The commutator-degree of a polynomial and images of multilinear polynomials

Joint work with Ivan Gonzales Gargate (arXiv:2106.12726)

Thiago Castilho de Mello

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Images of polynomials on algebras

If $f(x_1, \ldots, x_m) \in F\langle X \rangle$, then f defines a map (which will also be denoted by f)

$$\begin{array}{rccc} f: & M_k(F)^m & \longrightarrow & M_k(F) \\ & (A_1, \ldots, A_m) & \longmapsto & f(A_1, \ldots, A_m) \end{array}$$

Question (Kaplansky):

Which subsets of $M_k(F)$ are image of some polynomial $f \in F\langle X \rangle$?

Examples

- If $f(x_1, \ldots, x_m) = x_1$, then $Im(f) = M_k(F)$.
- (Amitsur-Levitzki) If $St_{2k}(x_1, \ldots, x_{2k}) = \sum_{\sigma \in S_{2k}} (-1)^\sigma x_{\sigma(1)} \cdots x_{\sigma(2k)}$, then $Im(St_{2k}) = \{0\}$ in $M_k(F)$.
- If $f(x_1, x_2) = [x_1, x_2]$, then $\mathrm{Im}(f) = sl_k(F)$, the set of trace zero matrices (Shoda, Albert and Muckenhoupt).
- If $h(x_1, x_2, x_3, x_4) = [x_1, x_2] \circ [x_3, x_4]$, then the image of h em $M_2(F)$ is F (the set of scalar matrices).

The structure of the image of a polynomial

Proposition
If $f \in F\langle X\rangle$, then $\mathrm{Im}(f)$ is invariant under conjugation by invertible matrices.

Proposition
If $f \in F\langle X\rangle$ is multilinear, then $\mathrm{Im}(f)$ is invariant under scalar multiplication.

Question (Lvov)
Let f be a multilinear polynomial over a field F. Is the image of f on the matrix algebra $M_k(F)$ a vector space?

The linear span of the image of a polynomial

Proposition
Let $f \in F\langle X\rangle$. Then the linear span of $\mathrm{Im}(f)$ is a Lie Ideal in $M_k(F)$.

Theorem (Herstein)
If F is an infinite field, any Lie ideal of $M_k(F)$ is one of the following:

$$M_k(F), sl_k(F), F \text{ or } \{0\}.$$

Corollary
Let $f \in F\langle X\rangle$. Then the linear span of $\mathrm{Im}(f)$ is one of the following:

$$M_k(F), sl_k(F), F \text{ or } \{0\}.$$

Conjecture (Lvov-Kaplansky)
Let $f(x_1, \ldots, x_m) \in F\langle X\rangle$ be a multilinear polynomial. Then $\mathrm{Im}(f)$ in $M_k(F)$ is one of the following:

$$M_k(F), sl_k(F), F \text{ or } \{0\}.$$

Example

Example

Let $f(x) = x^k$. What is the image of f on $M_k(F), k \geq 2$?

- $f(E_{ii}) = E_{ii}$.
- $f(E_{ii} + E_{ij}) = E_{ii} + E_{ij}$, if $i \neq j$.
- As a consequence, $\text{span}(\text{Im}(f)) = M_k(F)$.
- If $E_{ij} \in \text{Im}(f)$, $i \neq j$, then $E_{ij} = A^k$, for some $A \in M_k(F)$.
- Since E_{ij} is nilpotent, so is A.
- Then $A^k = 0$, and $E_{ij} = 0$.
- In particular, $\text{Im}(f)$ is not a vector subspace of $M_k(F)$.

Some known results

Theorem (Kanel-Belov, Malev and Rowen, 2012 and Malev, 2014)
If f is a multilinear polynomial evaluated on the matrix ring $M_2(F)$, where F is a quadratically closed field, or $F = \mathbb{R}$, then $\mathrm{Im}(f)$ is one of the following:

$$M_2(F), sl_2(F), F \text{ or } \{0\}.$$

Theorem (Malev, 2014)
If f is a multilinear polynomial evaluated on the matrix ring $M_2(F)$ (where F is an arbitrary field), then $\mathrm{Im}(f)$ is either 0, F, or $sl_2 \subseteq \mathrm{Im}(f)$.

Some known results

Theorem (Kanel-Belov, Malev and Rowen, 2016)
Let F be an algebraically closed field. Then the image of a multilinear polynomial $f \in F\langle X \rangle$ evaluated on $M_3(F)$ is one of the following:

- $\{0\}$,
- F;
- a dense subset of $sl_3(F)$;
- a dense subset of $M_3(F)$;
- the set of 3-scalar matrices, or
- the set of scalars plus 3-scalar matrices.

The non-subspace ones have not been showed to be the image of any polynomial.

Problem
Let A be an associative algebra (or Lie, Jordan, or some algebra in your favorite variety) and let $f(x_1, \ldots, x_m)$ be a multilinear polynomial (multilinear in the free algebra in your favorite variety). Is the image of f is a vector subspace of A?

Variations in the Lvov-Kaplansky Conjecture - Some results

Theorem (Kanel-Belov, Malev, Rowen, 2017)
For any algebraically closed field F of characteristic $\neq 2$, the image of any Lie polynomial f (not necessarily homogeneous) evaluated on $sl_2(F)$ is either $sl_2(F)$, or 0, or the set of trace zero non-nilpotent matrices.

Theorem (Špenko, 2012, Anzis, Emrich and Valiveti, 2015)
The image of multilinear Lie polynomials of degree ≤ 4 on sl_k, $su(k)$ and $so(k)$ are vector subspaces.

Theorem (Ma and Oliva, 2016)
The image of any degree-three multilinear Jordan polynomial over the Jordan subalgebra of symmetric elements in $M_k(F)$ is a vector space.

Theorem (Malev, Pines, 2020)
The image of a (nonassociative) multilinear polynomial evaluated on the rock-paper-scissors algebra is a vector subspace.

Theorem (Malev, 2021)
If p is a multilinear polynomial evaluated on the quaternion algebra $\mathbb{H}$, then Imp is either 0, or $\mathbb{R} \subseteq \mathbb{H}$ (the space of scalar quaternions), or V (the space of vector quaternions), or $\mathbb{H}$.

Variations in the Lvov-Kaplansky Conjecture - Some results

Theorem (Fagundes, de Mello, 2018)
Let f be a multilinear polynomial of degree ≤ 4. Then the image of f on $UT_k(F)$ is UT_k, J or J^2.

Theorem (Fagundes, de Mello, 2018)
If f is a multilinear polynomial, then the linear span of $\mathrm{Im}(f)$ on $UT_k(F)$ is $UT_k(F)$ or J^r, for some $r \geq 0$.

Theorem (Fagundes, 2018)
Let $k \geq 2$ and $m \geq 1$ be integers. Let $f(x_1, \ldots, x_m) \in F\langle X \rangle$ be a nonzero multilinear polynomial. Then the image of f on strictly upper triangular matrices is either 0 or J^m.

Image of multilinear polynomials on UT_n - the solution

The commutator-degree of polynomials

We have a strictly descending chain of T-ideals of $F\langle X\rangle$:

$$F\langle X\rangle \supsetneq \langle [x_1,x_2]\rangle^T \supsetneq \langle [x_1,x_2][x_3,x_4]\rangle^T \supsetneq \langle [x_1,x_2][x_3,x_4][x_5,x_6]\rangle^T \supsetneq \cdots$$

We say that f has *commutator-degree* r if
$f \in \langle [x_1,x_2][x_3,x_4]\cdots[x_{2r-1},x_{2r}]\rangle^T$ and
$f \notin \langle [x_1,x_2][x_3,x_4]\cdots[x_{2r+1},x_{2r+2}]\rangle^T$.

Proposition
Let $f \in F\langle X\rangle$. Then f has commutator-degree r if and only if
$f \in Id(UT_r)$ and $f \notin Id(UT_{r+1})$.

The commutator-degree of polynomials

Lemma
Let A be a unitary algebra over F and let

$$f(x_1, \dots, x_m) = \sum_{\sigma \in S_n} \alpha_\sigma x_{\sigma(1)} \cdots x_{\sigma(m)}.$$

be a multilinear polynomial in $F\langle X \rangle$.

1. If $\sum_{\sigma \in S_n} \alpha_\sigma \neq 0$, then $Im(f) = A$.
2. $f \in \langle [x_1, x_2] \rangle^T$ if and only if $\sum_{\sigma \in S_n} \alpha_\sigma = 0$.

Remark
The above characterizes the multilinear polynomials with commutator degree 0.

Suitable sets of permutations

If $k \geq 1$, let $T_1, \ldots, T_k \subseteq \{1, \ldots, m\}$ and $1 \leq t_1 < \cdots < t_k$ such that

$$\{1, \ldots, m\} = T_1 \dot{\cup} \cdots \dot{\cup} T_k \dot{\cup} \{t_1, \ldots, t_k\}$$

and let us denote by $S(k, T, t)$ the subset of S_m consisting of all permutations σ satisfying:

- $\sigma(\{1, 2, \cdots, h_1 - 1\}) = T_1$
- $\sigma(h_1) = t_1$
- if $i \in \{2, \ldots, k\}$, $\sigma(\{h_1 + \cdots + h_{i-1} + 1, \ldots, h_1 + \cdots + h_i - 1\}) = T_i$
- if $i \in \{2, \ldots, k\}$, $\sigma(h_1 + \cdots + h_i) = t_i$

where $h_i = |T_i| + 1$.

Suitable sums of coefficients

And now consider the following sum of coefficients of f:

$$\beta^{(k,T,t)} = \sum_{\sigma \in S(k,T,t)} \alpha_\sigma$$

Theorem

Let $f(x_1, \ldots, x_m) = \sum_{\sigma \in S_n} \alpha_\sigma x_{\sigma(1)} \cdots x_{\sigma(m)}$ be a multilinear polynomial in $F\langle X \rangle$. Then the following assertions are equivalent.

1. The polynomial f has commutator-degree $r \geq 1$
2. For all $k < r$, and for any T, and t, we have $\beta^{(k,T,t)} = 0$ and there exist T, and t such that $\beta^{(r,T,t)} \neq 0$

The Lvov-Kaplansky conjucture for UT_n is true

Theorem (I. Gonzales, T. C. de Mello)
Let $f \in F\langle X \rangle$ be a multilinear polynomial. Then the image of f on $UT_n(F)$ is J^r if and only if f has commutator-degree r.

Related problems

Let us consider the following descending chain of T-ideals in $F\langle X\rangle$

$$F\langle X\rangle \supsetneqq \langle St_2\rangle^T \supsetneqq \langle St_3\rangle^T \supsetneqq \langle St_4\rangle^T \supsetneqq \cdots$$

Define a polynomial f to be of *St-degree* k if

$$f \in \langle St_k\rangle^T \text{ and } f \notin \langle St_{k+1}\rangle^T.$$

Problem
Characterize multilinear polynomials of St-degree k by means of its coefficients.

Thank you!