Fine gradings on classical simple real Lie algebras

M. Kotchetov

Department of Mathematics and Statistics
Memorial University of Newfoundland

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Gradings and (semi)group gradings

Let $\mathcal{A}$ be a nonassociative algebra over a field $\mathbb{F}$.

Definition (Grading on an algebra)

A *grading* on $\mathcal{A}$ is a vector space decomposition $\Gamma : \mathcal{A} = \bigoplus_{s \in S} \mathcal{A}_s$ such that, whenever $\mathcal{A}_x \mathcal{A}_y \neq 0$, there exists a unique $z \in S$ such that $\mathcal{A}_x \mathcal{A}_y \subseteq \mathcal{A}_z$. This gives a partially defined operation on S: $x * y := z$.

Definition (G-graded algebra)

Let G be a (semi)group, written multiplicatively.

- A *G-grading* on $\mathcal{A}$ is a vector space decomposition $\Gamma : \mathcal{A} = \bigoplus_{g \in G} \mathcal{A}_g$ such that $\mathcal{A}_g \mathcal{A}_h \subseteq \mathcal{A}_{gh}$ for all $g, h \in G$.
- $(\mathcal{A}, \Gamma)$ is said to be a *G-graded algebra*, and $\mathcal{A}_g$ is its *homogeneous component* of degree g.

$\Gamma : \mathcal{A} = \bigoplus_{s \in S} \mathcal{A}_s$ is a *(semi)group grading* if there exists a (semi)group G and $\iota : S \hookrightarrow G$ such that $\mathcal{A}_x \mathcal{A}_y \neq 0 \;\Rightarrow\; \iota(x * y) = \iota(x)\iota(y)$.

Universal grading groups

Example (Gradings from matrix units)

$M_n(\mathbb{F}) = \bigoplus_{1\le i,j\le n} \mathbb{F}E_{ij}$ is a semigroup grading, but not a group grading.
$M_n(\mathbb{F}) = \mathrm{Span}\{E_{11},\dots,E_{nn}\} \oplus \bigoplus_{1\le i\ne j\le n} \mathbb{F}E_{ij}$ is an ab. group grading.

The *support* of a G-grading Γ is the set $\mathrm{Supp}\,\Gamma := \{g\in G \mid \mathcal{A}_g \ne 0\}$.

Fact: For any semigroup grading on a simple Lie algebra, the support generates an abelian group.

Elduque 2021: There exists a non-semigroup gradings on $\mathfrak{so}_{26}(\mathbb{C})$.

Definition (Universal group and universal abelian group)

The *universal (abelian) group* of $\Gamma : \mathcal{A} = \bigoplus_{s\in S} \mathcal{A}_s$, where all $\mathcal{A}_s \ne 0$, is the (abelian) group $U(\Gamma)$ with generating set S and defining relations $xy = z$ whenever $0 \ne \mathcal{A}_x\mathcal{A}_y \subseteq \mathcal{A}_z$ (i.e., $xy = x * y$ whenever defined).

$S \hookrightarrow U(\Gamma) \iff \Gamma$ is an (ab.) group grading. Then Γ is a $U(\Gamma)$-grading, and this is universal among realizations (G, ι).

Examples of abelian group gradings

Example

The following is a $\mathbb{Z}$-grading on $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$: $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$ where

$$\mathfrak{g}_{-1} = \mathrm{Span}\left\{\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\right\},\ \mathfrak{g}_0 = \mathrm{Span}\left\{\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\right\},\ \mathfrak{g}_1 = \mathrm{Span}\left\{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right\}.$$

This can also be regarded as a $\mathbb{Z}_m$-grading for any $m > 2$, but the universal group is $\mathbb{Z}$.

Example (Cartan grading)

Let $\mathfrak{g}$ be a s.s. Lie algebra over $\mathbb{C}$, $\mathfrak{h}$ a Cartan subalgebra. Then

$$\mathfrak{g} = \mathfrak{h} \oplus \Big(\bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha\Big)$$

can be viewed as a grading by the root lattice $\langle \Phi \rangle$.
$\mathrm{Supp}\,\Gamma = \{0\} \cup \Phi$; $U(\Gamma) = \langle \Phi \rangle \cong \mathbb{Z}^r$ where $r = \dim \mathfrak{h}$.

Examples continued

Example (Pauli grading on $\mathfrak{sl}_2(\mathbb{C})$)

The *Pauli matrices* $\sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, $\sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $\sigma_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$ define a grading on $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$ by $\mathbb{Z}_2 \times \mathbb{Z}_2$, namely, $\mathfrak{g} = \mathfrak{g}_a \oplus \mathfrak{g}_b \oplus \mathfrak{g}_c$ where

$$\mathfrak{g}_a = \operatorname{Span}\left\{\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\right\},\ \mathfrak{g}_b = \operatorname{Span}\left\{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\right\},\ \mathfrak{g}_c = \operatorname{Span}\left\{\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\right\};$$

$\operatorname{Supp}\Gamma = \{a, b, c\}$; $U(\Gamma) = \mathbb{Z}_2^2 = \{e, a, b, c\}$.

Example (Generalized Pauli grading on $M_n(\mathbb{F})$, $\mathfrak{gl}_n(\mathbb{F})$ and $\mathfrak{sl}_n(\mathbb{F})$)

If $\mathbb{F}$ contains a primitive n-th root of unity ε, then the matrices

$$X = \begin{bmatrix} \varepsilon^{n-1} & 0 & \dots & 0 & 0 \\ 0 & \varepsilon^{n-2} & \dots & 0 & 0 \\ \dots & & & & \\ 0 & 0 & \dots & \varepsilon & 0 \\ 0 & 0 & \dots & 0 & 1 \end{bmatrix} \text{ (“clock”) and } Y = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ \dots & & & & & \\ 0 & 0 & 0 & \dots & 0 & 1 \\ 1 & 0 & 0 & \dots & 0 & 0 \end{bmatrix} \text{ (“shift”).}$$

define a grading on $\mathcal{R} = M_n(\mathbb{F})$ by $\mathbb{Z}_n^2 = \langle a, b \rangle$, namely, $\mathcal{R}_{a^i b^j} = \mathbb{F} X^i Y^j$.

Gradings induced by group homomorphisms

Given $\Gamma : \mathcal{A} = \bigoplus_{g\in G} \mathcal{A}_g$, a group homomorphism $\alpha\colon G \to H$ induces ${}^{\alpha}\Gamma : \mathcal{A} = \bigoplus_{h\in H} \mathcal{A}'_h$ where $\mathcal{A}'_h = \bigoplus_{g\in\alpha^{-1}(h)} \mathcal{A}_g$.

Example (Gradings on polynomial algebra by assigning weights)

$\mathbb{F}[x_1,\dots,x_n] = \bigoplus_{h\in H} \mathcal{A}_h$ with $\mathcal{A}_h = \mathrm{Span}\,\{x_1^{k_1}\cdots x_n^{k_n} \mid w_1^{k_1}\cdots w_n^{k_n} = h\}$ is induced from the standard $\mathbb{Z}^n$-grading by $e_i \mapsto w_i \in H$ (ab. group).

Example ($\mathbb{Z}_2$-gradings on $\mathfrak{sl}_2(\mathbb{F})$)

Let $\Gamma : \mathfrak{sl}_2(\mathbb{F}) = \mathrm{Span}\,\{\left[\begin{smallmatrix}0&0\\1&0\end{smallmatrix}\right]\} \oplus \mathrm{Span}\,\{\left[\begin{smallmatrix}1&0\\0&-1\end{smallmatrix}\right]\} \oplus \mathrm{Span}\,\{\left[\begin{smallmatrix}0&1\\0&0\end{smallmatrix}\right]\}$ be the Cartan grading and $\alpha\colon \mathbb{Z} \to \mathbb{Z}_2$ be the quotient map. Then ${}^{\alpha}\Gamma : \mathfrak{sl}_2(\mathbb{F}) = \mathrm{Span}\,\{\left[\begin{smallmatrix}1&0\\0&-1\end{smallmatrix}\right]\} \oplus \mathrm{Span}\,\{\left[\begin{smallmatrix}0&0\\1&0\end{smallmatrix}\right], \left[\begin{smallmatrix}0&1\\0&0\end{smallmatrix}\right]\}$.

- $\mathbb{F}$ is a.c. $\Rightarrow$ any nontrivial homomorphisms $\mathbb{Z}_2^2 \to \mathbb{Z}_2$ induces from the Pauli grading on $\mathfrak{sl}_2(\mathbb{F})$ a $\mathbb{Z}_2$-grading isomorphic to the above.
- $\mathbb{F} = \mathbb{R} \Rightarrow$ one of the homomorphisms $\mathbb{Z}_2^2 \to \mathbb{Z}_2$ induces the $\mathbb{Z}_2$-grading $\mathfrak{sl}_2(\mathbb{F}) = \mathrm{Span}\,\{\left[\begin{smallmatrix}0&1\\-1&0\end{smallmatrix}\right]\} \oplus \mathrm{Span}\,\{\left[\begin{smallmatrix}1&0\\0&-1\end{smallmatrix}\right], \left[\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right]\}$, which is not isomorphic to the above (the identity comp. is not ad-diagonalizable).

Refinements, coarsenings, and fine group gradings

Definition

Consider a G-grading $\Gamma : \mathcal{A} = \bigoplus_{g \in S \subseteq G} \mathcal{A}_g$ and an H-grading $\Gamma' : \mathcal{A} = \bigoplus_{h \in S' \subseteq H} \mathcal{A}'_h$. We say that Γ' is a *coarsening* of Γ (or Γ is a *refinement* of Γ') if for any $g \in G$ there exists $h \in H$ such that $\mathcal{A}_g \subseteq \mathcal{A}'_h$. If we have $\neq$ for some $g \in S = \operatorname{Supp} \Gamma$, then Γ a *proper* refinement of Γ'. A grading is *fine* if it does not have proper refinements.

Example

$\mathfrak{sl}_2(\mathbb{C}) = \operatorname{Span}\left\{\left[\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right]\right\} \oplus \operatorname{Span}\left\{\left[\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right], \left[\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right]\right\}$ is a $\mathbb{Z}_2$-grading that is a proper coarsening of the Cartan grading and also of the Pauli grading.

Fact: If G is the universal group of Γ, then for any coarsening Γ' there exists a homomorphism $\alpha : G \to H$ such that $\Gamma' = {}^{\alpha}\Gamma$.

Example (Fine elementary grading on $M_n(\mathbb{F})$)

The group grading $M_n(\mathbb{F}) = \operatorname{Span}\{E_{11}, \ldots, E_{nn}\} \oplus \bigoplus_{1 \leq i \neq j \leq n} \mathbb{F} E_{ij}$ is fine. (But it has a proper refinement that is not a group grading.)

Isomorphism and equivalence

Definition (Homomorphism of graded algebras)

Let $\mathcal{A} = \bigoplus_{g\in G} \mathcal{A}_g$ and $\mathcal{A}' = \bigoplus_{g\in G} \mathcal{A}'_g$ be G-graded algebras.
A *homomorphism of graded algebras* (or *graded homomorphism*) is an algebra map $\psi : \mathcal{A} \to \mathcal{A}'$ such that $\psi(\mathcal{A}_g) \subseteq \mathcal{A}'_g$ for all $g \in G$.

In particular, $\mathcal{A}$ and $\mathcal{A}'$ are *isomorphic as G-graded algebras* (or *graded-isomorphic*) if there exists a graded isomorphism $\mathcal{A} \to \mathcal{A}'$.

Definition (Equivalence of graded algebras)

Let $\mathcal{A}$ be an algebra with a G-grading $\Gamma : \mathcal{A} = \bigoplus_{g\in S\subseteq G} \mathcal{A}_g$ and $\mathcal{A}'$ be an algebra with an H-grading $\Gamma' : \mathcal{A}' = \bigoplus_{h\in S'\subseteq H} \mathcal{A}'_h$. Then $\mathcal{A}$ and $\mathcal{A}'$ are *equivalent* if there exists an algebra isomorphism $\psi : \mathcal{A} \to \mathcal{A}'$ and a bijection $\alpha : S \to S'$ such that $\psi(\mathcal{A}_g) = \mathcal{A}'_{\alpha(g)}$ for all $g \in S$.

If G and H are universal groups, then α extends to a group isomorphism $G \to H$ and the condition on ψ says that it is a graded isomorphism $(\mathcal{A}, {}^{\alpha}\Gamma) \to (\mathcal{A}', \Gamma')$.

Classification problems

Definition

- Two G-gradings on $\mathcal{A}$, $\Gamma : \mathcal{A} = \bigoplus_{g\in G} \mathcal{A}_g$ and $\Gamma' : \mathcal{A} = \bigoplus_{g\in G} \mathcal{A}'_g$, are *isomorphic* if $\exists$ an algebra automorphism $\psi : \mathcal{A} \to \mathcal{A}$ such that $\psi(\mathcal{A}_g) = \mathcal{A}'_g$ for all $g \in G$ (i.e., $(\mathcal{A}, \Gamma) \cong (\mathcal{A}, \Gamma')$ as G-graded alg.)
- A G-grading $\mathcal{A} = \bigoplus_{g\in S\subseteq G} \mathcal{A}_g$ and an H-grading $\mathcal{A} = \bigoplus_{h\in S'\subseteq H} \mathcal{A}'_h$ are *equivalent* if $\exists$ an algebra automorphism $\psi : \mathcal{A} \to \mathcal{A}$ and a bijection $\alpha \colon S \to S'$ such that $\psi(\mathcal{A}_g) = \mathcal{A}'_{\alpha(g)}$ for all $g \in S$.

Given a "nice" algebra $\mathcal{A}$, classify

- fine (abelian) group gradings on $\mathcal{A}$ up to equivalence;
- all G-gradings on $\mathcal{A}$ up to isomorphism, for a fixed group G.

If we classified G-gradings on $\mathcal{A}$ for any G, it is still not trivial to determine which of them are fine and which of them are equivalent.

If $\dim \mathcal{A} < \infty$ then for any G-grading Γ on $\mathcal{A}$, $\exists$ a fine grading Δ on $\mathcal{A}$ and a homom. $\alpha : U(\Delta) \to G$ such that $\Gamma = {}^{\alpha}\Delta$, but it is often hard to determine which of the induced gradings are isomorphic.

A transfer theorem

Let $\mathbb{F}$ be an arbitrary field. Let $\mathcal{A}$ and $\mathcal{B}$ be f.d. algebras over $\mathbb{F}$, each with any number of multilinear operations.

Theorem

Suppose we have a homomorphism $\theta\colon \mathbf{Aut}_{\mathbb{F}}(\mathcal{A}) \to \mathbf{Aut}_{\mathbb{F}}(\mathcal{B})$.
Then, for any abelian group G, we have a mapping, $\Gamma \mapsto \theta(\Gamma)$, from G-gradings on $\mathcal{A}$ to G-gradings on $\mathcal{B}$.
If Γ and Γ' are isomorphic then $\theta(\Gamma)$ and $\theta(\Gamma')$ are isomorphic.

For any group homomorphism $\alpha\colon G \to H$, we have $\theta({}^{\alpha}\Gamma) = {}^{\alpha}(\theta(\Gamma))$.

Corollary

If θ is an isomorphism then $\mathcal{A}$ and $\mathcal{B}$ have the same classification of G-gradings up to isomorphism and fine gradings up to equivalence.

Type $A_1 \Leftrightarrow$ quaternion algebras

Let $\mathcal{Q}$ be a quaternion algebra over $\mathbb{F}$. Then $\mathbf{Aut}_{\mathbb{F}}(\mathcal{Q})$ is smooth.
Assume $\operatorname{char}\mathbb{F} \neq 2$. Then $\operatorname{Aut}_{\overline{\mathbb{F}}}(\mathcal{Q}_{\overline{\mathbb{F}}})$ is a simple alg. group of type A_1 and $\mathcal{L} := [\mathcal{Q}, \mathcal{Q}]$ is a simple Lie algebra of type A_1.
The "restriction" map $\mathbf{Aut}_{\mathbb{F}}(\mathcal{Q}) \to \mathbf{Aut}_{\mathbb{F}}(\mathcal{L})$ is an isomorphism.

$\mathbb{F} = \mathbb{R} \Rightarrow$ two simple Lie algebras of type A_1: the split real form $\mathfrak{sl}_2(\mathbb{R})$ and the compact real form $\mathfrak{so}_3(\mathbb{R})$, which correspond to $\mathcal{Q} = \mathbb{H}_s$ and $\mathbb{H}$.
Hence, $\mathfrak{sl}_2(\mathbb{R})$ has 2 fine gradings up to equivalence (Cartan and Pauli, with universal groups $\mathbb{Z}$ and $\mathbb{Z}_2^2$), while $\mathfrak{so}_3(\mathbb{R})$ has 1 (only Pauli).

The G-gradings induced from the Cartan grading on $\mathbb{H}_s$ are

$$\Gamma(g): \ \deg \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = g^{-1}, \ \deg \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = e, \ \deg \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = g^{-1}.$$

$\Gamma(g)$ and $\Gamma(g')$ are isomorphic $\Leftrightarrow g' \in \{g, g^{-1}\}$.

The remaining G-gradings are parametrized by (T, μ) where $T \leq G$, $T \cong \mathbb{Z}_2^r$ with $0 \leq r \leq 2$, and $\mu : T \to \{\pm 1\}$ is a character (trivial for $\mathbb{H}$).

Type $G_2 \Leftrightarrow$ octonion algebras

Let $\mathcal{C}$ be a Cayley algebra over $\mathbb{F}$. Then $\mathbf{Aut}_{\mathbb{F}}(\mathcal{C})$ is smooth.
Assume $\operatorname{char}\mathbb{F} \neq 2, 3$. Then $\operatorname{Aut}_{\overline{\mathbb{F}}}(\mathcal{C}_{\overline{\mathbb{F}}})$ is a simple alg. group of type G_2 and $\mathcal{L} := \operatorname{Der}_{\mathbb{F}}(\mathcal{C})$ is a simple Lie algebra of type G_2.
$\operatorname{Ad} : \mathbf{Aut}_{\mathbb{F}}(\mathcal{C}) \to \mathbf{Aut}_{\mathbb{F}}(\mathcal{L})$ is an isomorphism.

Hence, Ad gives a bijection between (isom. classes of) G-gradings on $\mathcal{C}$ and $\mathcal{L}$, also between (equiv. classes of) fine gradings on $\mathcal{C}$ and $\mathcal{L}$.
Ad maps a grading $\mathcal{C} = \bigoplus_{g \in G} \mathcal{C}_g$ to the following grading on $\mathcal{L}$:
$\mathcal{L}_g := \{D \in \operatorname{Der}_{\mathbb{F}}(\mathcal{C}) \mid D(\mathcal{C}_h) \subseteq \mathcal{C}_{gh} \ \forall h \in G\}$.

Theorem (Elduque 1998)

Any nontrivial grading on a Cayley algebra is, up to equivalence, either a grading induced by the Cayley–Dickson doubling process or a coarsening of the Cartan grading on the split Cayley algebra.

This leads to a classification of gradings on $\mathcal{C}$ (Elduque–K. 2018).

$\mathbb{F} = \mathbb{R}$ (CDM 2010) $\Rightarrow$ 2 fine gradings on $\mathbb{O}_s$ and split G_2 (with universal groups $\mathbb{Z}^2$ and $\mathbb{Z}_2^3$) and 1 on $\mathbb{O}$ and compact G_2 (only $\mathbb{Z}_2^3$).

$A, B, C, D \Leftrightarrow$ central simple assoc. alg. with involution

Assume $\mathrm{char}\,\mathbb{F} \neq 2$. Let $\mathcal{R}$ be a f.d. central simple associative algebra over $\mathbb{F}$, $\dim_{\mathbb{F}} \mathcal{R} = n^2$, and φ be an $\mathbb{F}$-linear involution on $\mathcal{R}$ such that

- B_r: $n = 2r + 1$ ($\Rightarrow \mathcal{R} \cong M_n(\mathbb{F})$ and φ is orthogonal), $r \geq 2$;
- C_r: $n = 2r$ and φ is symplectic, $r \geq 2$;
- D_r: $n = 2r$ and φ is orthogonal, $r \geq 3$.

Let $\mathcal{L} = \mathrm{Skew}(\mathcal{R}, \varphi)$. Then $\mathcal{L}$ is a simple Lie algebra of the indicated type, and the restriction map $\mathbf{Aut}_{\mathbb{F}}(\mathcal{R}, \varphi) \to \mathbf{Aut}_{\mathbb{F}}(\mathcal{L})$ is an isomorphism, except in the case D_4.

Let $\mathcal{R}$ to be a f.d. s.s. associative algebra with $Z(\mathcal{R}) = \mathbb{K}$, where $\mathbb{K}$ is a quadratic étale algebra over $\mathbb{F}$ (either $\mathbb{F} \times \mathbb{F}$ or a quadratic field extension of $\mathbb{F}$), and φ be an involution of the second kind (i.e., $\mathbb{F}$-linear but not $\mathbb{K}$-linear $\Leftrightarrow$ $(\mathcal{R}, \varphi)$ is central simple). Hence $\dim_{\mathbb{F}} \mathcal{R} = 2n^2$.

- A_r: $n = r + 1$, $r \geq 2$. Let $\mathcal{L}$ be the quotient of the derived algebra of $\mathrm{Skew}(\mathcal{R}, \varphi)$ modulo its center.

The "restriction" map $\mathbf{Aut}_{\mathbb{F}}(\mathcal{R}, \varphi) \to \mathbf{Aut}_{\mathbb{F}}(\mathcal{L})$ is an isomorphism, except in the case $n = 3 = \mathrm{char}\,\mathbb{F}$.

Graded-simple associative algebras

$\mathcal{D}$ is a *graded-division* algebra if all nonzero homogeneous elements are invertible ($\Rightarrow$ graded $\mathcal{D}$-modules have a graded basis).

Theorem ("Graded Wedderburn Theorem")

Let $\mathcal{R}$ be a G-graded algebra (or ring). Then $\mathcal{R}$ is graded-simple and satisfies d.c.c. on graded one-sided ideals $\Leftrightarrow$ there exists a graded-division algebra $\mathcal{D}$ and a graded right $\mathcal{D}$-module $\mathcal{V}$ of finite rank such that $\mathcal{R} \cong \mathrm{End}_{\mathcal{D}}(\mathcal{V})$ as G-graded algebras.

$\mathrm{End}^{\mathrm{gr}}_{\mathcal{D}}(\mathcal{V}) := \bigoplus_{g\in G} \mathrm{End}_{\mathcal{D}}(\mathcal{V})_g$ is a G-graded algebra where
$\mathrm{End}_{\mathcal{D}}(\mathcal{V})_g := \{T \in \mathrm{End}_{\mathcal{D}}(\mathcal{V}) \mid T(\mathcal{V}_h) \subseteq \mathcal{V}_{gh}\ \forall h \in G\}$.

Select a graded $\mathcal{D}$-basis $\{v_1, \ldots, v_k\}$ of $\mathcal{V}$, and let $\deg v_i = g_i$.
$\mathcal{R} \cong M_k(\mathbb{F}) \otimes \mathcal{D}$, where $\deg(E_{ij} \otimes d) = g_i(\deg d)g_j^{-1}$ for homog. $d \in \mathcal{D}$.

$\mathcal{R} = M_n(\mathbb{F}) \ \Rightarrow\ \mathcal{D} \cong M_\ell(\mathbb{F})$ with a *division grading*, $k\ell = n$.
If $\mathbb{F}$ is a.c. then $\mathcal{D}_e = \mathbb{F}$, hence, with any G-grading on $M_n(\mathbb{F})$, we have $M_n(\mathbb{F}) \cong M_k(\mathbb{F}) \otimes M_\ell(\mathbb{F})$ where all homog. components of $M_\ell(\mathbb{F})$ are 1-dim (Bahturin–Sehgal–Zaicev 2001).

Central simple graded-division algebras

Theorem (Havlíček–Patera–Pelantová 1998 and BSZ 2001 for $\operatorname{char}\mathbb{F}=0$; Bahturin–Zaicev 2003)

Let T be an ab. group and $\mathbb{F}$ an a.c. field. Then, for any division grading on $\mathcal{D}=M_\ell(\mathbb{F})$ with support T, there exists a decomposition $T=H_1\times\cdots\times H_r$ such that $H_i\cong\mathbb{Z}_{\ell_i}^2$ and $\mathcal{D}\cong M_{\ell_1}(\mathbb{F})\otimes\cdots\otimes M_{\ell_r}(\mathbb{F})$ where $M_{\ell_i}(\mathbb{F})$ has a generalized Pauli grading by H_i.

More generally, let $\mathcal{D}$ be a graded-division algebra with support T and $\mathcal{D}_e=\mathbb{F}$. Pick $0\neq X_t\in\mathcal{D}_t$ for any $t\in T$. Then $\mathcal{D}_t=\mathbb{F}X_t$ for any $t\in T$, so $\mathcal{D}$ is a *twisted group algebra* of T.

If T is abelian, we have $X_sX_t=\beta(s,t)X_tX_s$ where the mapping $\beta: T\times T\to\mathbb{F}^\times$ is an *alternating bicharacter*, i.e., multiplicative in each variable and satisfies $\beta(t,t)=1$ for all $t\in T$.

Assume $|T|<\infty$ and set $\operatorname{rad}\beta:=\{s\in T\mid \beta(s,t)=1\ \forall t\in T\}$.
$\mathcal{D}$ is central simple over $\mathbb{F}\Leftrightarrow\beta$ is *nondegenerate*, i.e., $\operatorname{rad}\beta=\{e\}$.

Central simple graded-division algebras continued

If β is nondegenerate, T admits a *symplectic basis*, i.e., a generating set of the form $\{a_1, b_1, \ldots, a_m, b_m\}$ with $o(a_i) = o(b_i) = n_i \geq 2$ such that $\beta(a_i, b_i) = \zeta_i$, with $\zeta_i \in \mathbb{F}$ a primitive root of unity of degree n_i, while $\beta(a_i, b_j) = 1$ for $i \neq j$ and $\beta(a_i, a_j) = \beta(b_i, b_j) = 1$ for all i, j.

The elements $X_i := X_{a_i}$ and $Y_i := X_{b_i}$ generate $\mathcal{D}$ as an $\mathbb{F}$-algebra and satisfy the following defining relations:

$$X_i^{n_i} = \mu_i,\ Y_i^{n_i} = \nu_i,\ X_i Y_i = \zeta_i Y_i X_i,$$
$$X_i X_j = X_j X_i,\ Y_i Y_j = Y_j Y_i,\ \text{and } X_i Y_j = Y_j X_i \text{ for } i \neq j,$$

so $\mathcal{D}$ is a tensor product of (graded) *cyclic* or *symbol algebras*:

$$\mathcal{D} \cong (\mu_1, \nu_1)_{\zeta_1^{-1}, \mathbb{F}} \otimes \cdots \otimes (\mu_m, \nu_m)_{\zeta_m^{-1}, \mathbb{F}}.$$

$\mathbb{F} = \mathbb{R} \Rightarrow$ all $n_i = 2 \Rightarrow T$ is an elementary abelian 2-group and $\mathcal{D}$ is a tensor product of (graded) quaternion algebras.

Simple f.d. graded-division algebras with abelian T and any $\mathcal{D}_e$ are classified (Bahturin–Zaicev 2016 and Rodrigo 2016).

Antiautomorphisms on $\mathrm{End}_{\mathcal{D}}(\mathcal{V})$

Theorem (Elduque 2010)

Let G be an abelian group and consider the G-graded algebra $\mathcal{R} = \mathrm{End}_{\mathcal{D}}(\mathcal{V})$ where $\mathcal{D}$ is a graded-division algebra and $\mathcal{V}$ is a nonzero graded right $\mathcal{D}$-module of finite rank.

(1) *If φ is an antiautomorphism of the graded algebra $\mathcal{R}$, then there exists an antiautomorphism φ_0 of the graded algebra $\mathcal{D}$ and a nondegenerate φ_0-sesquilinear form $B : \mathcal{V} \times \mathcal{V} \to \mathcal{D}$, by which we mean a nondegenerate $\mathbb{F}$-bilinear mapping that is φ_0-sesquilinear over $\mathcal{D}$, i.e.,*

   (i) $B(vd, w) = \varphi_0(d)B(v, w)$ *and* $B(v, wd) = B(v, w)d$,

   and homogeneous of some degree $g_0 \in G$, i.e.,

   (ii) $B(\mathcal{V}_a, \mathcal{V}_b) \subset \mathcal{D}_{g_0 ab}$ *for all* $a, b \in G$,

   such that φ is the adjunction with respect to B, i.e.,

   (iii) $B(rv, w) = B(v, \varphi(r)w)$ *for all* $r \in \mathcal{R}$ *and* $v, w \in \mathcal{V}$.

(2) *Another pair (φ_0', B') satisfies these conditions if and only if there exists $d \in \mathcal{D}_{\mathrm{gr}}^{\times}$ such that $B' = dB$ and $\varphi_0' = \mathrm{Int}(d) \circ \varphi_0$.*

Involutions on $\mathrm{End}_{\mathcal{D}}(\mathcal{V})$

Theorem (Elduque–K.–Rodrigo 2021)

(3) *If φ is an involution, then the pair (φ_0, B) as in part (1) can be chosen so that φ_0 is an involution and B is hermitian or skew-hermitian, by which we mean that $B(w, v) = \delta\varphi_0(B(v, w))$ for all $v, w \in \mathcal{V}$, where $\delta = 1$ (hermitian) or $\delta = -1$ (skew).*

(4) *Let (φ_0, B) be a pair chosen for φ as in part (3). Then:*
   - (i) *Any other such pair (φ_0', B') has the form $(\mathrm{Int}(d) \circ \varphi_0, dB)$ where $d \in \mathcal{D}_{\mathrm{gr}}^{\times}$ satisfies $\varphi_0(d) = d$ (symmetric) or $\varphi_0(d) = -d$ (skew).*
   - (ii) *If φ_0' is a degree-preserving involution of $\mathcal{D}$ such that $\varphi_0'\varphi_0^{-1}$ is an inner automorphism of $\mathcal{D}$, then there exists $d \in \mathcal{D}_{\mathrm{gr}}^{\times}$ such that $\varphi_0' = \mathrm{Int}(d) \circ \varphi_0$ and the pair (φ_0', dB) satisfies part (3).*

Corollary

Assume that $(\mathcal{R}, \varphi)$ is central simple as an algebra with involution. Then $\mathcal{D}$ admits a degree-preserving involution of the same kind as φ, and for any such involution φ_0, there exists B as in part (3).

The graded algebras with involution $\mathcal{M}^{\mathrm{ex}}(\mathcal{D}, k)$

Let $\mathcal{D}$ be a graded-division algebra with abelian support T and let $k \geq 1$ be an integer.

Definition

Let $\widetilde{G}(T, k) := F \times T$, where F is the free abelian group generated by the symbols $\tilde{g}_1, \ldots, \tilde{g}_k$.

(i) The $\widetilde{G}(T, k)$-grading on $M_k(\mathcal{D}) \cong M_k(\mathbb{F}) \otimes \mathcal{D}$ defined by

$$\deg(E_{ij} \otimes d) = \tilde{g}_i \tilde{g}_j^{-1} t \;\text{ for any } 0 \neq d \in \mathcal{D}_t$$

is denoted $\Gamma_{\mathcal{M}}(\mathcal{D}, k)$ and the resulting graded algebra $\mathcal{M}(\mathcal{D}, k)$.

(ii) Using the same grading on the opposite algebra, we obtain a $\widetilde{G}(T, k)$-graded algebra $\mathcal{M}(\mathcal{D}, k) \times \mathcal{M}(\mathcal{D}, k)^{\mathrm{op}}$ so that the exchange involution $\mathrm{ex} : (x, y) \mapsto (y, x)$ is degree-preserving. The resulting graded algebra with involution will be denoted by $\mathcal{M}^{\mathrm{ex}}(\mathcal{D}, k)$ and its grading by $\Gamma_{\mathcal{M}^{\mathrm{ex}}}(\mathcal{D}, k)$.

The graded algebras with involution $\mathcal{M}(\mathcal{D}, \varphi_0, q, s, \underline{d}, \delta)$

Let φ_0 be a degree-preserving involution on $\mathcal{D}$, let $q, s \geq 0$ be integers (not both zero), let $\delta \in \{\pm 1\}$ and let $\underline{d} = (d_1, \ldots, d_q)$ be a q-tuple of nonzero homogeneous elements of $\mathcal{D}$ such that $\varphi_0(d_i) = \delta d_i$ for all i.

Let $t_i := \deg d_i$ and let F be the free abelian group generated by the symbols $\tilde{g}_1, \ldots, \tilde{g}_k$ where $k := q + 2s$. Define $\widetilde{G} = \widetilde{G}(T, q, s, \underline{t})$ to be the quotient of $F \times T$ modulo the following relations:

$$\tilde{g}_1^2 t_1^{-1} = \ldots = \tilde{g}_q^2 t_q^{-1} = \tilde{g}_{q+1} \tilde{g}_{q+2} = \ldots = \tilde{g}_{q+2s-1} \tilde{g}_{q+2s}.$$

Definition

The $\widetilde{G}(T, q, s, \underline{t})$-graded algebra $M_k(\mathcal{D})$ with involution given by $\varphi(X) = \Phi^{-1} \varphi_0(X)^{\top} \Phi$ for all $X \in M_k(\mathcal{D})$ where

$$\Phi = \operatorname{diag}\left(d_1, \ldots, d_q, \begin{bmatrix} 0 & 1 \\ \delta & 0 \end{bmatrix}, \ldots, \begin{bmatrix} 0 & 1 \\ \delta & 0 \end{bmatrix} \right)$$

is denoted $\mathcal{M}(\mathcal{D}, \varphi_0, q, s, \underline{d}, \delta)$ and its grading $\Gamma_{\mathcal{M}}(\mathcal{D}, \varphi_0, q, s, \underline{d}, \delta)$.

Fine gradings on algebras with involution

Corollary

Let φ be an involution on an artinian algebra $\mathcal{R}$. If $(\mathcal{R},\varphi)$ is simple then, for any G-grading Γ on $(\mathcal{R},\varphi)$, exactly one of the following holds:

- *Γ is the image of some ${}^{\alpha}\Gamma_{\mathcal{M}^{\mathrm{ex}}}(\mathcal{D},k)$ under an isomorphism of algebras with involution, where $T := \operatorname{Supp}\mathcal{D}$ is a subgroup of G and $\alpha : \widetilde{G}(T,k) \to G$ is a homomorphism with $\alpha|_T = \mathrm{id}_T$;*
- *Γ is the image of some ${}^{\alpha}\Gamma_{\mathcal{M}}(\mathcal{D},\varphi_0,q,s,\underline{d},\delta)$ under an isomorphism of algebras with involution, where $T := \operatorname{Supp}\mathcal{D}$ is a subgroup of G, φ_0 is a degree-preserving involution on $\mathcal{D}$, and $\alpha : \widetilde{G}(T,q,s,\underline{t}) \to G$ is a homomorphism with $\alpha|_T = \mathrm{id}_T$.*

Theorem (Elduque–K.–Rodrigo 2021)

Assume $\mathcal{D}$ is finite-dimensional. If $(q,s) \neq (2,0)$ and the grading $\Gamma_{\mathcal{D}}$ on $\mathcal{D}$ is fine, then so is $\Gamma = \Gamma_{\mathcal{M}}(\mathcal{D},\varphi_0,q,s,\underline{d},\delta)$. Conversely, if $(\mathcal{D},\varphi_0)$ is central simple over $\mathbb{R}$ and Γ is fine, then so is $\Gamma_{\mathcal{D}}$.

Classification up to equivalence: central simple over $\mathbb{R}$

Theorem (Elduque–K.–Rodrigo 2021)

Let $\mathcal{R}$ be a f.d. central simple algebra over $\mathbb{R}$ and φ an involution on $\mathcal{R}$. Set $\delta = +1$ if φ is orthogonal and $\delta = -1$ if φ is symplectic. If $(\mathcal{R}, \varphi)$ is equipped with a group grading Γ, then Γ is fine if and only if $\mathcal{R}$ is equivalent as a graded algebra with involution to one of the following:

- $\mathcal{M}(2m; \mathbb{R}; q, s, \underline{d}, \delta) := \mathcal{M}\big(\mathcal{D}(2m; +1), *, q, s, \underline{d}, \delta\big)$ *where* $m \geq 0$, $X^* = X^\top$ *for all* $X \in \mathcal{D}(2m; +1) \cong M_{2^m}(\mathbb{R})$,
- $\mathcal{M}(2m; \mathbb{H}; q, s, \underline{d}, \delta) := \mathcal{M}\big(\mathcal{D}(2m; -1), *, q, s, \underline{d}, -\delta\big)$ *where* $m \geq 1$, $X^* = \overline{X}^\top$ *for all* $X \in \mathcal{D}(2m; -1) \cong M_{2^{m-1}}(\mathbb{H})$,

where in the case $(q, s) = (2, 0)$, the pair $\underline{d} = (d_1, d_2)$ satisfies $\deg d_1 \neq \deg d_2$. Moreover, the above graded algebras with involution are classified up to equivalence by the following invariants: m, q, s, δ, $\mathrm{signature}(\underline{d})$, and the orbit of the multiset $\{\deg d_1, \ldots, \deg d_q\}$ in $T \cong \mathbb{Z}_2^{2m}$ under the action of the orthogonal group $\mathrm{O}(T, \mu)$ where $\mu : T \to \{\pm 1\}$ is the quadratic form defined by $X_t^* = \mu(t) X_t$ for $X_t \in \mathcal{D}_t$.

Classification up to equivalence over $\mathbb{R}$ and $\mathbb{C}$

Let $\mathcal{D}$ be a f.d. graded-division algebra with support T and $\mathcal{D}_e = \mathbb{F}$. Suppose $\mathcal{D} \neq \mathbb{F}$ admits a degree-preserving involution that makes it central simple as an algebra with involution. Then $\operatorname{char} \mathbb{F} \neq 2$ and T is an elementary abelian 2-group, i.e., a vector space over $GF(2)$.

Any such involution maps $X_t \mapsto \eta(t) X_t$ where $\eta : T \to \{\pm 1\}$ is a *nonsingular* quadratic form with polarization β:

$$\eta(st) = \eta(s)\eta(t)\beta(s,t) \text{ for all } s, t \in T,$$

and either $\operatorname{rad}\beta = \{e\}$ or $\operatorname{rad}\beta = \{e, f\}$ with $\eta(f) = -1$.

• If $\mathbb{F} = \mathbb{R}$ then $X_t^2 \in \mu(t)\mathbb{R}_{>0}$ defines a quadratic form $\mu : T \to \{\pm 1\}$ and hence a *distinguished involution* on $\mathcal{D}$.
μ is nonsingular if $\mathbb{K} := Z(\mathcal{D})$ is $\mathbb{R}$ or $\mathbb{C}$ (but not $\mathbb{R} \times \mathbb{R}$).

• If $\mathbb{F}$ is a.c. and $\mathbb{K} = \mathbb{F}$ ($\Leftrightarrow \operatorname{rad}\beta = \{e\}$), let $Q(T, \beta)$ be the set of quadratic forms on T whose polarization is β. Then $\operatorname{Sp}(T, \beta)$ acts naturally on both T and $Q(T, \beta)$, and $Q(T, \beta)$ is a T-*torsor*, i.e., admits a simply transitive T-action compatible with the $\operatorname{Sp}(T, \beta)$-actions.

Classification up to equivalence: summary

Let $(\mathcal{R}, \varphi)$ be central simple as an algebra with involution over $\mathbb{F}$, $\mathrm{char}\,\mathbb{F} \neq 2$. Then fine gradings on $(\mathcal{R}, \varphi)$ are classified up to equivalence by a finite ab. group T, with $|T|$ a divisor of $\dim \mathcal{R}$, and an orbit of multisets in a vector space over $GF(2)$ as follows:

$\mathbb{K} := Z(\mathcal{R})$	$\mathbb{F}$ is real closed	$\mathbb{F}$ is alg. closed
$\mathbb{F}$	$\mathrm{O}(T, \mu)$ on T	$\mathrm{Sp}(T, \beta)$ on $Q(T, \beta)$
$\mathbb{F} \times \mathbb{F}$, nontriv.	$\mathrm{AO}(\overline{T}, \bar{\mu})$ on $\overline{T} := T/\langle f \rangle$	$\mathrm{ASp}(\overline{T}, \bar{\beta})$ on $\overline{T} := T/\langle f \rangle$
$\mathbb{F} \times \mathbb{F}$, triv. gr.	no multiset	no multiset
$\mathbb{F}[i]$, nontriv.	$\mathrm{Sp}(\overline{T}, \bar{\beta})$ on $\overline{T} := T/\langle f \rangle$	
$\mathbb{F}[i]$, triv. gr.	$\mathrm{Sp}(V, \mathcal{F})$ on $V := T/T^{[2]}$	

where T is 2-elementary except in the shaded cells ($\Rightarrow T \cong A \times A$).

The average number $\hat{N}(k)$ of fine gradings for $\deg \mathcal{R} \leq k$ satisfies $\exp(bk^{2/3}) \leq \hat{N}(k) \leq \exp(ck^{2/3})$ for a.c. $\mathbb{F}$ (K.–Parsons–Sadov 2013).