Metabelian varieties and left nilpotent varieties

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- Regev (1972) There exist costants C, $d \geq 1$ such that $c_n(\mathcal{V}) \leq Cd^n$, for all n.

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Giambruno-Shestakov-Zaicev (2014) for varieties of finite dimensional simple algebras $exp(\mathcal{V})$ exists and is integer

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In other words $\text{lím}_{n\to\infty} log_n c_n(\mathcal{V}_\alpha) = \alpha$

Problem

Classify all possible growth of varieties $\mathcal{V}$ such that $c_n(\mathcal{V}) \leq Cn^{\alpha}$, with $0 < \alpha < 3$, for some constant C.

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Theorem(M-V)

Let $\mathcal{V}$ be a variety of non necessarily associative algebras. If $c_n(\mathcal{V}) \leq Cn^{\alpha}$ for some constants $C > 0$ and $0 < \alpha < 1$, then, for n large, $c_n(\mathcal{V}) \leq 1$.

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Theorem(M-V)

Let $\mathcal{V}$ be a variety of commutative or anticommutative (non necessarily associative) algebras. If $c_n(\mathcal{V}) \leq Cn^\alpha$ for some constant $C > 0$ and $1 \leq \alpha < 2$, then either, for n large, $c_n(\mathcal{V}) \leq 1$ or $\text{lím}_{n\to\infty} \log_n c_n(\mathcal{V}) = 1$.

Theorem

Let $\mathcal{V} = {}_2\mathcal{N}$. If $c_n(\mathcal{V}) \leq Cn^{\alpha}$ for some constant $C > 0$ and $1 \leq \alpha < 2$, then $c_n(\mathcal{V}) \leq C_1 n$ for some constant $C_1 > 0$.

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Theorem

Let $\mathcal{V} = {}_2\mathcal{N}$. If $c_n(\mathcal{V}) \leq Cn^{\alpha}$ for some constant $C > 0$ and $2 \leq \alpha < 3$, then $c_n(\mathcal{V}) \leq C_1 n^2$ for some constant $C_1 > 0$.

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We constructed a metabelian commutative (or anticommutative) algebra and a left nilpotent algebras of index two that share the same behavior of the sequence of codimensions.

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We complete this basis to a basis $B = \{a_1, a_2, \dots \ b_1, b_2, \dots\}$ of the whole algebra A. Since $a_i \in A_0$ it follows that

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$$a_i a_j = b_i a_j = 0.$$

Let assume that in A holds "the special condition" $b_i b_j = 0\ \forall i, j$. From the identity $x(yz) \equiv 0$, it follows that

$$a_i b_j = \sum_k \alpha_{ij}^k a_k = c_{ij}.$$

Let $A^+, (A^-)$ be the algebra with basis $B = \{a_1, a_2, \dots\ b_1, b_2, \dots\}$ and with the following multiplication table: for all i, j

$$a_i a_j = b_i b_j = 0, \quad a_i b_j = \pm b_j a_i = \sum_k \alpha_{ij}^k a_k = c_{ij}, .$$

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$$\frac{1}{2} c_n(A) \leq c_n(A^\pm) \leq c_n(A).$$

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Since $A^{\pm}$ is metabelian it follows that $a_i a_j = 0$ for any i, j and

$$a_i b_j = \pm b_j a_i = \sum_k \alpha_{i,j}^k a_k = c_{i,j}, \quad b_i b_j = \pm b_j b_i = \sum_k \beta_{i,j}^k a_k = d_{i,j}.$$

Let A be the algebra with basis $\{a_1, a_2, \ldots, b_1, b_2, \ldots\}$ and with the following multiplication table

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Theorem

$$c_n(A^{\pm}) \leq c_n(A) \leq 2c_n(A^{\pm}).$$

As a consequence we obtain the following

Theorem

1. There are no varieties of commutative (or anticommutative) metabelian algebras such that, for some constants $C_1, C_2 > 0$,

$$C_1 n^\alpha \leq c_n(\mathcal{V}) \leq C_2 n^\alpha$$

with $1 < \alpha < 2$.

2. There are no varieties of commutative (or anticommutative) metabelian algebras such that, for some constants $C_1, C_2 > 0$,

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with $2 < \alpha < 3$.

Example of varieties of metabelian algebras with fractional polynomial growth α, $3 < \alpha < 4$.

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Let $w = w_1 w_2 \cdots w_m$ be an associative word over the alphabet $\{0, 1\}$.

Let $A(w)$ the algebra with basis $\{a, b, z_1, z_2, \ldots, z_{m+1}\}$ satisfying the following relations:

1. $z_i a = \pm a z_i = (1 - w_i) z_{i+1}, \;\; i = 1, 2, \ldots, m;$
2. $z_i b = \pm b z_i = w_i z_{i+1}, \;\; i = 1, 2, \ldots, m;$
3. $a^2 = b^2 = ab = ba = z_i z_j = 0, \;\; \forall i, j.$

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Let $\mathcal{V}_m = \mathrm{var}(A_m)$ be the variety generated by the algebra $A(m) = A(w(m,1)) \oplus A(w(m,2)) \oplus \cdots \oplus A(w(m,[\sqrt{m+1}]))$.

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This variety is a variety of commutative or anticommutative metabelian algebras it is possible to show that for any $n \geq 25$

$$\frac{1}{2}([\sqrt{n}]-2)\frac{n(n-1)(n-5)}{6} \leq c_n(\mathbf{V}) \leq n^3\sqrt{n}+n^2(2n+3\sqrt{n})+n^2.$$

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Giambruno-Mishchenko-Zaicev (2006) For any real number α, $0 < \alpha < 1$, there exists a variety $\mathcal{V}_\alpha \subseteq {}_2\mathcal{N}$, that

$$\lim_{n\to\infty} \log_n \log_n c_n(\mathcal{V}_\alpha) = \alpha,$$

i.e. sequence $c_n(\mathcal{V}_\alpha)$ behaves like n^{n^α}, $n = 1, 2, \ldots$.

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Giambruno-Mishchenko-Zaicev (2008) For any real number $\beta > 1$, there exists a variety $\mathcal{V}_\beta \subseteq {}_2\mathcal{N}$, such that $\exp(\mathcal{V}_\beta) = \beta$.

Since in the construction of the previous varieties were considered left nilpotent algebras of index two satisfying the required condition from the relation between $c_n(A^{\pm})$ and $c_n(A)$ it follows that exp $(A) =$exp$(A^{\pm})$

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Corollary

For any real number α, $0 < \alpha < 1$, there exists a variety $\mathcal{V}_\alpha$ of commutative (or anticommutative) metabelian algebras such that

$$\lim_{n\to\infty} \log_n \log_n c_n(\mathcal{V}_\alpha) = \alpha,$$

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For any real number $\beta > 1$, there exists a variety $\mathcal{V}_\beta$ of commutative (or anticommutative) metabelian algebras such that $\exp(\mathcal{V}_\beta) = \beta$.

Thank You!!