such that* $A^0 = \mathbb{C}$ *or* $A^0 = 0$. *The Hilbert series of* A *is the formal power series*

$$H(A, t) = \sum_{i \geq 0} (\dim A^i) t^i.$$

The Hilbert series $H(A, t)$ gives information about the lowest degree of the generators in a minimal generating set of A and the maximal number of generators in each degree.

Definitions and notations

- Let $\lambda = (\lambda_1 \geq \cdots \geq \lambda_n) \in (\mathbb{N}_0)^n$ be a non-negative integer partition. By V_λ we denote the irreducible $\mathrm{GL}(n)$-module with highest weight λ.

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Question

Determine $H(A^G, t)$, where G is $\mathrm{O}(n)$, $\mathrm{SO}(n)$, or $\mathrm{Sp}(2d)$ (for $n = 2d$).

Definitions and notations

First examples:

Let W be a polynomial $\mathrm{GL}(n)$-module.

- $S(W) = \bigoplus_{i \geq 0} S^i W$, the symmetric algebra of W. $S(W) \cong \mathbb{C}[W^*]$.

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More generally:

- $T(W)/I$, where

$$T(W) = \bigoplus_{i \geq 0} W^{\otimes i} = \mathbb{C} \oplus W \oplus (W \otimes W) \oplus (W \otimes W \otimes W) \oplus \cdots$$

is the tensor algebra of W and I is a $\mathrm{GL}(n)$-invariant ideal in $T(W)$.

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  - Relatively free algebras of varieties of associative algebras.

Multigraded algebras and Hilbert series

Definition

Let $M = \bigoplus_{\mu \in \mathbb{N}_0^n} M(\mu)$ be a finitely generated algebra (or a vector space) with an $\mathbb{N}_0^n$-grading. The Hilbert series of M with respect to this grading is the formal power series $H(M, x_1, \ldots, x_n) \in \mathbb{Z}[[x_1, \ldots, x_n]]$ given by

$$H(M, x_1, \ldots, x_n) = \sum_{\mu=(\mu_1,\ldots,\mu_n)\in\mathbb{N}_0^n} \dim M(\mu) x_1^{\mu_1} \ldots x_n^{\mu_n}.$$

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- The module V_λ has an $\mathbb{N}_0^n$-grading

$$V_\lambda = \bigoplus_{\mu \in \mathbb{N}_0^n} V_\lambda(\mu),$$

where $V_\lambda(\mu)$ denotes the weight space corresponding to the weight μ.

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$$V_\lambda = \bigoplus_{\mu \in \mathbb{N}_0^n} V_\lambda(\mu),$$

  where $V_\lambda(\mu)$ denotes the weight space corresponding to the weight μ.
- The Hilbert series of V_λ with respect to this grading has the form

$$H(V_\lambda, x_1, \dots, x_n) = \chi_{V_\lambda}(x_1, \dots, x_n) = S_\lambda(x_1, \dots, x_n),$$

  where $S_\lambda(x_1, \dots, x_n)$ is the Schur polynomial corresponding to λ.

Multigraded algebras and Hilbert series

- Any polynomial $\mathrm{GL}(n)$-module $W \cong \bigoplus_\lambda k(\lambda)V_\lambda$ has an $\mathbb{N}_0^n$-grading. For the Hilbert series of W we get

$$H(W, x_1, \ldots, x_n) = \sum_\lambda k(\lambda) S_\lambda(x_1, \ldots, x_n) = \chi_W(x_1, \ldots, x_n).$$

Multigraded algebras and Hilbert series

- Any polynomial $\mathrm{GL}(n)$-module $W \cong \bigoplus_\lambda k(\lambda)V_\lambda$ has an $\mathbb{N}_0^n$-grading. For the Hilbert series of W we get

$$H(W, x_1, \ldots, x_n) = \sum_\lambda k(\lambda)S_\lambda(x_1, \ldots, x_n) = \chi_W(x_1, \ldots, x_n).$$

- The algebra $A = \bigoplus_{i\geq 0} A^i = \bigoplus_{i\geq 0}\bigoplus_\lambda m_i(\lambda)V_\lambda$ has two gradings – an $\mathbb{N}_0$-grading and an $\mathbb{N}_0^n$-grading. Following a work of Benanti, Boumova, Drensky, Genov, and Koev for $S(W)$, we introduce a Hilbert series of A which takes into account both gradings:

$$H(A, x_1, \ldots, x_n, t) = \sum_{i\geq 0} H(A^i, x_1, \ldots, x_n)t^i =$$

$$\sum_{i\geq 0} \chi_{A^i}(x_1, \ldots, x_n)t^i = \sum_{i\geq 0}\left(\sum_\lambda m_i(\lambda)S_\lambda(x_1, \ldots, x_n)\right)t^i$$

Hilbert series and multiplicity series

$$H(A, x_1, \ldots, x_n, t) = \sum_{i \geq 0} \left(\sum_{\lambda} m_i(\lambda) S_\lambda(x_1, \ldots, x_n) \right) t^i$$
$$\in \mathbb{Z}[[x_1, \ldots, x_n]]^{S_n}[[t]],$$

where S_n denotes the symmetric group in n variables. Following BBDGK, we introduce the **multiplicity series** of A by

$$M(A, x_1, \ldots, x_n, t) = \sum_{i \geq 0} \left(\sum_{\lambda} m_i(\lambda) x_1^{\lambda_1} \cdots x_n^{\lambda_n} \right) t^i.$$

By a change of variables $v_1 = x_1$, $v_2 = x_1 x_2$, $\ldots$, $v_n = x_1 \cdots x_n$ one can rewrite the above series as

$$M'(A, v_1, \ldots, v_n, t) = \sum_{i \geq 0} \left(\sum_{\lambda} m_i(\lambda) v_1^{\lambda_1 - \lambda_2} \cdots v_{n-1}^{\lambda_{n-1} - \lambda_n} v_n^{\lambda_n} \right) t^i.$$

M and M' carry the information about the $\mathrm{GL}(n)$-structure of A.

The algebra A^G for $G = \mathrm{SL}(n)$, $\mathrm{O}(n)$, $\mathrm{SO}(n)$, or $\mathrm{Sp}(2d)$

Theorem (BBDGK, 2012)

For the Hilbert series of $A^{\mathrm{SL}(n)}$ we obtain

$$H(A^{\mathrm{SL}(n)}, t) = M'(A, 0, \ldots, 0, 1, t).$$

Theorem

Let $n = 2d$. For the Hilbert series of $A^{\mathrm{Sp}(2d)}$ we obtain

$$H(A^{\mathrm{Sp}(2d)}, t) = M'(A, 0, 1, 0, 1, \ldots, 0, 1, t).$$

The algebra A^G for $G = \mathrm{SL}(n)$, $\mathrm{O}(n)$, $\mathrm{SO}(n)$, or $\mathrm{Sp}(2d)$

Theorem

For the Hilbert series of $A^{\mathrm{O}(n)}$ we obtain

$$H(A^{\mathrm{O}(n)}, t) = M_n(t),$$

where

$$M_1(x_2, \dots, x_n, t) =$$
$$\frac{1}{2}\left(M(A, -1, x_2, \dots, x_n, t) + M(A, 1, x_2, \dots, x_n, t)\right),$$
$$M_2(x_3, \dots, x_n, t) = \frac{1}{2}(M_1(-1, x_3, \dots, x_n, t) + M_1(1, x_3, \dots, x_n, t)),$$
$$\dots\dots\dots$$
$$M_n(t) = \frac{1}{2}(M_{n-1}(-1, t) + M_{n-1}(1, t)).$$

The algebra A^G for $G = \mathrm{SL}(n)$, $\mathrm{O}(n)$, $\mathrm{SO}(n)$, or $\mathrm{Sp}(2d)$

Theorem

For the Hilbert series of $A^{\mathrm{SO}(n)}$ we obtain

$$H(A^{\mathrm{SO}(n)}, t) = M'_n(t),$$

where

$$M'_1(v_2, \ldots, v_n, t) =$$
$$\frac{1}{2}(M'(A, -1, v_2, \ldots, v_n, t) + M'(A, 1, v_2, \ldots, v_n, t)),$$
$$M'_2(v_3, \ldots, v_n, t) = \frac{1}{2}(M'_1(-1, v_3, \ldots, v_n, t) + M'_1(1, v_3, \ldots, v_n, t)),$$
$$\ldots\ldots\ldots$$
$$M'_{n-1}(v_n, t) = \frac{1}{2}(M'_{n-2}(-1, v_n, t) + M'_{n-2}(1, v_n, t)),$$
$$M'_n(t) = M'_{n-1}(1, t).$$

Applications and examples

Recall that if $A = \bigoplus_{i\geq 0} \bigoplus_{\lambda} m_i(\lambda) V_\lambda$ then

$$M(A, x_1, \ldots, x_n, t) = \sum_{i\geq 0} \left(\sum_{\lambda} m_i(\lambda) x_1^{\lambda_1} \cdots x_n^{\lambda_n} \right) t^i.$$

$$M'(A, v_1, \ldots, v_n, t) = \sum_{i\geq 0} \left(\sum_{\lambda} m_i(\lambda) v_1^{\lambda_1-\lambda_2} \cdots v_{n-1}^{\lambda_{n-1}-\lambda_n} v_n^{\lambda_n} \right) t^i.$$

I.) If we know the coefficients $m_i(\lambda)$ then we can determine M and M'.

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I.) If we know the coefficients $m_i(\lambda)$ then we can determine M and M'.

Examples:

Let $V = \mathbb{C}^n$ denote the natural $\mathrm{GL}(n)$-module. Then for A we can take:

- $S(S^2 V)$, $S(\Lambda^2 V)$, $S(V \oplus \Lambda^2 V)$;
- $\Lambda(S^2 V)$ and $\Lambda(\Lambda^2 V)$.

Applications: Hilbert series for some relatively free algebras

- Let $V = \mathbb{C}^n$ be the natural representation of $\mathrm{GL}(n)$ and let

$$A = T(V) / \langle [[u, v], w] : \text{ for all } u, v, w \in T(V) \rangle .$$

A is called the relatively free algebra of rank n in the variety generated by the Grassmann algebra.

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- Decomposition of A as a $\mathrm{GL}(n)$-module: Let

$$\mathcal{P} = \{\text{all partitions } \lambda \in \mathbb{N}_0^n : \lambda = (k, \underbrace{1,\ldots,1}_{s}, \underbrace{0,\ldots,0}_{t}), k,s,t \geq 0\}.$$

Hence, $\mathcal{P}$ contains all partitions λ with Young diagram consisting of one long row and one long column. Then

$$A \cong \bigoplus_{i\geq 0} \bigoplus_{\substack{\lambda\in\mathcal{P} \\ |\lambda|=i}} V_\lambda .$$

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$$A \cong \bigoplus_{i \geq 0} \bigoplus_{\substack{\lambda \in \mathcal{P} \\ |\lambda| = i}} V_\lambda .$$

- Hence, $M(A, x_1, \ldots, x_n, t) = \sum_{i \geq 0} (\sum_{\substack{\lambda \in \mathcal{P} \\ |\lambda| = i}} x_1^{\lambda_1} \cdots x_n^{\lambda_n}) t^i$.

Applications: Hilbert series for some relatively free algebras

- For the Hilbert series $H(A^{\mathrm{SL}(n)}, t)$ and $H(A^{\mathrm{Sp}(2d)}, t)$ we obtain:

$$H(A^{\mathrm{SL}(n)}, t) = 1 + t^n;$$

$$H(A^{\mathrm{Sp}(2d)}, t) = 1 + t^2 + t^4 + \cdots + t^{2d}, \quad \text{where } n = 2d.$$

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- Let $\{x_1, \dots, x_n\}$ be the standard basis for $V = \mathbb{C}^n$.
The algebra $A^{\mathrm{SL}(n)}$ is generated by the standard polynomial of degree n

$$f = St_n(x_1, \dots, x_n) = \sum_{\sigma \in S_n} \mathrm{sign}(\sigma) x_{\sigma(1)} \otimes \cdots \otimes x_{\sigma(n)}.$$

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- The algebra $A^{\mathrm{Sp}(2d)}$ is generated by

$$f = [x_1, x_{d+1}] + [x_2, x_{d+2}] + \cdots + [x_d, x_{2d}].$$

Applications: Hilbert series for some relatively free algebras

- For the Hilbert series $H(A^{\mathrm{O}(n)}, t)$ and $H(A^{\mathrm{SO}(n)}, t)$ we obtain:

$$H(A^{\mathrm{O}(n)}, t) = \frac{1}{1 - t^2};$$

$$H(A^{\mathrm{SO}(n)}, t) = \frac{1 + t^n}{1 - t^2}.$$

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- The algebra $A^{\mathrm{O}(n)}$ is generated by $f = x_1 \otimes x_1 + \cdots + x_n \otimes x_n$.
- The algebra $A^{\mathrm{SO}(n)}$ is generated by the elements f_1 and f_2, where

$$f_1 = x_1 \otimes x_1 + \cdots + x_n \otimes x_n,$$

$$f_2 = St_n(x_1, \ldots, x_n).$$

Let $V \cong \mathbb{C}^n$ and let I be an ideal in $T(V)$. I is called a **T-ideal** of $T(V)$ if I is closed under all endomorphisms of $T(V)$ as an algebra over $\mathbb{C}$.

Theorem (Domokos-Drensky, 1998)

Let $G \subset \mathrm{GL}(n)$ be a reductive group and let I be a T-ideal of $T(V)$. If the algebra $T(V)/I$ satisfies the polynomial identity $[x_1, \ldots, x_n] = 0$ for some n, then the algebra of invariants $(T(V)/I)^G$ is finitely generated.

Applications: Hilbert series for some relatively free algebras

- Let $V = \mathbb{C}^n$ with basis $\{x_1, \ldots, x_n\}$ and let

$$A = T(V)/\left\langle [u_1, u_2] \otimes [u_3, u_4] : \text{ for all } u_1, \ldots, u_4 \in T(V) \right\rangle .$$

A is called the relatively free algebra of rank n in the variety generated by the algebra of 2×2 upper triangular matrices.

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- The Hilbert series of $A^{\mathrm{Sp}(2d)}$ is

$$H(A^{\mathrm{Sp}(2d)}, t) = \frac{1}{1 - t^2}.$$

- $A^{\mathrm{Sp}(2d)}$ is not finitely generated. A set of generators can be defined inductively by

$$f_1 = [x_1, x_{d+1}] + [x_2, x_{d+2}] + \cdots + [x_d, x_{2d}] = \sum_{i=1}^{d} [x_i, x_{d+i}],$$

$$f_{m+1} = \sum_{i=1}^{d} x_i \otimes f_m \otimes x_{d+i} - x_{d+i} \otimes f_m \otimes x_i, \quad m = 1, 2, \ldots$$

Applications: Hilbert series for some relatively free algebras

- The Hilbert series of $A^{\mathrm{O}(n)}$ is

$$H(A^{\mathrm{O}(n)},t)=\frac{1-2t^2+2t^4}{(1-t^2)^3}.$$

- For the Hilbert series of $A^{\mathrm{SO}(n)}$ we obtain
  (i) If $n=2$, then

$$H(A^{\mathrm{SO}(2)},t)=\frac{1-t^2+2t^4}{(1-t^2)^3}.$$

  (ii) If $n=3$, then

$$H(A^{\mathrm{SO}(3)},t)=\frac{1-2t^2+t^3+2t^4}{(1-t^2)^3}.$$

  (iii) If $n>3$, then

$$H(A^{\mathrm{SO}(n)},t)=H(A^{\mathrm{O}(n)},t).$$

- The algebras $A^{\mathrm{O}(n)}$ and $A^{\mathrm{SO}(n)}$ are not finitely generated.

Applications: Computing $H(\Lambda(W)^G, t)$ and $H(S(W)^G, t)$

II.) If the coefficients $m_i(\lambda)$ are not known, we can try to determine first $H(A, x_1, \ldots, x_n, t)$ and then M and M'.

Applications: Computing $H(\Lambda(W)^G, t)$ and $H(S(W)^G, t)$

II.) If the coefficients $m_i(\lambda)$ are not known, we can try to determine first $H(A, x_1, \ldots, x_n, t)$ and then M and M'.
Let W be a p-dimensional polynomial $\mathrm{GL}(n)$-module. Let $\alpha_1 = (\alpha_{11}, \ldots, \alpha_{1n})$, $\ldots$, $\alpha_p = (\alpha_{p1}, \ldots, \alpha_{pn})$ denote the weights of W (with possible repetitions). Then,

$$H(\Lambda(W), x_1, \ldots, x_n, t) = \sum_{i \geq 0} \chi_{\Lambda^i(W)}(x_1, \ldots, x_n) t^i = \prod_{j=1}^{p} (1 + x_1^{\alpha_{j1}} \cdots x_n^{\alpha_{jn}} t).$$

$$H(S(W), x_1, \ldots, x_n, t) = \sum_{i \geq 0} \chi_{S^i(W)}(x_1, \ldots, x_n) t^i = \prod_{j=1}^{p} \frac{1}{1 - x_1^{\alpha_{j1}} \cdots x_n^{\alpha_{jn}} t}$$

Applications: Computing $H(\Lambda(W)^G, t)$ and $H(S(W)^G, t)$

A generalization of a lemma of Berele.

Lemma (Berele, 2006)

Let $X = \{x_1, \dots, x_n\}$ *and let* $H(A, X, t)$ *denote the Hilbert series of* A. *Let*

$$g(X,t) = H(A,X,t)\prod_{i<j}(x_i - x_j) = \sum_{i\geq 0}(\sum_{r_{i_j}\geq 0}\alpha_i(r_{i_1},\dots,r_{i_n})x_1^{r_{i_1}}\cdots x_n^{r_{i_n}})t^i,$$

for some $\alpha_i(r_{i_1},\dots,r_{i_n}) \in \mathbb{C}$. *Then the multiplicity series of* A *is given by*

$$M(A; x_1,\dots,x_n,t) = \frac{1}{x_1^{n-1}x_2^{n-2}\cdots x_{n-2}^2 x_{n-1}}\sum_{i\geq 0}(\sum_{r_{i_j}>r_{i_{j+1}}}\alpha_i(r_{i_1},\dots,r_{i_n})x_1^{r_{i_1}}\cdots x_n^{r_{i_n}})t^i,$$

where the sum is over all $r_i = (r_{i_1},\dots,r_{i_n})$ *such that* $r_{i_1} > r_{i_2} > \dots > r_{i_n}$.

Applications: Computing $H(\Lambda(W)^G, t)$

Table: Hilbert series for $n = 2$

k	$H(\Lambda(S^k V)^{\mathrm{Sp}(2)=\mathrm{SL}(2)}, t)$
3	$1 + t^2 + t^4$
4	$1 + t^5$
5	$1 + t^2 + t^4 + t^6$
6	$1 + t^3 + t^4 + t^7$
7	$1 + t^2 + t^4 + t^6 + t^8$
8	$1 + t^4 + t^5 + t^9$
9	$1 + t^2 + 2t^4 + 2t^6 + t^8 + t^{10}$
10	$1 + t^3 + t^4 + t^7 + t^8 + t^{11}$
11	$1 + t^2 + 2t^4 + 3t^6 + 2t^8 + t^{10} + t^{12}$
12	$1 + 2t^4 + 2t^5 + 2t^8 + 2t^9 + t^{13}$
13	$1 + t^2 + 2t^4 + 4t^6 + 4t^8 + 2t^{10} + t^{12} + t^{14}$
14	$1 + t^3 + 2t^4 + 4t^7 + 4t^8 + 2t^{11} + t^{12} + t^{15}$

Applications: Computing $H(\Lambda(W)^G, t)$

Table: Hilbert series for $n = 2$

k	$H(\Lambda(S^k V)^{\mathrm{O}(2)}, t)$	$H(\Lambda(S^k V)^{\mathrm{SO}(2)}, t)$
3	$1 + t^4$	$1 + 2t^2 + t^4$
4	$1 + t + t^4 + t^5$	$1 + t + 2t^2 + 2t^3 + t^4 + t^5$
5	$1 + 3t^4$	$1 + 3t^2 + 3t^4 + t^6$
6	$1 + t + t^3 + 4t^4 + 3t^5$	$1 + t + 3t^2 + 5t^3 + 5t^4 + 3t^5 + t^6 + t^7$
7	$1 + 7t^4 + t^8$	$1 + 4t^2 + 8t^4 + 4t^6 + t^8$
8	$1 + t + 2t^3 + 9t^4 + 9t^5 + 2t^6 + t^8 + t^9$	$1 + t + 4t^2 + 8t^3 + 12t^4 + 12t^5 + 8t^6 + 4t^7 + t^8 + t^9$
9	$1 + 14t^4 + 4t^6 + 5t^8$	$1 + 5t^2 + 18t^4 + 18t^6 + 5t^8 + t^{10}$
10	$1 + t + 4t^3 + 17t^4 + 21t^5 + 11t^6 + 7t^7 + 9t^8 + 5t^9$	$1 + t + 5t^2 + 13t^3 + 24t^4 + 32t^5 + 32t^6 + 24t^7 + 13t^8 + 5t^9 + t^{10} + t^{11}$
11	$1 + 24t^4 + 19t^6 + 24t^8 + t^{12}$	$1 + 6t^2 + 33t^4 + 58t^6 + 33t^8 + 6t^{10} + t^{12}$

Applications: Computing $H(\Lambda(W)^G, t)$

Table: Hilbert series for $n = 3$

k	$H(\Lambda(S^k V)^{\mathrm{O}(3)}, t)$	$H(\Lambda(S^k V)^{\mathrm{SO}(3)}, t)$
3	$1 + 2t^4 + 2t^6 + t^{10}$	$1 + 3t^3 + 2t^4 + 2t^6 + 3t^7 + t^{10}$
4	$1 + t + 3t^4 + 12t^5 + 15t^6 + 8t^7 + 8t^8 + 15t^9 + 12t^{10} + 3t^{11} + t^{14} + t^{15}$	$1 + t + 3t^4 + 12t^5 + 15t^6 + 8t^7 + 8t^8 + 15t^9 + 12t^{10} + 3t^{11} + t^{14} + t^{15}$
5	$1 + 10t^4 + 60t^6 + 158t^8 + 294t^{10} + 210t^{12} + 125t^{14} + 15t^{16} + 7t^{18}$	$1 + 7t^3 + 10t^4 + 15t^5 + 60t^6 + 125t^7 + 158t^8 + 210t^9 + 294t^{10} + 294t^{11} + 210t^{12} + 158t^{13} + 125t^{14} + 60t^{15} + 15t^{16} + 10t^{17} + 7t^{18} + t^{21}$

Similarly, we computed $H(\Lambda(S^3 V)^{\mathrm{O}(n)}, t)$, $H(\Lambda(S^3 V)^{\mathrm{SO}(n)}, t)$, and $H(\Lambda(S^3 V)^{\mathrm{Sp}(2k)}, t)$ for $n = 4$ and $n = 5$.

Applications: Computing $H(S(W)^G, t)$

We use an algorithm of Benanti, Boumova, Drensky, Genov, and Koev for computing M and M'. This algorithm is based on Berele's lemma.

Table: Hilbert series for $n = 2$

W	$H(S(W)^{\mathrm{O}(2)}, t)$	$H(S(W)^{\mathrm{SO}(2)}, t)$
$V = \mathbb{C}^2$	$\frac{1}{1-t^2}$	$\frac{1}{1-t^2}$
S^2V	$\frac{1}{(1-t)(1-t^2)}$	$\frac{1}{(1-t)(1-t^2)}$
S^3V	$\frac{1}{(1-t^2)^2(1-t^4)}$	$\frac{1+t^4}{(1-t^2)^2(1-t^4)}$
S^4V	$\frac{1}{(1-t)(1-t^2)^2(1-t^3)}$	$\frac{1+t^3}{(1-t)(1-t^2)^2(1-t^3)}$
S^5V	$\frac{1+t^2+3t^4+4t^6+5t^8+4t^{10}+3t^{12}+t^{14}+t^{16}}{(1-t^8)(1-t^6)(1-t^4)(1-t^2)^2}$	$\frac{1+t^2+6t^4+9t^6+12t^8+9t^{10}+6t^{12}+t^{14}+t^{16}}{(1-t^8)(1-t^6)(1-t^4)(1-t^2)^2}$
S^6V	$\frac{1+t^2+t^3+2t^4+t^5+2t^6+t^7+t^8+t^{10}}{(1-t)(1-t^2)^2(1-t^3)(1-t^4)(1-t^5)}$	$\frac{1+t^2+3t^3+4t^4+4t^5+4t^6+3t^7+t^8+t^{10}}{(1-t)(1-t^2)^2(1-t^3)(1-t^4)(1-t^5)}$
Λ^2V	$\frac{1}{1-t^2}$	$\frac{1}{1-t}$
$V_{(3,1)}$	$\frac{1}{(1-t^2)^2}$	$\frac{1}{(1-t)(1-t^2)}$
$V_{(5,1)}$	$\frac{1+t^4}{(1-t^2)^3(1-t^3)}$	$\frac{1+t^3}{(1-t)(1-t^2)^2(1-t^3)}$

Applications: Computing $H(S(W)^G, t)$

Table: Hilbert series for $n = 2$

W	$H(S(W)^{\mathrm{O}(2)}, t)$	$H(S(W)^{\mathrm{SO}(2)}, t)$
$V \oplus V$	$\frac{1}{(1-t^2)^3}$	$\frac{1+t^2}{(1-t^2)^3}$
$V \oplus S^2V$	$\frac{1}{(1-t)(1-t^2)^2(1-t^3)}$	$\frac{1+t^3}{(1-t)(1-t^2)^2(1-t^3)}$
$S^2V \oplus S^2V$	$\frac{1}{(1-t)^2(1-t^2)^3}$	$\frac{1+t^2}{(1-t)^2(1-t^2)^3}$
$V \oplus \Lambda^2V$	$\frac{1}{(1-t^2)^2}$	$\frac{1}{(1-t)(1-t^2)}$
$S^2V \oplus \Lambda^2V$	$\frac{1}{(1-t)(1-t^2)^2}$	$\frac{1}{(1-t)^2(1-t^2)}$
$\Lambda^2V \oplus \Lambda^2V$	$\frac{1+t^2}{(1-t^2)^2}$	$\frac{1}{(1-t)^2}$
$V \oplus V \oplus V$	$\frac{1+t^2+t^4}{(1-t^2)^5}$	$\frac{1+4t^2+t^4}{(1-t^2)^5}$
$V \oplus V \oplus S^2V$	$\frac{(1+t+t^2+t^3+t^4)(1+t^3)}{(1-t^2)^4(1-t^3)^2}$	$\frac{1+2t^2+4t^3+2t^4+t^6}{(1-t)(1-t^2)^3(1-t^3)^2}$
$V \oplus S^3V$	$\frac{1+t^2+3t^4+t^6+t^8}{(1-t^2)^3(1-t^4)^2}$	$\frac{1+2t^2+8t^4+2t^6+t^8}{(1-t^2)^3(1-t^4)^2}$
$V \oplus S^4V$	$\frac{(1+t^4)(1+t+t^2+t^3+t^4+t^5+t^6)}{(1-t^2)^3(1-t^3)^2(1-t^5)}$	$\frac{1+t^2+4t^4+t^5+t^7}{(1-t^2)^3(1-t^3)(1-t^5)(1-t^6)}$

Applications: Computing $H(S(W)^G, t)$

Let $n = 3$ and $W = S^3 V$. Then we obtain that

$$H(S(W)^{\mathrm{O}(3)}, t) = \frac{(1+t^4)(1+t^6)(1+t^2+t^4+3t^6+5t^8+3t^{10}+t^{12}+t^{14}+t^{16})}{(1-t^2)(1-t^4)^3(1-t^6)^2(1-t^{10})}.$$

and

$$H(S(W)^{\mathrm{SO}(3)}, t) = \frac{t^{14}+t^{13}-2t^{11}+t^9+5t^8+5t^7+5t^6+t^5-2t^3+t+1}{(1-t^3)^2(1-t^5)(1-t^2)^2(1-t^4)^2(1+t)}.$$

Applications: Computing $H(S(W)^G, t)$

Let $n = 3$ and $W = S^4 V$. Then,

$$H(S(W)^{\mathrm{O}(3)}, t) = H(S(W)^{\mathrm{SO}(3)}, t) = \frac{A(t)}{(1-t^7)(1-t^5)^2(1-t^4)^2(1-t^3)^4(1-t^2)^3},$$

where

$$\begin{aligned} A(t) =& t^{28} + t^{27} + 3t^{24} + 9t^{23} + 17t^{22} + 22t^{21} + 28t^{20} + 41t^{19} + 63t^{18} + \\ & + 85t^{17} + 107t^{16} + 118t^{15} + 121t^{14} + 118t^{13} + \\ & + 107t^{12} + 85t^{11} + 63t^{10} + 41t^9 + 28t^8 + 22t^7 + 17t^6 + \\ & + 9t^5 + 3t^4 + t + 1. \end{aligned}$$

Applications: Computing $H(S(W)^G, t)$

Let $n = 3$ and $W = V_{(3,1,0)}$. Then,

$$H(S(W)^{\mathrm{O}(3)}, t) = H(S(W)^{\mathrm{SO}(3)}, t) = \frac{A(t)}{(1-t^5)^2(1-t^4)^3(1-t^3)^4(1-t^2)^3(1+t)},$$

where

$$\begin{aligned} A(t) = & t^{26} + t^{25} + 9t^{22} + 22t^{21} + 50t^{20} + 79t^{19} + 120t^{18} + 160t^{17} + \\ & + 221t^{16} + 269t^{15} + 325t^{14} + 339t^{13} + 325t^{12} + 269t^{11} + \\ & + 221t^{10} + 160t^{9} + 120t^{8} + 79t^{7} + 50t^{6} + 22t^{5} + \\ & + 9t^{4} + t + 1. \end{aligned}$$

Applications: Coregular $\mathrm{O}(2)$- and $\mathrm{O}(3)$-representations

- Let G be a reductive complex linear algebraic group. A finite dimensional representation W of G is called *coregular* if the algebra of invariants $\mathbb{C}[W]^G$ is regular, i.e. isomophic to a polynomial algebra.
- The irreducible coregular representations of connected simple complex algebraic groups were classified by Kac, Popov and Vinberg in 1976.
- The reducible coregular representations of connected simple complex algebraic groups were classified by Schwarz in 1978.

Question: What can we say about $\mathbb{C}[W]^{\mathrm{O}(n)}$?

We use that if

$$H(\mathbb{C}[W]^{\mathrm{O}(n)}, t) = \frac{p(t)}{\prod_i (1 - t^{h_i})}, \quad \text{where } p(t) = \sum_j t^{l_j}.$$

and if $p(t) \neq 1$ then $\mathbb{C}[W]^{\mathrm{O}(n)}$ is not polynomial, hence W is not coregular.

Coregular $O(2)$- and $O(3)$-representations

Theorem

Let W be a polynomial $GL(2)$-module. If the algebra $S(W)^{O(2)}$ is polynomial, then up to an $O(2)$-isomorphism W is one of the following:

(1) V, S^2V, S^3V, S^4V, Λ^2V, $V_{(3,1)}$;

(2) $V \oplus V$, $V \oplus S^2V$, $S^2V \oplus S^2V$, $V \oplus \Lambda^2V$, $S^2V \oplus \Lambda^2V$.

Theorem

Let W be a polynomial $GL(3)$-module. If the algebra $S(W)^{O(3)}$ is polynomial, then up to an $O(3)$-isomorphism W is one of the following:

(1) V, S^2V, Λ^2V, Λ^3V;

(2) $V \oplus V$, $V \oplus S^2V$, $V \oplus \Lambda^2V$, $V \oplus \Lambda^3V$, $S^2V \oplus \Lambda^3V$, $\Lambda^2V \oplus \Lambda^2V$, $\Lambda^2V \oplus \Lambda^3V$;

(3) $V \oplus V \oplus V$, $V \oplus V \oplus \Lambda^3V$, $\Lambda^2V \oplus \Lambda^2V \oplus \Lambda^3V$.

The end

Thank you for your attention!