Quotient groups of IA-automorphisms of free metabelian groups of finite rank

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1 Preliminaries

Let G be a group. For subgroups X and Y of G, we write $[X,Y]$ for the subgroup of G generated by all commutators $[x,y] = x^{-1}y^{-1}xy$. For a positive integer c, let $\gamma_c(G)$ be the c-th term of the lower central series of G. We point out that $\gamma_2(G) = G'$; that is, the derived group of G, and $G'' = (G')'$.

We denote by $\mathrm{Aut}(G)$ the automorphism group of G. For $c \geq 2$, the natural epimorphism from G onto $G/\gamma_c(G)$ induces a group homomorphism, say $\pi_{c,G}$,

$$\pi_{c,G} : \mathrm{Aut}(G) \to \mathrm{Aut}(G/\gamma_c(G)),$$

Let $\mathrm{I}_c\mathrm{A}(G) = \mathrm{Ker}\pi_{c,G}$. For $c = 2$, we write $\mathrm{I}_2\mathrm{A}(G) = \mathrm{IA}(G)$. The elements of $\mathrm{IA}(G)$ are called IA-automorphisms of G. By an inductive argument on c, we get

$$\gamma_c(\mathrm{IA}(G)) \subseteq \mathrm{I}_{c+1}\mathrm{A}(G)$$

for all $c \geq 1$. For $c \geq 2$, we write

$$\mathcal{L}^c(\mathrm{IA}(G)) = \mathrm{I}_c\mathrm{A}(G)/\mathrm{I}_{c+1}\mathrm{A}(G).$$

In general, it is a very difficult problem to determine whether or not $\gamma_c(\mathrm{IA}(G)) = \mathrm{I}_{c+1}\mathrm{A}(G)$ for all $c \geq 2$. Moreover, it is quite difficult to study the structure of the quotient groups $\mathrm{gr}_c(\mathrm{IA}(G)) = \gamma_c(\mathrm{IA}(G))/\gamma_{c+1}(\mathrm{IA}(G))$ for all $c \geq 1$.

For a positive integer n, with $n \geq 2$, let F_n be a free group of rank n with a free generating set $\{f_1, \ldots, f_n\}$. Let G_n be a free non-abelian poly-nilpotent group of rank n.

1. G_n is residually nilpotent [Gruenberg 1957] and
2. Each quotient group $\mathrm{gr}_c(G_n) = \gamma_c(G_n)/\gamma_{c+1}(G_n)$ is a free abelian group of finite rank [Shmel'kin 1964].

For $c \geq 2$, let $G_{n,c} = G_n/\gamma_c(G_n)$. The natural group epimorphism from $G_{n,c+1}$ onto $G_{n,c}$ induces a group homomorphism

$$\psi_{c+1,G_n} : \mathrm{Aut}(G_{n,c+1}) \longrightarrow \mathrm{Aut}(G_{n,c}).$$

It is well known that ψ_{c+1,G_n} is onto. Define

$$A_{c+1}(G_n) = \mathrm{Ker}\psi_{c+1,G_n} \quad \text{and} \quad A^*_{c+1}(G_n) = \mathrm{Im}\pi_{c+1,G_n} \cap \mathrm{Ker}\psi_{c+1,G_n}.$$

It may be shown the following result.

Lemma 1.1 *With the previous notations, for positive integers $n, c \geq 2$,*

1. $A_{c+1}(G_n) \cong \underbrace{\mathrm{gr}_c(G_n) \oplus \cdots \oplus \mathrm{gr}_c(G_n)}_{n-\text{times}} = \mathrm{gr}_c(G_n)^{\oplus n}$ *as free abelian groups.*
2. $A^*_{c+1}(G_n) \cong \mathcal{L}^c(\mathrm{IA}(G_n)) = \mathrm{I}_c\mathrm{A}(G_n)/\mathrm{I}_{c+1}\mathrm{A}(G_n)$ *as free abelian groups.*

For any subgroup H of $\mathrm{IA}(G_n)$ and a positive integer r, with $r \geq 2$, let $H_r = H \cap \mathrm{I}_r\mathrm{A}(G_n)$. For $q \in \mathbb{N}$, let

$$\mathcal{L}^q_1(H) = \gamma_q(H)(\mathrm{I}_{q+2}\mathrm{A}(G_n))/\mathrm{I}_{q+2}\mathrm{A}(G_n),$$

and, for $q \geq 2$, let

$$\mathcal{L}^q_2(H) = H_q(\mathrm{I}_{q+1}\mathrm{A}(G_n))/\mathrm{I}_{q+1}\mathrm{A}(G_n).$$

Note that, for $q \in \mathbb{N}$, $\mathcal{L}^q_1(H) \subseteq \mathcal{L}^{q+1}_2(H)$.

The natural epimorphism from F_n onto G_n induces a group homomorphism

$$\rho_n : \mathrm{Aut}(F_n) \to \mathrm{Aut}(G_n).$$

Write $T_{n,G} = \mathrm{Im}\rho_n$. The elements of $T_{n,G}$ are called tame automorphisms of G_n.

For $n \geq 2$, we denote

$$H_{n,G} = T_{n,G} \cap \mathrm{IA}(G_n).$$

For a positive integer r, with $r \geq 2$, let

$$H_{n,G,r} = T_{n,G} \cap \mathrm{I}_r\mathrm{A}(G_n).$$

Note that $H_{n,G,r} = H_{n,G} \cap \mathrm{I}_r\mathrm{A}(G_n)$ for $r \geq 2$. Since $G_n/G'_n \cong F_n/F'_n$, ρ_n induces a group homomorphism

$$\rho'_n : \mathrm{IA}(F_n) \to \mathrm{IA}(G_n).$$

Lemma 1.2 *With the previous notations, for all n, with $n \geq 2$, $H_{n,G} = \rho'_n(\mathrm{IA}(F_n))$ and $\gamma_2(H_{n,G}) = H_{n,G,3} = \rho'_n(\mathrm{I}_3\mathrm{A}(F_n))$.*

Theorem 1.1 ([Kofinas-P. 2020]) *For a positive integer n, with $n \geq 2$, let G_n be a free non-abelian poly-nilpotent group of rank n. Then, for all integers n and r, with $n \geq 2$, $\mathcal{L}_1^r(H_{n,G}) = \mathcal{L}_1^r(\mathrm{IA}(G_n))$.*

2 History and Statement

2.1 Some history on F_n

For $G = F_n$, the above problem is known as the Andreadakis' conjecture.

- It has been proved in [Andreadakis 1965] that $\gamma_c(\mathrm{IA}(F_2)) = \mathrm{I}_{c+1}\mathrm{A}(F_2)$ for all $c \geq 2$, and that the first three terms of this series coincide in the case $n = 3$ (Andreadakis 1965]). That is, $\gamma_c(\mathrm{IA}(F_3)) = \mathrm{I}_{c+1}\mathrm{A}(F_3)$, $c = 1, 2, 3$.

- For $n \geq 2$, $\gamma_2(\mathrm{IA}(F_n)) = \mathrm{I}_3\mathrm{A}(F_n)$ (see, for example, Pettet 2005]).

- For $n \geq 3$, Pettet [2005] showed that $\gamma_3(\mathrm{IA}(F_n))$ has finite index in $\mathrm{I}_4\mathrm{A}(F_n)$, and Satoh [2019] proved that $\gamma_3(\mathrm{IA}(F_n)) = \mathrm{I}_4\mathrm{A}(F_n)$.

- The Andreadakis' conjecture is **not true in general**. It is shown [Bartholdi 2013, 2016] that

$$\mathrm{I}_5\mathrm{A}(F_3)/\gamma_4(\mathrm{IA}(F_3) \cong C_2^{\oplus 14} \oplus C_3^{\oplus 9}$$

and

$$\mathrm{I}_6\mathrm{A}(F_3)/\gamma_5(\mathrm{IA}(F_3) \otimes_{\mathbb{Z}} \mathbb{Q} \cong \mathbb{Q}^{\oplus 3}$$

by using a computer. For $n \geq 4$, the conjecture is still open.

- Darne [2017] proved that the **stable Andreadakis' conjecture** is true. Namely, the natural map

$$\mathrm{gr}_c(\mathrm{IA}(F_n)) = \gamma_c(\mathrm{IA}(F_n))/\gamma_{c+1}(\mathrm{IA}(F_n)) \longrightarrow \mathcal{L}^{c+1}(\mathrm{IA}(F_n)) = \mathrm{I}_{c+1}\mathrm{A}(F_n)/\mathrm{I}_{c+2}\mathrm{A}(F_n)$$

induced from the inclusion is surjective for sufficient large n.

2.2 Statement

For $n \geq 2$, let $M_n = F_n/F_n''$, that is, M_n is a free metabelian group of rank n. The set $\{x_1, \ldots, x_n\}$, where $x_i = f_i F_n''$ and $i \in \{1, \ldots, n\}$, is a free generating set of M_n.

The natural group epimorphism from F_n onto M_n induces a group homomorphism α_n from $\mathrm{Aut}(F_n)$ into $\mathrm{Aut}(M_n)$ and let $T_n = \mathrm{Im}\alpha_n$. In a series of papers (Andreadakis 1965, Bachmuth 1965, Bachmuth and Mochizuki 1982, 1985, Chein 1968, Romankov 1985), it has been shown that α_n, with $n \neq 3$, is onto and α_3 is not onto.

For $n, c \geq 2$, the set

$$\mathcal{M}_{n,c} = \{[x_{i_1}, x_{i_2}, \ldots, x_{i_c}]\gamma_{c+1}(M_n) : i_1 > i_2 \leq i_3 \leq \cdots \leq i_c; i_1, \ldots, i_c \in [n]\}$$

is a $\mathbb{Z}$-basis of $\mathrm{gr}_c(M_n) = \gamma_c(M_n)/\gamma_{c+1}(M_n)$ and the rank of $\mathrm{gr}_c(M_n)$ is $(c-1)\binom{n+c-2}{n-2}$ and so, the rank of $A_{c+1}(M_n)$ is $n(c-1)\binom{n+c-2}{n-2}$.

• It follows, by a result of Andreadakis [1965] (see, also, [Bachmuth 1966]), that the rank of $A_3^*(M_n)$ is $n\binom{n}{2}$.

• By a result of Bachmuth [1966], we have the rank of $A_{c+1}^*(M_n)$, with $c \geq 3$, is $n(c-1)\binom{n+c-2}{n-2} - \binom{n+c-2}{n-1}$.

Theorem 2.1 ([Kofinas-P. 2020]) *Let M_n be a free metabelian group of finite rank n, with $n \geq 2$. Then,*

1. *For $c \geq 1$, $\gamma_c(\mathrm{IA}(M_2)) = \mathrm{I}_{c+1}\mathrm{A}(M_2))$. In particular, for $c \geq 2$,*

$$\mathcal{L}^c(\mathrm{IA}(M_2)) = \gamma_{c-1}(\mathrm{IA}(M_2))/\gamma_c(\mathrm{IA}(M_2)).$$

2. *$\mathcal{L}^4(\mathrm{IA}(M_3)) \neq \gamma_3(\mathrm{IA}(M_3))\mathrm{I}_5\mathrm{A}(M_3)/\mathrm{I}_5\mathrm{A}(M_3)$ and so, $\gamma_3(\mathrm{IA}(M_3)) \neq \mathrm{I}_4\mathrm{A}(M_3)$.*

3. *For $n, c \geq 4$, $\mathcal{L}^c(\mathrm{IA}(M_n)) = \gamma_{c-1}(\mathrm{IA}(M_n))\mathrm{I}_{c+1}\mathrm{A}(M_n)/\mathrm{I}_{c+1}\mathrm{A}(M_n)$.*

3 Proof of Theorem 2.1

3.1 The case $n = 2$

By combining results from [Andeadakis 1965] and [Bachmuth 1965], we have Theorem 2.1 for $n = 2$. That is, $\gamma_c(\mathrm{IA}(M_2)) = \mathrm{I}_{c+1}\mathrm{A}(M_2)$ for all c.

3.2 The case $n = 3$

We write $H_3 = T_3 \cap \mathrm{IA}(M_3) = \rho_3'(\mathrm{IA}(F_3))$ (by Lemma 1.2) and, for all $r \in \mathbb{N}$, with $r \geq 2$, $H_{3,r} = T_3 \cap \mathrm{I}_r\mathrm{A}(M_3)$. Note that $H_3 = H_{3,2}$. It can be deduced from [Chein 1968, Proof of Theorem 8] that $\mathcal{L}_2^4(H_3) \neq \mathcal{L}^4(\mathrm{IA}(M_3)) \cong A_5^*(M_3)$.

By a result of Andreadakis [Andreadakis 1965], we have

$$\gamma_3(H_3) = \gamma_3(\rho_3'(\mathrm{IA}(F_3))) = \rho_3'(\gamma_3(\mathrm{IA}(F_3))) = \rho_3'(\mathrm{I}_4\mathrm{A}(F_3)).$$

But $\gamma_3(H_3) = H_{3,4}$ and so,

$$\mathcal{L}_1^3(H_3) = \mathcal{L}_2^4(H_3). \tag{4.1}$$

By Theorem 1.1, we have $\mathcal{L}_1^3(H_3) = \mathcal{L}_1^3(\mathrm{IA}(M_3))$. By (4.1) and since $\mathcal{L}_2^4(H_3) \neq \mathcal{L}^4(\mathrm{IA}(M_3))$, we obtain $\mathcal{L}_1^3(\mathrm{IA}(M_3)) \neq \mathcal{L}^4(\mathrm{IA}(M_3))$. Hence, $\gamma_3(\mathrm{IA}(M_3)) \neq \mathrm{I}_4\mathrm{A}(M_3)$.

3.3 The case $n \geq 4$

For a group G, we write $\mathbb{Z}G$ for the integral group ring of G. For elements x and y of a group G, y^x denotes the conjugate $x^{-1}yx$.

Since M_n' is abelian, it may be regarded as a right $\mathbb{Z}(M_n/M_n')$-module in the usual way, where the module action comes from conjugation in M_n.

Write $A_n = F_n/F_n'$ for the free abelian group of rank n and $a_i = f_iF_n'$, $i \in \{1, \ldots, n\}$. The natural epimorphism $\pi : M_n \to A_n$ induces an isomorphism from M_n/M_n' to A_n. So, we may regard M_n' as a right $\mathbb{Z}A_n$-module. For $w \in M_n'$ and $s \in \mathbb{Z}A_n$, we write w^s to denote the image of w under the action of s. This notation is consistent with our notation for conjugation.

A set of generators of M_n' is given by $[x_i, x_j]^{a_1^{\kappa_1} \cdots a_n^{\kappa_n}}$, where $i < j$ and $\kappa_\mu \in \mathbb{Z}$, $\mu \in \{1, \ldots, n\}$. Since M_n' is abelian, any element of M_n' may be written in the form $\prod_{1 \leq i < j \leq n} [x_i, x_j]^{P_{ij}}$, where $P_{ij} \in \mathbb{Z}A_n$.

The natural epimorphism from M_n onto M_n/M_n' induces a group homomorphism

$$\tilde{\sigma}_n : \mathrm{Aut}(M_n) \to \mathrm{GL}_n(\mathbb{Z}) = \mathrm{Aut}(M_n/M_n').$$

Since the natural homomorphism from $\mathrm{Aut}(F_n)$ into $\mathrm{GL}_n(\mathbb{Z})$ is surjective [Andreadakis 1965] and $M_n/M_n' \cong F_n/F_n'$, we have $\tilde{\sigma}_n$ is surjective and the kernel of $\tilde{\sigma}_n$ is equal to $\mathrm{IA}(M_n)$.

We define an action of $\mathrm{GL}_n(\mathbb{Z})$ on $\mathcal{L}^c(\mathrm{IA}(M_n))$ by

$$g * \overline{\phi} = \overline{t_g \phi t_g^{-1}}$$

for all $g \in \mathrm{GL}_n(\mathbb{Z})$, $\overline{\phi} = \phi\ \mathrm{I}_{c+1}\mathrm{A}(M_n)$, with $\phi \in \mathrm{I}_c\mathrm{A}(M_n)$, and $\tilde{\sigma}_n(t_g) = g$.

The above action is a left action of $\mathrm{GL}_n(\mathbb{Z})$ on $\mathcal{L}^c(\mathrm{IA}(M_n))$. The action of $\mathrm{GL}_n(\mathbb{Z})$ commutes with the multiplication of the elements of $\mathbb{Z}$. So, $\mathcal{L}^c(\mathrm{IA}(M_n))$ is a $\mathbb{Z}\mathrm{GL}_n(\mathbb{Z})$-module for all $c \geq 2$. Clearly, for all $c \geq 2$, $\gamma_{c-1}(\mathrm{IA}(M_n))\mathrm{I}_{c+1}\mathrm{A}(M_n)/\mathrm{I}_{c+1}\mathrm{A}(M_n)$ is a $\mathbb{Z}\mathrm{GL}_n(\mathbb{Z})$-submodule of $\mathcal{L}^c(\mathrm{IA}(M_n))$.

Write

$$\operatorname{gr}_{1,\mathbb{Q}}(M_n) = \mathbb{Q} \otimes_{\mathbb{Z}} \operatorname{gr}_1(M_n).$$

Since $\operatorname{gr}_1(M_n)$ is a free $\mathbb{Z}$-module with a free generating set $\{\overline{x}_1, \ldots, \overline{x}_n\}$, where $\overline{x}_i = x_i M_n'$ for $i \in [n] = \{1, \ldots, n\}$, we may regard $\operatorname{gr}_1(M_n) \subseteq \operatorname{gr}_{1,\mathbb{Q}}(M_n)$.

Identify $\operatorname{Aut}(\operatorname{gr}_{1,\mathbb{Q}}(M_n))$ with $\operatorname{GL}_n(\mathbb{Q})$ with respect to the $\mathbb{Q}$-basis $\{\overline{x}_1, \ldots, \overline{x}_n\}$. The group $\operatorname{GL}_n(\mathbb{Q})$ acts naturally on $\operatorname{gr}_{1,\mathbb{Q}}(M_n)$ by

$$g\overline{x}_j = \sum_{i=1}^{n} g_{ij}\overline{x}_i$$

for $j \in [n]$, where $g = (g_{ij}) \in \operatorname{GL}_n(\mathbb{Q})$ for $i, j \in [n]$. For $c \geq 2$, let $\operatorname{gr}_{c,\mathbb{Q}}(M_n)$ denote the tensor product of $\mathbb{Q}$ with $\operatorname{gr}_c(M_n)$ over $\mathbb{Z}$.

Since $\operatorname{gr}_c(M_n)$ is a free $\mathbb{Z}$-module with a free generating set $\mathcal{M}_{n,c}$, we may regard $\operatorname{gr}_c(M_n) \subseteq \operatorname{gr}_{c,\mathbb{Q}}(M_n)$. The action of $\operatorname{GL}_n(\mathbb{Q})$ on $\operatorname{gr}_{1,\mathbb{Q}}(M_n)$ can be extended diagonally on $\operatorname{gr}_{c,\mathbb{Q}}(M_n)$, for $c \geq 2$, subject to

$$g\overline{[x_{j_1}, \ldots, x_{j_c}]} = g([x_{j_1}, \ldots, x_{j_c}] + \gamma_{c+1}(M_n)) = [g\overline{x}_{j_1}, \ldots, g\overline{x}_{j_c}] + \gamma_{c+1}(M_n) = \overline{[g\overline{x}_{j_1}, \ldots, g\overline{x}_{j_c}]}$$

with $j_1 > j_2 \leq j_3 \leq \cdots \leq j_c$. Each $\operatorname{gr}_{c,\mathbb{Q}}(M_n)$ is a $\mathbb{Q}\operatorname{GL}_n(\mathbb{Q})$-module.

The group $\mathrm{GL}_n(\mathbb{Q})$ acts on

$$\mathrm{gr}_{c,\mathbb{Q}}(M_n)^{\oplus n} = \underbrace{\mathrm{gr}_{c,\mathbb{Q}}(M_n) \oplus \cdots \oplus \mathrm{gr}_{c,\mathbb{Q}}(M_n)}_{n-\text{factors}}$$

by

$$g \bullet (\overline{u}_1, \ldots, \overline{u}_n) = (g\overline{u}_1, \ldots, g\overline{u}_n)g^{-1},$$

where $g \in \mathrm{GL}_n(\mathbb{Q})$ and $\overline{u}_i \in \mathrm{gr}_{c,\mathbb{Q}}(M_n)$ for $i \in [n]$. Here $g\overline{u}_i$ means the canonical action of g on $\mathrm{gr}_{c,\mathbb{Q}}(M_n)$ and the multiplication of a $1 \times n$ and an $n \times n$ matrix. Each $\mathrm{gr}_{c,\mathbb{Q}}(M_n)^{\oplus n}$ is a $\mathbb{Q}\mathrm{GL}_n(\mathbb{Q})$-module.

For $c \geq 2$, let $\chi_c : \mathrm{I}_c\mathrm{A}(M_n) \to \mathrm{gr}_{c,\mathbb{Q}}(M_n)^{\oplus n}$ defined by

$$\chi_c(\phi) = (\overline{u}_1, \ldots, \overline{u}_n)$$

for all $\phi \in \mathrm{I}_c\mathrm{A}(M_n)$, with $\phi(x_i) \equiv x_i u_i \pmod{\gamma_{\mathrm{c+1}}(\mathrm{M_n})}$, $u_i \in \gamma_c(M_n)$, $i \in [n]$.

- χ_c is a group homomorphism and $\mathrm{Ker}\chi_c = \mathrm{I}_{c+1}\mathrm{A}(M_n)$.

- via the monomorphism $\overline{\chi}_c$ induced by χ_c, $\mathcal{L}^c(\mathrm{IA}(M_n))$ is isomorphic to a subgroup of $\mathrm{gr}_{c,\mathbb{Q}}(M_n)^{\oplus n}$.

- $\overline{\chi}_c(g * \overline{\phi}) = g \bullet (\overline{u}_1, \ldots, \overline{u}_n)$ for all $g \in \mathrm{GL}_n(\mathbb{Z})$ and $\phi \in \mathrm{I}_c\mathrm{A}(M_n)$, with $\phi(x_i) \equiv x_i u_i \gamma_{c+1}(M_n)$, $u_i \in \gamma_c(M_n)$, $i \in [n]$.

• Let $n \geq 4$ and $c \geq 3$. For $i \in [n]$ and $i_1, \ldots, i_c \in [n] \setminus \{i\}$, with $i_1 \neq i_2$, let $\tau_{i,(i_1,\ldots,i_c)}$ be the IA-endomorphism of M_n satisfying the conditions $\tau_{i,(i_1,\ldots,i_c)}(x_i) = x_i[x_{i_1}, \ldots, x_{i_c}]$ and $\tau_{i,(i_1,\ldots,i_c)}(x_r) = x_r$, $r \neq i$. Clearly, $\tau_{i,(i_1,\ldots,i_c)}$ is an automorphism of M_n. Furthermore, for $i \in [n]$ and $i_1, \ldots, i_c \in [n] \setminus \{i\}$, with $i_1 \neq i_2$, we write $\overline{u}_i(i_1, \ldots, i_c) = [\overline{x}_{i_1}, \ldots, \overline{x}_{i_c}] + \gamma_{c+1}(M_n)$. Then, $\overline{\chi}_c(\overline{\tau}_{i,(i_1,\ldots,i_c)})$ is the n-tuple with $\overline{u}_i(i_1, \ldots, i_c)$ in the i-th position and the rest are zero. Let P_c be the $\mathbb{Q}\mathrm{GL}_n(\mathbb{Q})$-submodule of $\mathrm{gr}_{c,\mathbb{Q}}(M_n)^{\oplus n}$ generated by the set $\mathcal{P}_c = \{\overline{\chi}_c(\overline{\tau}_{1,(i_1,\ldots,i_c)}) : i_1 \neq i_2; i_1, \ldots, i_c \in [n] \setminus \{1\}\}$.

• For $u \in M_n'$, write $\xi_u \in \mathrm{IA}(M_n)$ which satisfies the conditions $\xi_u(x_i) = x_i[x_i, u]$, $i \in [n]$. Note that $\xi_{u^{-1}} = (\xi_u)^{-1}$ for all $u \in M_n'$. Let $\mathcal{Q}_c = \{\overline{\chi}_c(\overline{\xi}_u) = (\overline{[x_1, u]}, \ldots, \overline{[x_n, u]}) : u \in \gamma_{c-1}(M_n)\}$. Note that if $c \geq 3$, $\psi \in \mathrm{Aut}(M_n)$, $u, u_1, u_2 \in \gamma_{c-1}(M_n)$ and $a \in \mathbb{Z}$, then $\psi \xi_u \psi^{-1} = \xi_{\psi(u)}$, $\xi_{u_1}\xi_{u_2} = \xi_{u_1 u_2}$ and $a\overline{\xi}_u = \overline{\xi}_{u^a}$. Let Q_c be the $\mathbb{Q}\mathrm{GL}_n(\mathbb{Q})$-submodule of $\mathrm{gr}_{c,\mathbb{Q}}(M_n)^{\oplus n}$ generated by the set $\mathcal{Q}_c$.

• Let η be the IA-endomorphism of M_n satisfying the conditions $\eta(x_j) = x_j[x_j, {}_{(c-1)}x_1]$ for all $j \in [n]$. Since $\det(J(\eta)) \neq a_1^{k_1} \cdots a_n^{k_n}$ for all $k_1, \ldots, k_n \in \mathbb{Z}$, we have $\eta \notin \mathrm{I}_c\mathrm{A}(M_n)$. Let R_c be the $\mathbb{Q}\mathrm{GL}_n(\mathbb{Q})$-submodule of $\mathrm{gr}_{c,\mathbb{Q}}(M_n)^{\oplus n}$ generated by $(0, \overline{[x_2, {}_{(c-1)}x_1]}, \ldots, \overline{[x_n, {}_{(c-1)}x_1]})$.

• It follows, by [Bryant and Drensky, Section 2, Proposition 3.5 (i)], that $\mathrm{gr}_{c,\mathbb{Q}}(M_n)^{\oplus n} = P_c \oplus Q_c \oplus R_c$ and P_c, Q_c and R_c are irreducible (rational) $\mathbb{Q}\mathrm{GL}_n(\mathbb{Q})$-modules.

• We point out that

$$\dim(P_c \oplus Q_c) = \mathrm{rank}(\mathcal{L}^c(\mathrm{IA}(M_n))) = n(c-1)\binom{n+c-2}{n-2} - \binom{n+c-2}{n-1}$$

for all $c \geq 3$.

3.3.1 A generating set for $\mathcal{L}^c(\mathrm{IA}(M_n))$

• For distinct $i, j, k \in [n]$ and non-negative integers $r_1, \ldots, r_n$, with $r_1 + \cdots + r_n = c - 2$, let $\tau_{ijk,(r_1,\ldots,r_n)}$ be the IA-automorphism of M_n satisfying the conditions

$$\tau_{ijk,(r_1,\ldots,r_n)}(x_i) = x_i[x_j, x_k]^{(a_1-1)^{r_1}\cdots(a_n-1)^{r_n}} \text{ and } \tau_{ijk,(r_1,\ldots,r_n)}(x_r) = x_r, \quad r \neq i.$$

Each $\tau_{ijk,(r_1,\ldots,r_n)} \in \mathrm{I}_c\mathrm{A}(M_n)$.

• For $j_1, \ldots, j_{c-1} \in [n]$, with $j_1 \neq j_2$, we write

$$u(j_1, \ldots, j_{c-1}) = [x_{j_1}, \ldots, x_{j_{c-1}}],$$

and let $\xi_{u(j_1,\ldots,j_{c-1})}$ be the inner automorphism of M_n on $u(j_1, \ldots, j_{c-1})$. Each $\xi_{u(j_1,\ldots,j_{c-1})} \in \mathrm{I}_c\mathrm{A}(M_n)$.

• For $c \geq 3$, let $P, Q \in \mathbb{Z}A_n$ be monomials in the $a_1 - 1, \ldots, a_n - 1$ of total degree $c - 2$ and $c - 3$, respectively. For distinct $i, j, k \in [n]$, let $B_{ikj}(P)$, and $B_{ij}(Q)$ be the IA-endomorphisms of M_n satisfying the conditions

$$\begin{aligned} B_{ikj}(P)(x_i) &= x_i[x_i, x_j]^{a_i^{-1}a_j^{-1}a_k^{-1}P}[x_k, x_j]^{a_k^{-2}a_j^{-1}P}, \\ B_{ikj}(P)(x_k) &= x_k[x_i, x_j]^{-a_i^{-2}a_j^{-1}P}[x_k, x_j]^{-a_i^{-1}a_j^{-1}a_k^{-1}P} \text{ and} \\ B_{ikj}(P)(x_r) &= x_r, \text{with } r \neq i, k, \end{aligned}$$

and

$$\begin{aligned} B_{ij}(Q)(x_i) &= x_i[x_i, x_j]^{-a_i^{-2}a_j^{-2}(a_i-1)Q}, \\ B_{ij}(Q)(x_j) &= x_j[x_i, x_j]^{-a_i^{-2}a_j^{-2}(a_j-1)Q} \text{ and} \\ B_{ij}(Q)(x_s) &= x_s, \text{ with } s \neq i, j. \end{aligned}$$

It is proved that $B_{ikj}(P), B_{ij}(Q) \in \mathrm{I}_c\mathrm{A}(M_n)$.

By using a result of Bachmuth [Bach., 1969, Proof of Lemma 7], we obtain the following result.

Lemma 3.1 *For $c \geq 3$, $\mathcal{L}^c(\mathrm{IA}(M_n))$ is generated as a $\mathbb{Z}\mathrm{GL}_n(\mathbb{Z})$-module by all $\overline{\tau}_{123,(r_1,\ldots,r_n)}$, $\overline{B}_{123}(P)$ and $\overline{B}_{12}(Q)$ with $r_1+\cdots+r_n = c-2$, P and Q are monomials in $a_1-1,\ldots,a_n-1$ of total degree $c-2$ and $c-3$, respectively.*

For the proof of Theorem 2.1 (3), the following result gives us a suitable generating set for $\mathcal{L}^c(\mathrm{IA}(M_n))$ as a $\mathbb{Z}\mathrm{GL}_n(\mathbb{Z})$-module.

Proposition 3.1 *Let M_n be a free metabelian group of finite rank n, with $n \geq 4$. Then, for any $c \geq 4$, the $\mathbb{Z}$-module $\mathcal{L}^c(\mathrm{IA}(M_n))$ is generated by $\overline{\tau}_{123,(r_1,\ldots,r_n)}$ and $\overline{\xi}_{v(k_1,\ldots,k_{c-1})}$, with $v(k_1,\ldots,k_{c-1}) = [x_{k_1},\ldots,x_{k_{c-1}}]$, $r_1+\cdots+r_n = c-2$, $k_1,\ldots,k_{c-1} \in \{1,\ldots,n\}$ and $k_1 > k_2 \leq k_3 \leq \cdots \leq k_{c-1}$ as a $\mathbb{Z}\mathrm{GL}_n(\mathbb{Z})$-module.*

Finally, we need the following result.

Lemma 3.2 *Let M_n be a free metabelian group of rank n, with $n \geq 4$ and let $c \geq 2$. Then,*

1. *For $i_1,\ldots,i_c \in \{1,\ldots,n\} \setminus \{1\}$, $\tau_{1,(i_1,\ldots,i_c)} \in \gamma_{c-1}(\mathrm{IA}(M_n))$.*
2. *For non-negative integers $r_1,r_2,\ldots,r_n$, with $r_1+r_2+\cdots+r_n = c-2$, $\tau_{123,(r_1,r_2,\ldots,r_n)} \in \gamma_{c-1}(\mathrm{IA}(M_n))$.*

In the next few lines, for any group G, we denote by $\mathrm{Z}(G)$ the center of G. Let $\mathrm{Inn}(G)$ be the group of inner automorphisms of G. For a positive integer c, with $c \geq 2$, let $\mathrm{Inn}_c(G) = \mathrm{Inn}(G) \cap \mathrm{I}_c\mathrm{A}(G)$.

Lemma 3.3 *Let G be a group. If $\mathrm{Z}(G/\gamma_{c+1}(G)) = \gamma_c(G)/\gamma_{c+1}(G)$ for all c, then, $\gamma_d(\mathrm{Inn}(G))$ $= \mathrm{Inn}_{d+1}(G)$ for all d.*

We recall the statement of Theorem 2.1 (3): For $n, c \geq 4$,

$$\mathcal{L}^c(\mathrm{IA}(M_n)) = \gamma_{c-1}(\mathrm{IA}(M_n))\mathrm{I}_{c+1}\mathrm{A}(M_n)/\mathrm{I}_{c+1}\mathrm{A}(M_n).$$

4 Proof of Theorem 2.1 (3)

For $n, c \geq 4$, by Proposition 3.1, the $\mathbb{Z}\mathrm{GL}_n(\mathbb{Z})$-module $\mathcal{L}^c(\mathrm{IA}(M_n))$ is generated by $\overline{\tau}_{123,(r_1,\ldots,r_n)}$ and $\overline{\xi}_{v(k_1,\ldots,k_{c-1})}$. Since $\xi_{v(k_1,\ldots,k_{c-1})}$ is an inner automorphism of M_n and since $Z(M_n/\gamma_{c+1}(M_n)) = \gamma_c(M_n)/\gamma_{c+1}(M_n)$ for all c, it follows, by Lemma 3.3, that $\xi_{v(k_1,\ldots,k_{c-1})} \in \gamma_{c-1}(\mathrm{IA}(M_n))$. Furthermore, by Lemma 3.2 (2), we have $\tau_{123,(r_1,r_2,\ldots,r_n)} \in \gamma_{c-1}(\mathrm{IA}(M_n))$ and so, we obtain the desired result.

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