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Derivations and automorphisms of the endomorphism semiring of an infinite chain

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Introduction

The study of derivations in semirings has a short history. In [a] canadian linguist Gabriel Thierrin first considers differential semirings. He proved that the semiring of languages over some alphabet forms an additively idempotent semiring under the operations of union as the addition and catenation as the product. The endomorphism semirings of a semilattices are well-established. The first author investigated derivations in endomorphism semirings of a finite chain, see [b]. In [c] the authors obtained some results for nilpotent and idempotent elements of the endomorphism semiring of an infinite chain with least element.

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[a] Thierrin, G.: Insertion of languages and differential semirings. in Where Mathematics, Computer Science, Linguistics and Biology Meet, Dordrecht, Kluwer Academic (2001)

[b] Vladeva, D.: Projections on right and left ideals of endomorphism semiring which are derivations. Journal Algebra Appl. Vol. 19, No. 11 (2020)

[c] Vladeva D., Trendafilov I. Nilpotent and Idempotent Elements of Subsemirings of the Endomorphism Semiring of an Infinite Chain, Amer. Inst. of Phys. Conf. Proc. 2172 (2019)

Preliminaries

An additively idempotent semiring $(S, +, .)$ is an additive idempotent Abelian semigroup $(S, +)$ and multiplicative monoid $(S, ., 1_S)$ satisfying the usual distributive laws.
For a join-semilattice (idempotent commutative semigroup) $(\mathcal{M}, \vee)$ the map $\alpha : \mathcal{M} \to \mathcal{M}$ is called endomorphism if $\alpha(x \vee y) = \alpha(x) \vee \alpha(y)$, where $x, y \in \mathcal{M}$.
The set $\mathcal{E}_{\mathcal{M}}$ of the endomorphisms of $\mathcal{M}$ is an additively idempotent semiring with respect to the addition and multiplication defined with:
$\alpha = \beta + \gamma,$ if $\alpha(x) = \beta(x) \vee \gamma(x)$,
$\alpha = \beta.\gamma,$ if $\alpha(x) = \gamma\left(\beta(x)\right)$ for all $x \in \mathcal{M}$.

Most of endomorphism semirings of a finite or of an infinite chain and also a semirings investigated in [a] are semirings S without zero, but with identity 1_S.

[a] L. H. Rowen, Algebras with a negation maps,arXiv:1602.00353v5 [math.RA] 11 May 2018

Endomorphisms of an infinite chain. Increasing endomorphisms

In the set Z of integers we consider the binary operation

$$k \vee \ell = \max\{k, \ell\}, \text{ where } k, \ell \in \mathrm{Z}.$$

Then $(\mathrm{Z}, \vee)$ is an infinite chain (without least element). A map $\alpha : (\mathrm{Z}, \vee) \to (\mathrm{Z}, \vee)$ such that $\alpha(k \vee \ell) = \alpha(k) \vee \alpha(\ell)$ is called Z-endomorphism[a].

[a] The idea of studying such endomorphisms was suggested by **acad. Vesselin Drensky** in the Annual meeting of IMI-BAN at the end of 2018.

It follows that $\alpha(k \vee \ell) = \max\{\alpha(k), \alpha(\ell)\}$ and if $k \leq \ell$, then $\alpha(k) \leq \alpha(\ell)$, i.e. α is an order-preserving map.
Any Z-endomorphism α can be expressed by the images $\alpha(k) = a_k$, where $k \in \mathrm{Z}$:

$$\alpha = (, \ldots, a_{-n}, \ldots, a_{-1}, a_0, a_1, \ldots, a_n, \ldots,) \text{ or briefly } \alpha = \{a_k\}_{k \in \mathrm{Z}}.$$

The integers a_k, $k \in \mathrm{Z}$, are called coordinates of the Z-endomorphism α.
In the set $\mathcal{E}_{\mathrm{Z}}$ consisting of Z-endomorphisms we define the operations: for $\alpha = \{a_k\}_{k\in\mathrm{Z}}$ and $\beta = \{b_k\}_{k\in\mathrm{Z}}$,

$$\alpha + \beta = \{c_k\}_{k\in\mathrm{Z}}, \text{ where } c_k = \max\{a_k, b_k\}, k \in \mathrm{Z} \qquad (1)$$

$$\alpha.\beta = \{b_{a_k}\}_{k\in\mathrm{Z}}, \text{ where } b_{a_k} = \beta(a_k), k \in \mathrm{Z} \qquad (2)$$

It is easy to prove that $(\mathcal{E}_{\mathrm{Z}}, +, .)$ is an additively idempotent semiring. The identity map i such that $\mathrm{i}(k) = k$ for all $k \in \mathrm{Z}$ is the identity of $\mathcal{E}_{\mathrm{Z}}$.

The endomorphism $\alpha = \{a_k\}_{k\in\mathrm{Z}}$ is called an increasing if $a_{k+1} > a_k$ for each $k \in \mathrm{Z}$. The identity i is an increasing endomorphism. The set of the increasing endomorphisms of $\mathcal{E}_{\mathrm{Z}}$ is denoted by $\mathcal{IE}_{\mathrm{Z}}$. From (1) and (2) it follows that sum and product of increasing endomorphisms are also increasing. Hence, the set of increasing endomorphisms $\mathcal{IE}_{\mathrm{Z}}$ is a subsemiring of $\mathcal{E}_{\mathrm{Z}}$. The endomorphisms which are increasing can be described as follows

Proposition 1. *The endomorphism $\alpha \in \mathcal{IE}_{\mathsf{Z}}$ if and only if it has a right inverse.*

Jacobson, see [a] proved the next theorem and noted that this result was proved firstly by I. Kaplansky using structure theory.

Theorem **(Kaplansky-Jacobson)** *If a is an element of a ring R with identity such that a has more than one right inverses, then a has infinitely many right inverses.*

As in Kaplansky-Jacobson theorem we construct infinitely many right inverses of any edomorphism of $\mathcal{IE}_{\mathsf{Z}}$.

[a] Jacobson, N.: Some remarks on one-sided inverses. Proc. Amer. Math. Soc. **1**, 352 – 355 (1950).

Derivations

We define a map $\delta_\ell : \mathcal{IE}_Z \to \mathcal{IE}_Z$ such that for any endomorphism $\alpha = \{a_k\}_{k \in Z} \in \mathcal{IE}_Z$

$$\delta_\ell(\alpha)(k) = a_{k+1}, \text{ for all } k \in Z.$$

Proposition 2. *The map $\delta_\ell : \mathcal{IE}_Z \to \mathcal{IE}_Z$ is a derivation in the semiring $\mathcal{IE}_Z$.*

Since $\delta_\ell(\mathrm{i})(k) = k + 1$, it follows $\delta_\ell(\mathrm{i})\alpha = \delta_\ell(\alpha)$. An immediate consequence is

Corollary *Any left ideal of the semiring $\mathcal{IE}_Z$ is closed under the derivation δ_ℓ.*

For the positive powers of the derivation δ_ℓ we prove

Theorem 1. *The map $\delta_\ell^m : \mathcal{IE}_Z \to \mathcal{IE}_Z$, where m is a positive integer, is a derivation in semiring $\mathcal{IE}_Z$.*

Let S_ℓ be the subset of $\mathcal{IE}_Z$, consisting of the endomorphisms α, such that $\mathrm{i} \leq \alpha$. From $\mathrm{i} \leq \alpha$ and $\mathrm{i} \leq \beta$, it follows $\mathrm{i} \leq \alpha + \beta$ and $\mathrm{i} \leq \alpha\beta$. Hence S is a subsemiring of $\mathcal{IE}_Z$. Since $\mathrm{i} \leq \alpha \leq \delta_\ell(\alpha)$, it follows that S is closed under the derivation δ_ℓ.

The inequality $\mathrm{i} \leq \alpha$ implies $\alpha \leq \alpha^2$. But from $\alpha \leq \alpha^2$, using Proposition 1, it follows $\mathrm{i} \leq \alpha$. This means that the semiring S is consisting of the endomorphisms α such that $\alpha \leq \alpha^2 \iff \alpha + \alpha^2 = \alpha^2$. In [a] these elements are called almost idempotent. Authors characterized the subsemiring generated by the set of all almost idempotent elements of k-regular additively idempotent semiring.

[a] Bhuniya, A. K., Sekh, S.: On the subsemiring generated by almost idempotents of a k-regular semiring. Semigroup Forum **97**(2), 268 – 277 (2018).

Now we define a map $\delta_r : \mathcal{IE}_{\mathrm{Z}} \to \mathcal{IE}_{\mathrm{Z}}$ such that for any $\alpha = \{a_k\}_{k \in \mathrm{Z}} \in \mathcal{IE}_{\mathrm{Z}}$, it follows

$$\delta_r(\alpha)(k) = a_k - 1, \text{ for all } k \in \mathrm{Z}.$$

Proposition 3. *The map $\delta_r : \mathcal{IE}_{\mathrm{Z}} \to \mathcal{IE}_{\mathrm{Z}}$ is a derivation in $\mathcal{IE}_{\mathrm{Z}}$.*
Since $\delta_r(\mathrm{i})(k) = k - 1$, it follows $\alpha\delta_r(\mathrm{i}) = \delta_r(\alpha)$. Hence
Corollary *Any right ideal of the semiring $\mathcal{IE}_{\mathrm{Z}}$ is closed under the derivation δ_r.*

For the positive powers of the derivation δ_r we have

Theorem 2 *The map* $\delta_r^m : \mathcal{IE}_Z \to \mathcal{IE}_Z$, *where* m *is a positive integer, is a derivation in semiring* $\mathcal{IE}_Z$.

Let S_r be the subset of $\mathcal{IE}_Z$, consisting of the endomorphisms α, such that $\alpha \leq \mathrm{i}$. From $\alpha \leq \mathrm{i}$ and $\beta \leq \mathrm{i}$, it follows $\alpha + \beta \leq \mathrm{i}$ and $\alpha\beta \leq \mathrm{i}$. Hence S_r is a subsemiring of $\mathcal{IE}_Z$. Since $\delta_r(\alpha) < \alpha \leq \mathrm{i}$, it follows that S_r is closed under the derivation δ_r.

An additively idempotent semiring S, such that $x + xy = x$, $x, y \in S$ is called an incline [a]. If S has identity 1, for $x = 1$, it follows $1 + y = 1$ for all $y \in S$, that is 1 is the biggest element of S. Conversely, if 1 is the biggest element of the semiring S, then from $1 + y = 1$, it follows $x + xy = x$ for all $x, y \in S$, i.e. S is an incline.

Proposition 4. *The subsemiring* S_r *of* $\mathcal{IE}_Z$ *is an incline.*

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[a] Cao, Z.-Q, Kim, K.H., Roush, F.W.: Incline algebra and applications. John Wiley & Sons, NY (1984).

From the definitions of $\delta_\ell(\alpha)$ and $\delta_r(\alpha)$ and the proofs of Theorem 1 and Theorem 2, it follows that for any $\alpha \in \mathcal{IE}_Z$, $m \geq 2$ we have

$$\cdot < \delta_r^m(\alpha) < \cdots < \delta_r(\alpha) < \alpha < \delta_\ell(\alpha) < \cdots < \delta_\ell^m(\alpha) < \cdot. \quad (3)$$

Let S be a semiring, $X \subseteq S$ and $\delta : X \to S$ a derivation. Following [a] we say that δ is commuting on X if $\delta(x)x = x\delta(x)$ for all $x \in X$. All derivations, considered by authors for endomorphism semiring of a finite chain or of infinite chain with least element are commuting. In $\mathcal{IE}_Z$ for $\alpha = \{2k+1\}_{k\in Z}$, $\delta_\ell(\alpha) = \{2k+3\}_{k\in Z}$, $\delta_r(\alpha) = \{2k\}_{k\in Z}$ we calculate

$$\alpha\delta_\ell(\alpha) = \{4k+5\}_{k\in Z} \neq \{4k+7\}_{k\in Z} = \delta_\ell(\alpha)\alpha,$$

$$\alpha\delta_r(\alpha) = \{4k+3\}_{k\in Z} \neq \{4k+1\}_{k\in Z} = \delta_r(\alpha)\alpha.$$

Hence δ_ℓ and δ_r are not commuting derivations.

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[a] Brešar, M.: Commuting maps: a survey. Taiwanese journal of mathematics **8**(3), 361-397 (2014)

Automorphisms and inverse maps

Proposition 5. *The derivations δ_ℓ and δ_r commute.*

For an arbitrary endomorphism $\alpha \in \mathcal{IE}_{\mathsf{Z}}$ we obtain

$$\delta_\ell\delta_r(\alpha) = \delta_r(\delta_\ell(\alpha)) = \delta_\ell(\alpha)\delta_r(\mathsf{i}) = \delta_\ell(\alpha\delta_r(\mathsf{i})) = \delta_\ell(\delta_r(\alpha)) = \delta_r\delta_\ell(\alpha).$$

For any $\alpha \in \mathcal{IE}_{\mathsf{Z}}$ we obtain

$$\delta_r(\alpha) < \alpha \leq \delta_\ell\delta_r(\alpha) < \delta_\ell(\alpha). \qquad (4)$$

Proposition 6. *The product of derivations δ_ℓ and δ_r is an automorphism of the semiring $\mathcal{IE}_{\mathsf{Z}}$.*

Surprisingly the automorphism has the following separating property

Proposition 7. *For an arbitrary $\alpha, \beta \in \mathcal{IE}_{\mathsf{Z}}$, it follows $\delta_\ell\delta_r(\alpha\beta) = \delta_\ell(\alpha)\delta_r(\beta)$.*

Now we define a map $\delta_\ell^{-1} : \mathcal{IE}_{\mathsf{Z}} \to \mathcal{IE}_{\mathsf{Z}}$ such that for any endomorphism $\alpha = \{a_k\}_{k\in \mathsf{Z}} \in \mathcal{IE}_{\mathsf{Z}}$, it follows

$$\delta_\ell^{-1}(\alpha)(k) = a_{k-1}, \text{ for all } k \in \mathsf{Z}.$$

Evidently $\delta_\ell^{-1}(\delta_\ell(\alpha)) = \delta_\ell(\delta_\ell^{-1}(\alpha)) = \alpha$ for $\alpha \in \mathcal{IE}_{\mathsf{Z}}$. Since the k-th coordinate of $\delta_r(\mathrm{i})\alpha$ is a_{k-1}, it follows $\delta_\ell^{-1}(\alpha) = \delta_r(\mathrm{i})\alpha$ for $\alpha \in \mathcal{IE}_{\mathsf{Z}}$. δ_ℓ^{-1} is a linear map, but it is not a derivation. We have only $\delta_\ell^{-1}(\alpha\beta) = \delta_\ell^{-1}(\alpha)\beta$. In particular $\delta_\ell^{-1}(\alpha) = \delta_\ell^{-1}(\mathrm{i})\alpha$. Hence any left ideal of the semiring $\mathcal{IE}_{\mathsf{Z}}$ is closed under the map δ_ℓ^{-1}.

We define a map $\delta_r^{-1} : \mathcal{IE}_{\mathsf{Z}} \to \mathcal{IE}_{\mathsf{Z}}$ such that for any endomorphism $\alpha = \{a_k\}_{k\in \mathsf{Z}} \in \mathcal{IE}_{\mathsf{Z}}$, it follows

$$\delta_r^{-1}(\alpha)(k) = a_k + 1, \text{ for all } k \in \mathsf{Z}.$$

Evidently $\delta_r^{-1}(\delta_r(\alpha)) = \delta_r(\delta_r^{-1}(\alpha)) = \alpha$ for any $\alpha \in \mathcal{IE}_{\mathsf{Z}}$. As above δ_r^{-1} is a linear map, but is not a derivation. We have only $\delta_r^{-1}(\alpha\beta) = \alpha\delta_r^{-1}(\beta)$. In particular $\delta_r^{-1}(\alpha) = \alpha\delta_r^{-1}(\mathrm{i})$.

An immediate consequence of this equality is that any right ideal of the semiring $\mathcal{IE}_Z$ is closed under the map δ_r^{-1}.

Another properties of the maps δ_ℓ^{-1} and δ_r^{-1} are summarized in
Corollary *For identity* $\mathrm{i} \in \mathcal{IE}_Z$ *and arbitrary* $\alpha \in \mathcal{IE}_Z$*, it follows:*
a) $\delta_\ell^{-1}(\mathrm{i}) = \delta_r(\mathrm{i})$, b) $\delta_r^{-1}(\mathrm{i}) = \delta_\ell(\mathrm{i})$, c) $\delta_r(\alpha)\delta_\ell(\mathrm{i}) = \alpha$,
d) $\delta_r(\mathrm{i})\delta_\ell(\alpha) = \alpha$, e) $\delta_\ell(\alpha)\delta_r(\mathrm{i}) = \delta_\ell.\delta_r(\alpha)$,
f) $\delta_\ell(\mathrm{i})\delta_r(\alpha) = \delta_\ell.\delta_r(\alpha)$.

Similarly to Proposition 5 we obtain
Proposition 8. *The maps* δ_ℓ^{-1} *and* δ_r^{-1} *commute.*
Then, it follows $(\delta_\ell\delta_r)^{-1} = \delta_\ell^{-1}\delta_r^{-1}$ which implies that the product $\delta_\ell^{-1}\delta_r^{-1}$ is an automorphism of the semiring $\mathcal{E}_Z$.

Now we can extend the equality (4) to

$$\delta_r(\alpha) < \delta_\ell^{-1}\delta_r^{-1}(\alpha) \leq \alpha \leq \delta_\ell\delta_r(\alpha) < \delta_\ell(\alpha). \tag{5}$$

By the similar reasoning we obtain the following result:
Corollary *For an arbitrary integer m the map* $(\delta_\ell\delta_r)^m$ *is an automorphism of the semiring* $\mathcal{IE}_{\mathsf{Z}}$.
For the elements of the infinite cyclic group generated by the automorphism $\delta_\ell\delta_r$ we find inequalities similar to (5). Since $a_{k+m} - m \leq a_{k+m+1} - m - 1$ for any $m \in \mathsf{Z}$, it follows $(\delta_\ell\delta_r)^m(\alpha) \leq (\delta_\ell\delta_r)^{m+1}(\alpha)$. Hence

$$\cdots < (\delta_\ell\delta_r)^{-1}(\alpha) < \alpha < \delta_\ell\delta_r(\alpha) < \cdots < (\delta_\ell\delta_r)^m(\alpha) \cdots , \tag{6}$$

where $\alpha \neq \mathsf{i}$.

Semirings of derivations and automorphisms

In this last section we shall construct:

* new derivations, generated by δ_ℓ^m, where m is a nonnegative integer;
* new derivations, generated by δ_r^m, where m is a nonnegative integer;
* new derivations, generated by δ_ℓ^m and δ_r^n, where m and n are nonnegative integers;
* new automorphisms, generated by $(\delta_\ell \delta_r)^m$, where m is a nonnegative integer.

We define a map $d_\ell : \mathcal{IE}_{\mathsf{Z}} \to \mathcal{IE}_{\mathsf{Z}}$. Let $\kappa = \{k_n\}, k_n \in \mathsf{N}$ be a strictly increasing sequence, called a configuration of d_ℓ. The terms of κ are called jump points of d_ℓ. Let k_1, k_2 and k_3 be jump points of d_ℓ, which are consecutive terms of κ. Then for $\alpha \in \mathcal{IE}_{\mathsf{Z}}$

$$d_\ell(\alpha)(i) = \begin{cases} \delta_\ell^{m_{k_1}}(\alpha)(i), & \text{if } k_1 \leq i < k_2 \\ \delta_\ell^{m_{k_2}}(\alpha)(i), & \text{if } k_2 \leq i < k_3 \end{cases}, \qquad (7)$$

where m_{k_1} and m_{k_2} are nonnegative integers such that $m_{k_1} \leq m_{k_2}$.

When $m_k = 0$ we have $\delta_\ell^{m_k} = \mathrm{i}$. If $m_{k_1} = m_{k_2}$ the point k_2 is called a point of zero jump. The configuration of d_ℓ is called trivial, if all their terms are points of zero jump. So, any derivation of the type δ_ℓ^m, $m \in \mathrm{Z}$, $m \geq 0$, is a map with trivial configuration.
In the general case when the map d_ℓ have a configuration $\kappa = \{k_n\}, n \in \mathrm{Z}$, which is not trivial, in all of the intervals $[k_j, k_{j+1})$, $j \in \mathrm{Z}$, the map d_ℓ is a derivation.
Corollary *The map $d_\ell : \mathcal{IE}_\mathrm{Z} \to \mathcal{IE}_\mathrm{Z}$ is a derivation in $\mathcal{IE}_\mathrm{Z}$.*
The set of derivations d_ℓ of the type (7) is denoted by $\mathcal{D}_\ell$.
Let $d_{\ell 1}, d_{\ell 2} \in \mathcal{D}_\ell$ be derivations with different configurations.
Let $\{k_n\}, n \in \mathrm{Z}$, be a configuration of $d_{\ell 1}$ and h_1, h_2 and h_3 be jump points of $d_{\ell 2}$, which are consecutive terms of the given configuration of $d_{\ell 2}$. Let $k_{10}, k_{11}, \dots, k_{1p_1}, k_{21}, \dots, k_{2p_2}, k_{31}$ be consecutive terms of the sequence $\{k_n\}$ such that

$$k_{10} < h_1 < k_{11} < \cdots < k_{1p_1} < h_2 < k_{21} < \cdots < k_{2p_2} < h_3 < k_{31}. \tag{8}$$

Then for $\alpha \in \mathcal{IE}_Z$ we define $(d_{\ell 1} + d_{\ell 2})(\alpha)(i) =$

$$\begin{cases} \delta_\ell^{m_{k_{10}}}(\alpha)(i), & \text{if } k_{10} \le i < h_1 \\ \delta_\ell^{m_{s_{10}}}(\alpha)(i), & \text{if } h_1 \le i < k_{11} \\ \delta_\ell^{m_{s_{11}}}(\alpha)(i), & \text{if } k_{11} \le i < k_{12} \\ \dots\dots\dots\dots\dots\dots & \\ \delta_\ell^{m_{s_{1p_1}}}(\alpha)(i), & \text{if } k_{1p_1} \le i < h_2 \\ \delta_\ell^{m_{s_{20}}}(\alpha)(i), & \text{if } h_2 \le i < k_{21} \\ \delta_\ell^{m_{s_{21}}}(\alpha)(i), & \text{if } k_{21} \le i < k_{22} \\ \dots\dots\dots\dots\dots\dots & \\ \delta_\ell^{m_{s_{2p_2}}}(\alpha)(i), & \text{if } k_{2p_2} \le i < h_3 \\ \delta_\ell^{m_{s_{30}}}(\alpha)(i), & \text{if } h_3 \le i < k_{31} \end{cases}, \qquad (9)$$

where $m_{s_{10}} = \max\{m_{h_1}, m_{k_{10}}\}$,
$m_{s_{11}} = \max\{m_{h_1}, m_{k_{11}}\}, \dots,\ m_{s_{1p_1}} = \max\{m_{h_1}, m_{k_{1p_1}}\}$,
$m_{s_{20}} = \max\{m_{h_2}, m_{k_{1p_1}}\},\ m_{s_{21}} = \max\{m_{h_2}, m_{k_{21}}\}, \dots,$
$m_{s_{2p_2}} = \max\{m_{h_2}, m_{k_{2p_2}}\}$ and $m_{s_{30}} = \max\{m_{h_3}, m_{k_{2p_2}}\}$.

There are two another (except (8)) possibilities for the jump points of the derivations $d_{\ell 1}$ and $d_{\ell 2}$:
Case 1. some of the numbers h_1, h_2 and h_3 are jump points of $d_{\ell 1}$;
Case 2. between two of numbers h_1, h_2 and h_3 there are no jump points of $d_{\ell 1}$.
In the first case the value $(d_{\ell 1} + d_{\ell 2})(\alpha)(i)$ from (9) have the same type, for example, if $h_1 = k_{11}$, $m_{s_{11}} = m_{k_{11}}, \dots, m_{s_{1p_1}} = m_{k_{1p_1}}$.
In the second case the value $(d_{\ell 1} + d_{\ell 2})(\alpha)(i)$ from (9) have similar, but shorter type
From (7) and (9) it follows

$$m_{k_{10}} \le m_{s_{10}} \le \cdots \le m_{s_{1p_1}} \le m_{s_{20}} \le m_{s_{21}} \le \cdots \le m_{s_{2p_2}} \le m_{s_{30}},$$

thus $d_{\ell 1} + d_{\ell 2} \in \mathcal{D}_\ell$.

Configuration of the map $d_{\ell 1} + d_{\ell 2}$ is a part of the union of jump points of $d_{\ell 1}$ and $d_{\ell 2}$. It is possible some of the jump points of $d_{\ell 1}$ and $d_{\ell 2}$ to be points of zero jump of the sum $d_{\ell 1} + d_{\ell 2}$, for example, if in (9) we have $m_{h_1} \leq m_{k_{10}}$, the point h_1 is a point of zero jump of $d_{\ell 1} + d_{\ell 2}$.
From the definition of $d_{\ell 1} + d_{\ell 2}$, it follows
$(d_{\ell 1} + d_{\ell 2})(\alpha) = d_{\ell 1}(\alpha) + d_{\ell 2}(\alpha)$ for an arbitrary endomorphism $\alpha \in \mathcal{IE}_{\mathsf{Z}}$.

The product of derivations $d_{\ell 1}$ and is $d_{\ell 2}$ defined similarly. Let us suppose that the jump points of the derivations $d_{\ell 1}$ and $d_{\ell 2}$ are arranged as in (8).

Then for $\alpha \in \mathcal{IE}_{\mathsf{Z}}$ we define $(d_{\ell 1} d_{\ell 2})(\alpha)(i) = d_{\ell 2}(d_{\ell 1}(\alpha))(i) =$

$$\begin{cases} \delta_\ell^{m_{k_{10}}}(\alpha)(i), & \text{if } k_{10} \leq i < h_1 \\ \delta_\ell^{m_{h_1}+m_{k_{10}}}(\alpha)(i), & \text{if } h_1 \leq i < k_{11} \\ \delta_\ell^{m_{h_1}+m_{k_{11}}}(\alpha)(i), & \text{if } k_{11} \leq i < k_{12} \\ \dots\dots\dots\dots\dots\dots & \\ \delta_\ell^{m_{h_1}+m_{k_{1p_1}}}(\alpha)(i), & \text{if } k_{1p_1} \leq i < h_2 \\ \delta_\ell^{m_{h_2}+m_{k_{1p_1}}}(\alpha)(i), & \text{if } h_2 \leq i < k_{21} \\ \delta_\ell^{m_{h_2}+m_{k_{21}}}(\alpha)(i), & \text{if } k_{21} \leq i < k_{22} \\ \dots\dots\dots\dots\dots\dots & \\ \delta_\ell^{m_{h_2}+m_{k_{2p_2}}}(\alpha)(i), & \text{if } k_{2p_2} \leq i < h_3 \\ \delta_\ell^{m_{h_3}+m_{k_{2p_2}}}(\alpha)(i), & \text{if } h_3 \leq i < k_{31} \end{cases}. \qquad (10)$$

From (10) follows that $d_{\ell 1} d_{\ell 2} \in \mathcal{D}_\ell$ and any two derivations of $\mathcal{D}_\ell$ commute. Thus we have proved

Theorem 3. *The set of derivations $\mathcal{D}_\ell$ is a commutative additively idempotent semiring*

From the last theorem, it follows

Corollary *Any left ideal of the semiring $\mathcal{IE}_{\mathsf{Z}}$ is closed under an arbitrary derivation $d_\ell \in \mathcal{D}_\ell$.*

If $d_{\ell 1}, d_{\ell 2} \in \mathcal{D}_\ell$ have the same configuration from (9) and (10) follows that $d_{\ell 1} + d_{\ell 2}$ and $d_{\ell 1} d_{\ell 2}$ have also this configuration. If we denote by $\mathcal{D}_\ell^\kappa$ the set of derivations of $\mathcal{D}_\ell$, with the fixed configuration $\kappa = \{k_n\}$, $n \in \mathsf{Z}$, so we obtain

Proposition 9. *The set of derivations $\mathcal{D}_\ell^\kappa$ for an arbitrary configuration κ is a subsemiring of $\mathcal{D}_\ell$.*

Analogously we construct a map $d_r : \mathcal{IE}_Z \to \mathcal{IE}_Z$. Let k_t, k_{t+1} and k_{t+2} be jump points of d_r, which are consecutive numbers of the fixed configuration of d_r. Then for an arbitrary endomorphism $\alpha \in \mathcal{IE}_Z$ and integer i we define

$$d_r(\alpha)(i) = \begin{cases} \delta_r^{m_{k_t}}(\alpha)(i), & \text{if } k_t \le i < k_{t+1} \\ \delta_r^{m_{k_{t+1}}}(\alpha)(i), & \text{if } k_{t+1} \le i < k_{t+2} \end{cases}, \qquad (11)$$

where m_{k_t} and $m_{k_{t+1}}$ are nonnegative integers such that $m_{k_t} \ge m_{k_{t+1}}$. Configuration of d_r is called trivial if all their numbers are points of zero jump. Hence any derivation of the type δ_r^m, $m \in Z$, $m \ge 0$, is a map with trivial configuration.

From (3) we obtain that the derivations δ_r^m, $m \in Z$, $m \ge 0$, forms a decreasing sequence when the degree m is increasing. There is a point of nonzero jump, denoted by k_{-1}, such that all terms of configuration, greater than k_{-1} are points of zero jump.

So we can consider a configuration κ as a strictly decreasing sequence of integers with first term k_{-1}. In general the map d_r has a configuration $\{k_{-n}\}, n \in \mathrm{N}$, which is not trivial. In each of the intervals $[k_{-i}, k_{-i-1})$, $i \in \mathrm{N}$, d_r is a derivation. Thus we obtain that the map $d_r : \mathcal{IE}_\mathrm{Z} \to \mathcal{IE}_\mathrm{Z}$ is a derivation in the semiring $\mathcal{IE}_\mathrm{Z}$. The set of derivations d_r defined in (11) is denoted by $\mathcal{D}_r$.

In the same way as in the proof of the last theorem we obtain

Theorem 4. *The set of derivations $\mathcal{D}_r$ is a commutative additively idempotent semiring.*

As a consequence we find that any right ideal of the semiring $\mathcal{IE}_\mathrm{Z}$ is closed under an arbitrary derivation $d_r \in \mathcal{D}_r$.

Similarly to the previous reasoning we find that if $d_{r1}, d_{r2} \in \mathcal{D}_r$ have a same configuration, it follows that $d_{r1} + d_{r2}$ and $d_{r1}d_{r2}$ have also this configuration. If we denote by $\mathcal{D}_r^\kappa$ the set of derivations of $\mathcal{D}_r$, having the same configuration $\kappa = \{k_{-n}\}$, $n \in \mathrm{N}$, we obtain

Proposition 10. *The set of derivations $\mathcal{D}_r^\kappa$ for an arbitrary configuration κ is a subsemiring of $\mathcal{D}_r$.*

We construct a new map $d_{\ell r} : \mathcal{IE}_{\mathsf{Z}} \to \mathcal{IE}_{\mathsf{Z}}$. Let k_0 be a point of nonzero jump of $d_{\ell r}$. For an arbitrary endomorphism $\alpha \in \mathcal{IE}_{\mathsf{Z}}$ and integer i, using (7) and (11) we define

$$d_{\ell r}(\alpha)(i) = \begin{cases} d_r(\alpha)(i), & \text{if } i < k_0 \\ d_\ell(\alpha)(i), & \text{if } k_0 \leq i \end{cases}. \quad (12)$$

In the each interval with endpoints from a given configuration of $d_{\ell r}$ the map $d_{\ell r}$ is a derivation. Thus, it follows that the map $d_{\ell r} : \mathcal{IE}_{\mathsf{Z}} \to \mathcal{IE}_{\mathsf{Z}}$ is a derivation in the semiring $\mathcal{IE}_{\mathsf{Z}}$.
Let k_0 be a fixed integer. The set of derivations $d_{\ell r}$, such that this k_0 appears in (12) is denoted by $\mathcal{D}_{\ell r}^{k_0}$.
Corollary *For an arbitrary integer k_0 the set of derivations $\mathcal{D}_{\ell r}^{k_0}$ is a commutative additively idempotent semiring.*
If we denote by $\mathcal{D}_{\ell r}^{\kappa}$ the set of derivations with the same configuration κ, containing the k_0, from the last corollary it follows
Proposition 11. *The set of derivations $\mathcal{D}_{\ell r}^{\kappa}$ for an arbitrary configuration κ, containing the point k_0 is a subsemiring of $\mathcal{D}_{\ell r}^{k_0}$.*

By analogous construction we consider an automorphism $A_{\ell r} : \mathcal{IE}_Z \to \mathcal{IE}_Z$. Let $\kappa = \{k_n\}, k_n \in N$ be a strictly increasing sequence of integers which is called a configuration of the map and their terms are called jump points of $A_{\ell r}$. Let k_1, k_2 and k_3 be consecutive jump points. For an arbitrary endomorphism $\alpha \in \mathcal{IE}_Z$ and integer i we define

$$A_{\ell r}(\alpha)(i) = \begin{cases} (\delta_\ell \delta_r)^{m_{k_1}}(\alpha)(i), & \text{if } k_1 \leq i < k_2 \\ (\delta_\ell \delta_r)^{m_{k_2}}(\alpha)(i), & \text{if } k_2 \leq i < k_3 \end{cases}, \qquad (13)$$

where m_{k_1} and m_{k_2} are an arbitrary integers such that $m_{k_1} \leq m_{k_2}$. If $m_{k_1} = m_{k_2}$ the number k_2 is called a point of zero jump

The configuration of $A_{\ell r}$ is called trivial if all their terms are points of zero jump. So any automorphism considered in (6) is a map with a trivial configuration. Thus the set of automorphisms with a trivial configuration is an additively idempotent semifield $\mathcal{AUT}^0_{\ell r} = \{(\delta_\ell \delta_r)^m \mid m \in \mathrm{Z}\}$. If $A_{\ell r}$ has configuration $\kappa = \{k_n\}, n \in \mathrm{Z}$, which is not a trivial, in any of the intervals $[k_j, k_{j+1}), j \in \mathrm{Z}$, the map $A_{\ell r}$ is an automorphism.

Corollary *The map $A_{\ell r} : \mathcal{IE}_\mathrm{Z} \to \mathcal{IE}_\mathrm{Z}$ is an automorphism of the semiring $\mathcal{IE}_\mathrm{Z}$.*

The set of automorphisms $A_{\ell r}$ defined in (13) is denoted by $\mathcal{AUT}_{\ell r}$.

In a similar way as we prove Theorem 3, it follows

Theorem 5. *The set of automorphisms $\mathcal{AUT}_{\ell r}$ is an additively idempotent semifield.*

The additively idempotent semifield $\mathcal{AUT}^0_{\ell r}$ is a subsemifield of $\mathcal{AUT}_{\ell r}$.

If two automorphisms of $\mathcal{AUT}_{\ell r}$ has the same configuration, their sum and product also has the same configuration. We denote by $\mathcal{AUT}^{\kappa}_{\ell r}$ the set of automorphisms from $\mathcal{AUT}_{\ell r}$ with same configuration $\kappa = \{k_n\}$, $n \in \mathrm{Z}$. .
Now we construct the inverse of a given automorphism $A_{\ell r} \in \mathcal{AUT}_{\ell r}$.
Let $\alpha = \{a_k\}_{k \in \mathrm{Z}} \in \mathcal{IE}_{\mathrm{Z}}$. Let κ be a configuration with point k_0 of nonzero jump. Let

$$A_{\ell r}(\alpha)(i) = \begin{cases} (\delta_\ell \delta_r)^m(\alpha)(i), & \text{if } i < k_0 \\ (\delta_\ell \delta_r)^{m+p}(\alpha)(i), & \text{if } k_0 \leq i \end{cases}, \tag{14}$$

where $m, p \in \mathrm{Z}$. The integer $p > 0$ is called size of the jump of the point k_0. Then for a given configuration κ, it follows $A_{\ell r} \in \mathcal{AUT}^{\kappa}_{\ell r}$. For an arbitrary endomorphism β, with configuration κ containing jump point k_0, this jump point has a size $p > 0$ if and only if the difference of values of β before and after k_0 is equal to p. For any endomorphism β of this type we define

$$A_{\ell r}^{-1}(\beta)(i) = \begin{cases} (\delta_\ell \delta_r)^{-m}(\beta)(i), & \text{if } i < k_0 \\ (\delta_\ell \delta_r)^{-(m+p)}(\beta)(i), & \text{if } k_0 \leq i \end{cases}.$$

Then for an arbitrary α we obtain $A_{\ell r}^{-1}(A_{\ell r}(\alpha))(i) =$

$$\begin{cases} (\delta_\ell \delta_r)^{-m}((\delta_\ell \delta_r)^{m}(\alpha))(i) = \alpha(i), & \text{if } i < k_0 \\ (\delta_\ell \delta_r)^{-(m+p)}((\delta_\ell \delta_r)^{m+p}(\alpha))(i) = \alpha(i), & \text{if } k_0 \leq i \end{cases}.$$

Hence, $A_{\ell r}^{-1} A_{\ell r} = \mathrm{i}$ which implies

Proposition 12. *The set of automorphisms $\mathcal{AUT}_{\ell r}^{\kappa}$ for an arbitrary configuration κ, containing k_0 is a subsemifield of $\mathcal{AUT}_{\ell r}$.*

THANK YOU!