Some PI-results on superalgebras with pseudoinvolution

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Polynomial identities

- Consider a fixed field F.
- Let A be an associative F-algebra.
- Fix $X = \{x_1, x_2, \dots\}$, a countable set of variables.

$F\langle X\rangle$ is the free algebra on X over F.

Definition

A polynomial $f = f(x_1, \dots, x_n) \in F\langle X\rangle$ is a polynomial identity of A, and we write $f \equiv 0$, if, for any $a_1, \dots, a_n \in A$,

$$f(a_1, \dots, a_n) = 0.$$

Example

1. A nilpotent: $x_1 \cdots x_m \equiv 0$ on A (m is the nilpotency index).
2. C commutative: $[x_1, x_2] \equiv 0$ on C.

T-ideals and the Specht's problem

The set of all polynomial identities satisfied by A is

$$\mathrm{Id}(A) = \{f \in F\langle X\rangle : f \equiv 0 \text{ on } A\}.$$

- $\mathrm{Id}(A)$ is a T-ideal of $F\langle X\rangle$.
- Any T-ideal I is of this form: $I = \mathrm{Id}(A)$, for some PI-alg. A.

Theorem (Kemer, 1987)

Every T-ideal is finitely generated, as a T-ideal.

From now on, the field F has characteristic 0.

The codimension sequence

The n-th codimension of a PI-algebra A is the non-negative integer:

$$c_n(A) = \dim_F \frac{P_n}{P_n \cap \mathrm{Id}(A)}.$$

- **Regev, 1972**: $c_n(A)$ is exponentially bounded.
- **Kemer, 1979**: characterizes alg. with polynomial growth.
- **Giambruno, Zaicev, 1999**: $\exists \lim_{n\to\infty} \sqrt[n]{c_n(A)} \in \mathbb{Z}$.

PI-theory for algebras with additional structure

- Superalgebras.

  A superalgebra A is an algebra graded by the cyclic group $\mathbb{Z}_2$:
  $$A = A_0 \oplus A_1, \quad \text{with } A_i A_j \subseteq A_{i+j (\text{mod } 2)}.$$

Example

$M_2(F)$ is a superalgebra with grading: $\left\{\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}\right\} \oplus \left\{\begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix}\right\}$.

They allow reduction to the finite dimensional case.

Theorem (Kemer, 1988)

For any algebra A there exists a finite dimensional superalgebra $B = B_0 \oplus B_1$ such that $\mathrm{Id}(A) = \mathrm{Id}(G(B))$.

Superalgebras with pseudoinvolution

Let A be a superalgebra over an alg. closed field F, char$F = 0$.

Definition

A pseudoinvolution $*$ on $A = A_0 \oplus A_1$ is a linear map $* : A \to A$:

1. $A_i^* \subseteq A_i$, $i = 0, 1$,
2. $(a^*)^* = (-1)^{|a|} a$, for all $a \in A_0 \cup A_1$,
3. $(ab)^* = (-1)^{|a||b|} b^* a^*$, $a, b \in A_0 \cup A_1$.

- $A_0 = A_0^+ \oplus A_0^- = \{a \in A_0 : a^* = a\} \oplus \{a \in A_0 : a^* = -a\}$.
- $A_1 = A_1^i \oplus A_1^{-i} = \{b \in A_1 : b^* = ib\} \oplus \{b \in A_1 : b^* = -ib\}$.

Superalgebras with pseudoinvolution

Example

Consider the superalgebra $M_{1,1}(F) = \left\{ \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \right\} \oplus \left\{ \begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix} \right\}$.

On $M_{1,1}(F)$ it is possible to define the following pseudoinvolutions:

- Pseudo-transpose pt: $\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{pt} = \begin{pmatrix} a & c \\ -b & d \end{pmatrix}$.
- Pseudo-symplectic ps: $\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{ps} = \begin{pmatrix} d & ib \\ ic & a \end{pmatrix}$.

- **Jaber, 2008**: pseudoinvolutions on finite dim. CSS.
- **Martinez, Zelmanov, 2010**: classification of irreducible bimodules over finite dimensional simple Jordan superalgebras via pseudoinvolutions.

Some results in this setting

Let A be a superalgebra with pseudoinvolution and write $c_n^p(A)$ to denote its sequence of pseudo-codimensions.

- **I., Martino, 2018**: analogous of Regev's theorem.

If A is a superalg. with pseudoinv., then $c_n^p(A)$ is exp. bounded.

- **I., Martino, 2018**: analogous of Kemer's theorem.

Characterization of superalgebras with pseudoinvolution of PG.

- **I., 2020**: positive answer to Amitsur's conjecture.

If A is a superalg. with pseudoinv., then $\exists \lim_{n\to\infty} \sqrt[n]{c_n^p(A)} \in \mathbb{Z}$.

More results in this setting

In **I., Martino, 2021** we proved the following.

- The codimension sequence $c_n^p(A)$ is eventually non-decreasing
- Multiplicities of the pseudo-cocharacter:
    - bounded by a constant;
    - bounded by 1.
- Pseudo-colengths bounded by 3.
- $l_n^p(A) \leq k$ if and only if $c_n^p(A)$ grows at most polynomially.

Work in progress

Let $A = A_0 \oplus A_1$ be a superalgebra.

Definition

An involution $\sharp : A \to A$ is said to be graded if $A_i^* \subseteq A_i$, $i = 0, 1$.

Goal

Relation between pseudoinvolutions and graded involutions on A.

In this direction see [**I., Martino, 2018**]: relations between graded involutions and superinvolutions on UT_n.

Thank you for your attention.