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IDENTITIES GENERATED BY STANDARD POLYNOMIALS FOR SOME MATRIX ALGEBRAS WITH GRASSMANN ENTRIES

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1. Introduction

Finding the minimal identities for matrix algebras has been an important problem in PI-theory for more than 70 years. In 1950 Amitsur and Levitzki proved that the standard polynomial $St_{2n}(x_1,\ldots,x_n)=\sum_{\sigma\in Sym(2n)}(-1)^{\sigma}x_{\sigma(1)}\ldots x_{\sigma(2n)}$ is a polynomial identity of minimal degree for the algebra $M_n(K)$ over a field of characteristic zero.

A good survey on the problem in the involution (*) case is [Bessades et al. 2019], where the best up to now partial solution is given, namely that the smallest degree of a standard *-polynomial identity in symmetric variables in $(M_{2^m}(K),s)$ for s being the symplectic involution is indeed $2^{m+1}-2$ [Theorem 5.5].

The general situation of the matrix $M_n(E)$ for E being the Grassmann algebra (sometimes $M_n(E)$ is called a matrix algebra with Grassmann entries) is scarcely investigated. Berele and Regev proved

Proposition 1 [Berele and Regev 2001, Corollary 6.6] *The algebra* $M_n(E)$ *does not satisfy the identity* $S_m^n(X_1,\ldots,X_m)=0$ for any m.

In [Szigeti 1997] the identity of “algebraicity” for matrices over the Grassmann algebra was defined and the following proposition proved:

Proposition 2 [Szigeti 1997, Theorem 5.1] *The polynomial* $St_{2n^2}([X^{2n^2},Y],[X^{2n^2-1},Y],...,[X^2,Y],[X,Y])=0$ *is an identity for* $M_n(E)$.

In [Marki et al. 2015] an embedding of the m-generated Grassmann algebra $E^{(m)}$ into a $2^{(m-1)}\times 2^{(m-1)}$ matrix algebra over a factor of a commutative polynomial algebra in m variables was introduced. Applying this procedure it was proved

Proposition 3 [Marki at al. 2015, Theorem 3.7] *The standard identity* $St_{2^m n}=0$ *of degree* $2^m n$ *is a polynomial identity for* $M_n(E^{(m)})$*.*

Due to [Frenkel 2017] the following statements are valid:

Proposition 4 [Frenkel 2017, Theorem 7] *The standard identity of degree* $k=2n([m/2]+1)$ *holds in* $M_n(E^{(m)})$*.*

Proposition 5 [Frenkel 2017, Proposition 8] *The standard identity of degree* 6 *holds in* $M_2(E^2)$*.*

Proposition 6 [Frenkel 2017, Proposition 9] *The standard identity of degree* $k=2(n+[m/2])-1$ *does not hold in* $M_n(E^{(m)})$ *if the base ring is a field of characteristic either zero or a prime* $p>2[m/2]$*.*

In [Frenkel 2017] a question was asked about the degree function $k=k(m,n)$ of the standard identity $St_k=0$ for $M_n(E^{(m)})$, namely

Problem 1 [Frenkel 2017, Problem 10] *Does the standard identity of degree* $2(n+[m/2])$ *hold in* $M_n(E^{(m)})$?

Balazs and Meszaros [2019] showed that the minimal degree of a standard identity for $M_n(E^{(2)})$ and $M_n(E^{(3)})$ is $4n-2$ if $n\geq 2$. This is done by combinatorial arguments based on computing sums of signs corresponding to Eulerian trails in directed graphs.

Di Vincenzo and Koshlukov [2011] studied the superalgebra $M_{1,1}(E)=\begin{pmatrix} a & b \\ c & d \end{pmatrix}; a,d\in E_0; b,c\in E_1$ with the involution $\begin{pmatrix} a & b \\ c & d \end{pmatrix}^* = \begin{pmatrix} d & b \\ -c & a \end{pmatrix}$. We recall that $E=E_0\oplus E_1$, where E_0 is the space of the elements of even order and E_1 is the space of the elements of odd order. For the *- symmetric elements $Y_1,Y_2,...$ and *- skew symmetric ones $Z_1,Z_2,...$ in $M_{1,1}(E)$ the authors proved

Proposition 7 [Vincenzo and Koshlukov 2011, p.265] *The algebra* $M_{1,1}(E)$ *satisfies the identities of minimal degree* $St_2(Y_1,Y_2)=0$ *and* $St_2^2(Z_1,Z_2)=0$.

These results show that finding the minimal degree of a standard identity could be naturally generalized and identities which are generated by standard polynomials to be investigated as well.

2. Author's results related to the considered problem

We recall the definition of the infinite dimensional Grassmann algebra E as

$$E = E(V) = K\langle e_1, e_2, \ldots \mid e_i e_j + e_j e_i = 0 \quad i, j = 1,2,\ldots\rangle,$$

where the field K has characteristic zero.

Many of the PI-properties of E and $M_n(E)$ could be found in [Krakowski and Regev 1973; Berele and Regev 2001; Drensky and Formanek 2004]. Here we formulate only two of them:

Proposition 8 [Krakowski and Regev 1973, Corollary, p. 437] *The* T *-ideal* $Id(E)$ *is generated by the identity* $[x_1,x_2,x_3]=[x_1,x_2]x_3-x_3[x_1,x_2]=0$ (*called the Grassmann identity*).

Proposition 9 [Berele and Regev 2001, Lemma 6.1] *The algebra* E *satisfies* $St_n^k(x_1,\ldots,x_n)=0$ *for all* $n,k\geq 2$.

2.1. Results for some special matrix algebras with Grassmann entries

We point that there are classes of algebras satisfying identities of the considered type, namely the next Propositions 10, 12 and 13, respectively. We give by one example only in any of the three considered cases.

We follow the exposition in [Rashkova 2015]:

Let $\alpha_2,...,\alpha_n$ be fixed elements of the field K and $AM_n(E)$ be the n-th dimensional matrix algebra of the matrices of type

$$\begin{pmatrix} x_1 & x_2 & ... & x_n \\ \alpha_2 x_1 & \alpha_2 x_2 & ... & \alpha_2 x_n \\ ... & ... & ... & ... \\ \alpha_n x_1 & \alpha_n x_2 & ... & \alpha_n x_n \end{pmatrix}.$$

Proposition 10. *The algebra* $AM_n(E)$ *satisfies the identities*

$$St_2(X_1,X_2)St_3(X_3,X_4,X_5)=0;$$
$$St_3(X_1,X_2,X_3)St_2^2(X_4,X_5)=0;$$
$$St_3^2(X_1,X_2,X_3)=0.$$

Proof: Applying [Rashkova 2015, Proposition 5] we get that the above identities hold in any algebra, satisfying the identity $[x_1,x_2,x_3]x_4=0$.

Theorem 1 [Rashkova 2015] gives that the algebra $AM_n(E)$ satisfies the last identity.

Proposition 11 [Rashkova 2015, Proposition 7] For $AM_3(E)$ the identity $St_2^3(X_1,X_2)=0$ holds as well.

Let us consider the $2n^2$-th dimensional matrix algebra $BM_{2n}(E)$ of the matrices of type $\begin{pmatrix} a_1 & a_{12} & \dots & \dots & \dots & \dots & a_{1,2n-1} & a_1 \\ 0 & a_2 & a_{23} & \dots & \dots & a_{2,2n-2} & a_2 & 0 \\ \dots & \dots & \dots & \dots & .. & \dots & \dots & \dots \\ 0 & \dots & 0 & a_n & a_n & 0 & \dots & 0 \\ 0 & \dots & 0 & a_{n+1} & a_{n+1} & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & a_{2n-1} & a_{2n-1,3} & \dots & \dots & a_{2n-1,2n-2} & a_{2n-1} & 0 \\ a_{2n} & a_{2n,2} & \dots & \dots & \dots & \dots & a_{2n,2n-1} & a_{2n} \end{pmatrix}$.

Proposition 12 [Rashkova 2015, Proposition 11] *The algebra $BM_{2n}(E)$ satisfies the identities:*

$$St_2^{3n}(X_1, X_2) = 0;$$

$$(St_3^2(X_1, X_2, X_3)St_2^2(X_4, X_5))^n = 0;$$

$$(St_3(X_1, X_2, X_3)St_2(X_4, X_5))^k = 0, \quad n = 2k;$$

$$(St_3(X_1, X_2, X_3)St_2(X_4, X_5))^k St_3(X_6, X_7, X_8)St_2(X_9, X_{10}) = 0, \quad n = 2k+1.$$

The last special algebra considered is the $(4n+1)$-th dimensional matrix algebra $CM_{2n+1}(E)$ of the matrices of type $\begin{pmatrix} a_{11} & 0 & \dots & 0 \\ a_{21} & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ a_{n1} & 0 & \dots & 0 \\ a_{n+1,1} & a_{n+1,2} & \dots & a_{n+1,2n+1} \\ a_{n+2,1} & 0 & \dots & 0 \\ a_{n+3,1} & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ a_{2n+1,1} & 0 & \dots & 0 \end{pmatrix}$.

Proposition 13. *The algebra* $CM_{2n}(E)$ *satisfies the identities*

$$St_2(X_1,X_2)St_3(X_3,X_4,X_5)St_2^2(X_6,X_7)=0;$$
$$St_2^2(X_1,X_2)St_3(X_3,X_4,X_5)St_2(X_6,X_7)=0.$$

Proof: Proposition 12 from [Rashkova 2015] proves that the above identities hold in any algebra, satisfying the identity $[x_1,x_2,x_3]x_4[x_5,x_6,x_7]=0$. Due to Theorem 3 [Rashkova 2015] the algebra $CM_{2n}(E)$ satisfies the last identity.

2.2 Author's investigations related to Problem 1

For better understanding we introduce in more details the embedding of the *m*-generated Grassmann algebra $E^{(m)}$ into a $2^{(m-1)}\times 2^{(m-1)}$ matrix algebra over a factor of a commutative polynomial algebra in m variables as it is done in (Marki et al. 2015)

Let ${}_K R$ be an arbitrary and ${}_K\Omega$ be a commutative (associative) algebra over a field K. For an integer $n\geq 1$ we consider representations of R over Ω which are injective K-algebra homomorphisms (K-embeddings) $\varepsilon: R\to M_n(\Omega)$.

Definition 1. *We call* ε *a constant trace* (CT-) *representation if* $tr(\varepsilon(r))\in K$ *for all* $r\in R$ *(here* $tr(\varepsilon(r))$ *is the sum of the diagonal entries of the* $n\times n$ *matrix* $\varepsilon(r)\in M_n(\Omega)$*).*

The following representation, namely

$$1\to\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix},\ v_1\to m(v_1)=\begin{bmatrix}z_1 & 0\\ 0 & -z_1\end{bmatrix},\ v_2\to m(v_2)=\begin{bmatrix}0 & z_2\\ z_2 & 0\end{bmatrix}$$

is a CT-representation $\varepsilon^{(2)}:E^{(2)}\to M_2(K[z_1,z_2]/(z_1^2,z_2^2))$ as

$$\varepsilon^{(2)}(c_0+c_1v_1+c_2v_2+c_3v_1v_2)=\begin{bmatrix}c_0+c_1z_1+(z_1^2,z_2^2) & c_2z_2+c_3z_1z_2+(z_1^2,z_2^2)\\ c_2z_2-c_3z_1z_2+(z_1^2,z_2^2) & c_0-c_1z_1+(z_1^2,z_2^2)\end{bmatrix}$$

where $c_0, c_1, c_2, c_3 \in K$ and (z_1^2, z_2^2) is the ideal of the commutative polynomial ring $K[z_1, z_2]$ generated by the monomials z_1^2, z_2^2.

Proposition 14 [Marki et al. 2015, Theorem 3.1] *For some integers* $m,n \geq 2$, *let* $\varepsilon^{(m)} : E^{(m)} \to M_n(\Omega)$ *be a* CT*-representation of* $E^{(m)}$ *over a commutative* K*-algebra* Ω. *Then the assignments* $1 \to \begin{bmatrix} I_n & 0 \\ 0 & I_n \end{bmatrix}$, $v_i \to \begin{bmatrix} \varepsilon^{(m)}(v_i) & 0 \\ 0 & -\varepsilon^{(m)}(v_i) \end{bmatrix}$ *for* $1 \leq i \leq m$ *and* $v_{m+1} \to \begin{bmatrix} 0 & \hat{z}I_n \\ \hat{z}I_n & 0 \end{bmatrix}$ (*with* $\hat{z} = z + (z^2)$ *in* $\Omega(z)/(z^2))$ *define a* CT*-representation* $\varepsilon^{(m+1)} : E^{(m+1)} \to M_{2n}(\Omega[z]/(z^2))$.

The notation I_n stands for the unit matrix of order n.

Applying Proposition 14 we form the CT-representation $\varepsilon^{(3)} : E^{(3)} \to M_4(\Omega[z]/(z^2))$, namely,

$$v_1 \to M(v_1) = \begin{bmatrix} z_1 & 0 & 0 & 0 \\ 0 & -z_1 & 0 & 0 \\ 0 & 0 & -z_1 & 0 \\ 0 & 0 & 0 & z_1 \end{bmatrix},$$

$$v_2 \to M(v_2) = \begin{bmatrix} 0 & z_2 & 0 & 0 \\ z_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & -z_2 \\ 0 & 0 & -z_2 & 0 \end{bmatrix},$$

$$v_3 \to M(v_3) = \begin{bmatrix} 0 & 0 & z_3 & 0 \\ 0 & 0 & 0 & z_3 \\ z_3 & 0 & 0 & 0 \\ 0 & z_3 & 0 & 0 \end{bmatrix}.$$

2.2.1 The case $m = n = 2$

Using the CT-representation of $E^{(2)}$ we form the following 4×4 matrices (instead of z_1, z_2 we use the letters a, b):

A1={{a,0,0,0},{0,-a,0,0},{0,0,0,0},{0,0,0,0}};

A2={{0,0,0,0},{0,0,0,0},{0,0,0,b},{0,0,b,0}};

A3={{1,0,a,0},{0,1,0,-a},{0,0,0,0},{0,0,0,0}};

A4={{0,0,0,a b},{0,0,-a b,0},{0,0,0,0},{0,0,0,0}};

A5={{0,b,0,0},{b,0,0,0},{0,0,1,0},{0,0,0,1}};

A6={{1,0,1,0},{0,1,0,1},{1,0,1,0},{0,1,0,1}}.

Here we give the matrices in a way suitable for the system *Mathematica* used for our calculations. The application of Proposition 14 becomes more clear if we write them due to its language. Then the matrix A3 for example in a block way has the presentation $A3 = \begin{bmatrix} I_2 & m(v_1) \\ 0 & 0 \end{bmatrix}$.

Proposition 15 [Rashkova 2017, Proposition 11] $St_6(A1, A2, A3, A4, A5, A6) = 0$ *in the algebra* $M_2(E^{(2)})$.

Proof: We give a part of the program in *Mathematica* evaluating $T6 = St_6(A1, A2, A3, A4, A5, A6) = 0$. We define the standard polynomial recurrently and give the last two steps:

$$T5[x\_, y\_, z\_, t\_, u\_] := x.T4[y, z, t, u] + y.T4[z, t, u, x] + z.T4[t, u, x, y] + t.T4[u, x, y, z] + u.T4[x, y, z, t];$$

$$T6[x\_, y\_, z\_, t\_, u\_, v\_] := x.T5[y, z, t, u, v] - y.T5[z, t, u, v, x] + z.T5[t, u, v, x, y] - t.T5[u, v, x, y, z] + u.T5[v, x, y, z, t] - v.T5[x, y, z, t, u].$$

Then we calculate it the above matrices being its variables. We get

T6[A1,A2,A3,A4,A5,A6]=

{{0,0,4a^{3}b^2,4a^{3}b^2},{0,0,-4a^{3}b^2,-4a^{3}b^2},{0,0,0,0}, {0,0,0,0}}.

As we are working in the T-ideal, generated by a^2 and b^2, we get the desired result. It confirms Proposition 5 in the considered partial case.

2.2.2 The case $m=2, n=3$

Now Problem 1 asks if $St_8 = 0$ is an identity in $M_3(E^{(2)})$.

Using again the above CT-representation of $E^{(2)}$ we give a non-affirmative answer to Problem 1.

We form the following 8 matrices of type 6×6:

X1={{1,0,0,0,1,0},{0,1,0,0,0,1},{0,0,a,0,0,0},

{0,0,0,-a,0,0},{0,b,0,0,0,b},{b,0,0,0,b,0}};

X2={{a,0,0,0,a,0},{0,-a,0,0,0,-a},{0,0,1,0,0,0},

{0,0,0,1,0,0},{a,0,0,0,a,0},{0,-a,0,0,0,-a}};

X3={{0,0,1,0,1,0},{0,0,0,1,0,0},{1,0,0,0,1,0},

{0,1,0,0,0,1},{0,0,0,0,1,0},{0,0,0,0,0,1}};

X4={{a,0,0,0,a,0},{0,-a,0,0,0,-a},{0,0,1,0,1,0},

{0,0,0,1,0,1},{0,0,1,0,1,0},{0,0,0,1,0,1}};

X5={{1,0,0,0,0,0},{0,1,0,0,0,0},{1,0,a,0,a,0},

{0,1,0,-a,0,-a},{1,0,0,0,0,0},{0,1,0,0,0,0}}

X6={{1,0,0,0,0,0},{0,1,0,0,0,0},{1,0,a,0,a,0},

{0,1,0,-a,0,-a},{1,0,a,0,a,0},{0,1,0,-a,0,-a}};

X7={{1,0,1,0,1,0},{0,1,0,1,0,1},{1,0,1,0,1,0},

{0,1,0,1,0,1},{1,0,1,0,0,0},{0,1,0,1,0,0}};

X8={{0,0,1,0,1,0},{0,0,0,1,0,1},{1,0,1,0,1,0},

{0,1,0,1,0,1},{1,0,1,0,0,0},{0,1,0,1,0,0}}.

Using the CT-representation of $E^{(2)}$ we give the form of $X1=\begin{bmatrix} I_2 & 0 & I_2 \\ 0 & m(v_1) & 0 \\ m(v_2) & 0 & m(v_2) \end{bmatrix}$ only.

Proposition 16 [9, Proposition 12]. *The standard polynomial* $St_8(X_1,...,X_8)$ *is not an identity in the algebra* $M_3(E^{(2)})$*.*

Proof: Evaluating $St_8(X1,X2,X3,X4,X5,X6,X7,X8)=(x_{ij})$ in the system *Mathematica* we get that modulo the ideal generated by a^2 and b^2 the result is not a zero matrix as $x_{52}=-x_{61}=4ab$.

The above result leads to the following interpretation of Problem 1:

Proposition 17. *If the standard identity* $St_k(X_1,...,X_k)=0$ *holds in* $M_n(E^{(m)})$ *for* $n>2$*, then* $k>2(n+[m/2])$*.*

2.3 In [10] the author studied the algebra $(M_2(E),t)$ for "*t*" being the transpose involution and $(M_2(E),s)$ for "*s*" being the symplectic involution, namely $\begin{pmatrix} a & b \\ c & d \end{pmatrix}^* = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$. We recall that we use the notations $Y_1,Y_2,...$ and $Z_1,Z_2,...$ for the symmetric and skew-symmetric elements due to the corresponding involutions. Using Proposition 9 we get the following two propositions:

Proposition 18 [10, Proposition 10]. *The algebra* $(M_2(E),t)$ *satisfies the identities* $St_2^2(Z_1^2,Z_2^2)=0$*,* $St_3^2(Z_1^2,Z_2^2,Z_3^2)=0$ *and* $St_2(St_2(Z_1^2,Z_2^2),Z_3^2)=0$.

Proposition 19 [10, Proposition 11]. *The algebra* $(M_2(E), s)$ *satisfies the identities*

$St_2^2(Y_1, Y_2) = 0$;

$St_3^2(Y_1, Y_2, Y_3) = 0$;

$St_2(St_2(Y_1, Y_2), Y_3) = 0$;

$St_2(St_2(Y_1 \circ Y_2, Y_3 \circ Y_4), Y_5 \circ Y_6) = 0.$

2.4 We continue with another involution case

Let A be the algebra of the matrices over E of type $\begin{pmatrix} y_1 & 0 & z_1 & 0 \\ 0 & y_2 & 0 & z_2 \\ z_3 & 0 & y_3 & 0 \\ 0 & z_4 & 0 & y_4 \end{pmatrix}$ where $\begin{pmatrix} y_1 & 0 \\ 0 & y_2 \end{pmatrix}$ and $\begin{pmatrix} y_3 & 0 \\ 0 & y_4 \end{pmatrix}$ have even entries, while the matrices $\begin{pmatrix} z_1 & 0 \\ 0 & z_2 \end{pmatrix}$ and $\begin{pmatrix} z_3 & 0 \\ 0 & z_4 \end{pmatrix}$ have odd entries.

The algebra A is a subalgebra of $M_{2,2}(E)$.

In the algebra A we introduce the following mapping φ:

$$\begin{pmatrix} y_1 & 0 & z_1 & 0 \\ 0 & y_2 & 0 & z_2 \\ z_3 & 0 & y_3 & 0 \\ 0 & z_4 & 0 & y_4 \end{pmatrix}^{\varphi} = \begin{pmatrix} y_3 & 0 & z_1 & 0 \\ 0 & y_4 & 0 & z_2 \\ -z_3 & 0 & y_1 & 0 \\ 0 & -z_4 & 0 & y_2 \end{pmatrix}.$$

Proposition 20. *The mapping* φ *is an involution in the algebra* A (namely the transpose superinvolution in $M_{2,2}(E)$).

Proof: For any matrices A as above and $B=\begin{pmatrix} y_5 & 0 & z_5 & 0 \\ 0 & y_6 & 0 & z_6 \\ z_7 & 0 & y_7 & 0 \\ 0 & z_8 & 0 & y_8 \end{pmatrix}$ we form

$$(AB)^{\varphi}=\begin{pmatrix} z_3z_5+y_3y_7 & 0 & y_1z_5+z_1y_7 & 0 \\ 0 & z_4z_6+y_4y_8 & 0 & y_2z_6+z_2z_8 \\ -z_3y_5-y_3z_7 & 0 & y_1y_5+z_1z_7 & 0 \\ 0 & -z_4y_6-y_4z_8 & 0 & y_2y_6+z_2z_8 \end{pmatrix}$$ and

$$B^{\varphi}A^{\varphi}=\begin{pmatrix} y_7y_3-z_5z_3 & 0 & y_7z_1+z_5y_1 & 0 \\ 0 & y_8y_4-z_6z_4 & 0 & y_8z_2+z_6y_2 \\ -z_7y_3-y_5z_3 & 0 & -z_7z_1+y_5y_1 & 0 \\ 0 & -z_8y_4-y_6z_4 & 0 & -z_8z_2+y_6y_2 \end{pmatrix}.$$

As all y's are even and all z's are odd we get that $(AB)^{\varphi}=B^{\varphi}A^{\varphi}$.

We give the form of the symmetric due to the involution variables Y_i of the algebra A and of the skew - symmetric ones Z_i, namely

$$Y_i=\begin{pmatrix} \alpha_{i1} & 0 & \beta_{i1} & 0 \\ 0 & \alpha_{i2} & 0 & \beta_{i2} \\ 0 & 0 & \alpha_{i1} & 0 \\ 0 & 0 & 0 & \alpha_{i2} \end{pmatrix};\ Z_i=\begin{pmatrix} a_{i1} & 0 & 0 & 0 \\ 0 & a_{i2} & 0 & 0 \\ c_{i1} & 0 & -a_{i1} & 0 \\ 0 & c_{i2} & 0 & -a_{i2} \end{pmatrix} \quad (1)$$

for $\alpha_{ij}, a_{ij} \in E_0,\ \beta_{ij}, c_{ij} \in E_1$.

Proposition 21. *The algebra* A *satisfies the* φ*-polynomial identities* $St_2(Y_1, Y_2) = 0$ *and* $St_2^2(Z_1, Z_2) = 0$*.*

Proof: By direct calculations using the above given presentation (1).

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