Union of Bulgarian Mathematicians

Sava Grozdev Emil Kolev

BULGARIAN
MATHEMATICAL COMPETITIONS
2000

Sofia, 2000

Winter Mathematical Competition
Russe, 4-6 February 2000

Problem 8.1 Given the inequality $(n^2-1)x < -3n^3-4n^2+n+2$, where n is an integer.

a) Factorize the expression $-3n^3-4n^2+n+2$.

b) Find all n, for which the inequality holds true for any positive number x.

Solution: a) $-3n^3-4n^2+n+2 = (n+1)^2(2-3n)$.

b) Since $0.x < 0$ is not true for any x it follows that $n \neq -1$. If $n > -1$ then the inequality is equivalent to $(n-1)x < (n+1)(2-3n)$. If $n = 1$ then $0.x < -2$ which is not true for any x. Let $n \neq 1$. If $n-1 > 0$, then $\dfrac{(n+1)(2-3n)}{n-1} < 0$ and the inequality is not true for any positive x. If $n-1 < 0$, then the inequality becomes $x > \dfrac{(n+1)(2-3n)}{n-1}$ and $n = 0$ is the only solution. If $n < -1$ then $\dfrac{(n+1)(2-3n)}{n-1} > 0$ and there exists $x > 0$, which is not a solution. Therefore the inequality has no solution when $n < -1$.

Problem 8.2 In an isosceles $\triangle ABC (AC = BC)$ the points A_1, B_1 and C_1 are midpoints of BC, AC and AB respectively. Points A_2 and B_2 are symmetric points of A_1 and B_1 with respect to AB. Let M be the intersecting point of CA_2 and A_1C_1, and let N be the intersecting point of CB_2 and B_1C_1. The intersecting point of AN and BM is denoted by P. Prove that $AP = BP$.

Solution: Since $CC_1 \parallel A_1A_2$ and $CC_1 = A_1A_2$, we have that $CC_1A_2A_1$ is a parallelogram. Thus, $A_1M = C_1M$. But $A_1B_1C_1B$ is also a parallelogram and therefore the intersecting point of BM and AC is B_1. Hence P lies on the median BB_1. Analogously P lies on the median AA_1. In the isosceles $\triangle ABC$ the medians AA_1 and BB_1 are of the same length. Therefore $AP = \frac{2}{3}AA_1 = \frac{2}{3}BB_1 = BP$.

Problem 8.3 Find all pairs of prime numbers p and q, such that $p^2 + 3pq + q^2$ is:

a) a perfect square;

b) a power of 5.

Solution: a) Let $p^2 + 3pq + q^2 = r^2$, where p and q are prime numbers. If $p \neq 3, q \neq 3$, then $p^2 + 3pq + q^2 \equiv 2 (\text{mod } 3)$ and $r^2 \equiv 2 (\text{mod } 3)$, a contradiction. Without loss of generality $p = 3$ and we get that $q^2 + 9q + 9 = r^2$ and $4q^2 + 36q + 36 = (2r)^2$. Therefore $(2q - 2r + 9)(2q + 2r + 9) = 45$. We may assume that $r > 0$ and so $2q + 2r + 9 = 15$ or $2q + 2r + 9 = 45$. In the first case $q + r = 3$, which is impossible and in the second case solving the system

$$\left| \begin{array}{rcl} q + r & = & 18 \\ 2q - 2r + 9 & = & 1, \end{array} \right.$$

we find $q = 7$. Because of the symmetry the only solutions are $p = 3, q = 7$ and $p = 7, q = 3$.

b) Let $p^2 + 3pq + q^2 = 5^n$, where n is a natural number. Since $p \geq 2, q \geq 2$, we have $p^2 + 3pq + q^2 \geq 20$ and so $n \geq 2$. It follows now that $25/(p^2 + 3pq + q^2)$ and $5/(p^2 + 3pq + q^2) = (p - q)^2 + 5pq$. Thus, $5/(p - q)^2$ and $25/(p - q)^2$. Therefore $25/5pq$, showing that $p = 5$ or $q = 5$. But if $p = 5$ then $q = 5$ (and vice versa). We obtain $p^2 + 3pq + q^2 = 125 = 5^3$. The only solution of the problem is $p = q = 5$.

Problem 9a.1 Given the equation $\dfrac{1}{|x-2|} = \dfrac{1}{|x-52a|}$, where a is a parameter.

a) Solve the equation.

b) If a is the square of a prime number prove that the equation has a solution which is a compose integer.

Solution: After squaring and simple calculations we obtain $(26a-1)x = (26a-1)(26a+1)$.

a) If $a = \dfrac{1}{26}$ then every $x \neq 2$ is a solution. If $a \neq \dfrac{1}{26}$ then the only solution is $x = 26a+1$.

b) Let $a = p^2$ where p is a prime number. If $p = 3$ then $x = 235$ is not a prime. If $p \neq 3$ then $p = 3k \pm 1$ and $x = 26(3k \pm 1)^2 + 1 = 3A + 27$, which is divisible by 3.

Problem 9a.2 The quadrilateral $ABCD$ is inscribed in a circle with diameter BD. Let M be the symmetric point of A with respect to BD and let N be the intersecting point of the straight lines AM and BD. The line passing through N, which is parallel to AC, intersects CD and BC in P and Q respectively. Prove that the points P, C, Q and M are vertices of a rectangle.

Solution: It follows from the condition of the problem that M lies on the circumcircle of $ABCD$. Since $\sphericalangle MAC = \sphericalangle MBC = \frac{\overset{\frown}{MC}}{2}$ and $\sphericalangle MNQ = \sphericalangle MAC$ i.e. $\sphericalangle MNQ = \sphericalangle MBC$, we get that the points M, N, B and Q lie on a circle. Since $\sphericalangle MNB = 90^\circ$ we conclude that $\sphericalangle BQM = 90^\circ$. Also, since $\triangle BDC$ is a right angle triangle we have that $MQ \parallel PC$. From the other hand $\sphericalangle MDC = \sphericalangle MAC = \frac{\overset{\frown}{MC}}{2}$, and therefore $\sphericalangle MDC = \sphericalangle MNQ$, so the points N, P, M and D lie on a circle. Thus, $\sphericalangle MPD = \sphericalangle MND = 90^\circ$ and $MP \parallel CQ$. Therefore P, C, Q and M are vertices of a rectangle.

Problem 9a.3 See Problem 8.3.

Problem 9b.1 Given the system:

$$\left| \begin{array}{rcl} \dfrac{1}{x+y} + x & = & a-1 \\ \\ \dfrac{x}{x+y} & = & a-2, \end{array} \right.$$

where a is a real parameter.

a) Solve the system if $a = 0$.

b) Find all values of a, such that the system has an unique solution.

c) If $a \in (2, 3)$ and (x, y) is a solution of the system find all values of a such that the expression $\dfrac{x}{y} + \dfrac{y}{x}$ takes its minimal value.

Solution: It follows easily that x and $\dfrac{1}{x+y}$ are roots of the equation $t^2 - (a-1)t + a - 2 = 0$. There are two cases to be considered:

$$(1) \quad \left| \begin{array}{rcl} x & = & 1 \\ \\ \dfrac{1}{x+y} & = & a-2 \end{array} \right. \iff \left| \begin{array}{rcl} x & = & 1 \\ \\ y & = & \dfrac{3-a}{a-2}, \end{array} \right. \quad \text{when} \quad a \neq 2$$

and

$$(2) \quad \left| \begin{array}{rcl} x & = & a-2 \\ \\ \dfrac{1}{x+y} & = & 1 \end{array} \right. \iff \left| \begin{array}{rcl} x & = & a-2 \\ \\ y & = & 3-a. \end{array} \right.$$

a) When $a = 0$ we have $(x, y) = (1; -\dfrac{3}{2})$ or $(x, y) = (-2; 3)$.

b) The sytem has an unique solution $(0,1)$ when $a=2$ and $(1,0)$ when $a=3$.

c) If (x,y) is a solution and $a \in (2;3)$ then $\dfrac{x}{y}$ and $\dfrac{y}{x}$ are positive. Further $\dfrac{x}{y}+\dfrac{y}{x} \geq 2$, and equality occures when $\dfrac{x}{y}=\dfrac{y}{x}$. It follows from (1) and (2) that $\dfrac{x}{y}=\dfrac{a-2}{3-a}$. Using the equality $\dfrac{a-2}{3-a}=\dfrac{3-a}{a-2}$ we find the only value of $a=\dfrac{5}{2}$.

Problem 9b.2 Given an acute $\triangle ABC$. The bisector of $\sphericalangle ACB$ intersects AB at point L. The feet of the perpendiculars from L to AC and BC are denoted by M and N respectively. Let P be the intersecting point of AN and BM. Prove that $CP \perp AB$.

Solution: Let l be the line through C which is parallel to AB. Let F and E be respectively the intersecting points of AN and BM with l. The intersecting point of CP and AB is denoted by D. We obtain $\dfrac{AD}{CF}=\dfrac{PD}{PC}=\dfrac{BD}{CE}$, so $\dfrac{AD}{BD}=\dfrac{CF}{CE}$. From the other hand $\dfrac{AM}{CM}=\dfrac{AB}{CE}$ and $\dfrac{BN}{CN}=\dfrac{AB}{CF}$. But $CM=CN$ and we get $\dfrac{AM}{BN}=\dfrac{CF}{CE}$. Therefore $\dfrac{AD}{BD}=\dfrac{AM}{BN}$, which implies

$$\frac{AM}{AD}=\frac{BN}{BD} \tag{1}$$

Further if $CH \perp AB (H \in AB)$ then $\triangle ALM \sim \triangle AHC$ and so $\dfrac{AL}{AC}=\dfrac{AM}{AH}$. In the same manner $\dfrac{BL}{BC}=\dfrac{BN}{BH}$. But CL is a bisector and therefore $\dfrac{AL}{AC}=\dfrac{BL}{BC}$, so $\dfrac{AM}{AH}=\dfrac{BN}{BH}$. The last equation combined with (1) gives $D \equiv H$ which implies $CP \perp AB$.

Problem 9b.3 Prove that the digit of the hundreds of $2^{1999}+2^{2000}+$

2^{2001} is even.

Solution: Write the number $2^{1999}+2^{2000}+2^{2001}$ in the form $2^{1999}(1+2+4) = 7.2^9.2^{1990} = 7.2^9.2^{10}.2^{1980} = 7.2^9.2^{10}.(2^{20})^{99}$. Since $2^9 = 512, 2^{10} = 1024$ and $2^{20} = (2^{10})^2$ we have that the last two digits of 2^{20} coincide with the last two digits of 24^2, so the last two digits of 2^{20} are 76. Moreover the last two digits of 76.76 are also 76. Therefore the last two digits of the given number are the last two digits of the product 7.12.24.76, which are 1 and 6. Since $2^{1999} + 2^{2000} + 2^{2001}$ is divisible by 8 and it ends by 16, the digit of the hundreds is even.

Problem 10.1 Find all values of the real parameter a such that the nonnegative solutions of the equation $(2a-1)\sin x + (2-a)\sin 2x = \sin 3x$ form an infinite arithmetic progression.

Solution: Since $\sin 2x = 2\sin x\cos x$ and $\sin 3x = \sin x(4\cos^2 x - 1)$, we may write the equation in the form
$\sin x\,(2\cos^2 x - (2-a)\cos x - a) = 0$. Thus $\sin x = 0, \cos x = 1$ or $\cos x = -\dfrac{a}{2}$. The nonnegative solutions of the equations $\sin x = 0$ and $\cos x = 1$ are $x = k\pi$ and $x = 2k\pi, k = 0, 1, 2, \dots$ respectively. Let $|a| > 2$. In this case the equation $\cos x = -\dfrac{a}{2}$ has no solution and therefore the nonnegative solutions of the initial equation are $0, \pi, 2\pi, \dots$, which form an arithmetic progression. Let now $|a| \le 2$ and let x_0 be the only solution of the equation $\cos x = -\dfrac{a}{2}$ in the interval $[0, \pi]$. In this case the nonnegative solutions of the last equation are $x = x_0 + 2k\pi$ and $x = 2\pi - x_0 + 2k\pi$. It is clear now that the nonnegative solutions form an arithmetic progression only when $x_0 = 0, x_0 = \dfrac{\pi}{2}$ and $x_0 = \pi$, so giving $a = -2, a = 0$ and $a = 2$. The values of a are $a = 0$ and $|a| > 2$.

Problem 10.2 Let O, I and H be respectively the circumcenter, incenter and orthocenter for an acute nonequilateral $\triangle ABC$. Prove that if the circumcircle of $\triangle OIH$ passes through one of the vertices of $\triangle ABC$ then it passes through another vertex of $\triangle ABC$.

Solution: Assume that O, I, H and C lie on a circle. It is well known that CI is the bisector of $\sphericalangle HCO$. Thus $\sphericalangle IHO = \sphericalangle ICO = \sphericalangle ICH = \sphericalangle HOI$ and it follows from $\triangle IHO$ that $IH = IO = t$.

We shall prove that if $\sphericalangle BAC \neq 60°$ then O, I, H and A lie on a circle. Denote by M and N the projection points of I respectively on AO and AH. Let O_1 and O_2 be such that $IO_1 = IO_2 = t$ and O_1 lies between A and M, and M lies between O_1 and O_2. Analogously let H_1 and H_2 be such that $IH_1 = IH_2 = t$ and H_1 lies between A and N, and N lies between H_1 and H_2. If $O \equiv O_1; H \equiv H_1$ or $O \equiv O_2; H \equiv H_2$, then $\triangle AIO \sim \triangle AIH$ and therefore $AO = AH$. But $AH = 2AO \cos \sphericalangle BAC$ and so $\sphericalangle BAC = 60°$. If $O \equiv O_1; H \equiv H_2$ or $O \equiv O_2; H \equiv H_1$, then it follows from $\sphericalangle IO_1O_2 = \sphericalangle IO_2O_1 = \sphericalangle IH_1H_2 = \sphericalangle IH_2H_1$ that $AOIH$ is inscribed.

Suppose now that A and B do not lie on the circumcircle of $\triangle OIH$. In this case $\sphericalangle BAC = \sphericalangle ABC = 60°$ and therefore $\triangle ABC$ is equilateral which is a contradiction.

Problem 10.3 In each of the cells of a 3×3 table is written a real number. The element in the $i-$th row and $j-$th column equals to the modulus of the difference of the sum of the elements from the $i-$th row and the sum of the elements from the $j-$th column. Prove that every element of the table equals either to the sum or to the difference of two other elements of the table.

Solution: Let p_1, p_2, p_3 and q_1, q_2, q_3 be the sum of the elements in the first, second and third row and in the first, second and third column respectively. It is clear that $p_1 + p_2 + p_3 = q_1 + q_2 + q_3$. Therefore the element in the first row and the first column equals to $|p_1 - q_1|$. From the other hand $|p_1 - q_1| = |p_2 + p_3 - q_2 - q_3|$ which implies $|p_1 - q_1| = \epsilon_1|p_2 - q_2| + \epsilon_2|p_3 - q_3|$ where $\epsilon_1, \epsilon_2 \in +1, -1$. Since

$|p_1 - q_1| \geq 0$ it is clear that $\epsilon_1 = \epsilon_2 = -1$ is impossible. Therefore $|p_1 - q_1|$ is either the sum or the difference of $|p_2 - q_2|$ and $|p_3 - q_3|$. By analogy every element of the table is the sum or the difference of two other elements.

Problem 11.1. Prove that for every positive number a the sequence $\{x_n\}_{n=1}^{\infty}$, such that $x_1 = 1, x_2 = a, x_{n+2} = \sqrt[3]{x_{n+1}^2 x_n}, n \geq 1$, is convergent and find its limit.

Solution: It follows by induction that the terms of the sequence $\{x_n\}_{n=1}^{\infty}$ can be expressed as $x_n = a^{\alpha_n}$, where $\{\alpha_n\}_{n=1}^{\infty}$, is a siquence defined by $\alpha_1 = 0, \alpha_2 = 1, \alpha_{n+2} = \dfrac{2\alpha_{n+1} + \alpha_n}{3}, n \geq 1$. Thus, $\alpha_{n+2} - \alpha_{n+1} = -\dfrac{1}{3}(\alpha_{n+1} - \alpha_n)$ and therefore $\alpha_{n+2} - \alpha_{n+1} = \left(-\dfrac{1}{3}\right)^n (\alpha_2 - \alpha_1) = \left(-\dfrac{1}{3}\right)^n$. Adding the equalities $\alpha_{k+2} - \alpha_{k+1} = \left(-\dfrac{1}{3}\right)^k$ for $k = 0, 1, \ldots, n$ after simple calculations we obtain $\alpha_{n+2} - \alpha_0 = \left(-\dfrac{1}{3}\right)^0 + \left(-\dfrac{1}{3}\right)^1 + \ldots + \left(-\dfrac{1}{3}\right)^n = \dfrac{1 - \left(-\frac{1}{3}\right)^{n+1}}{1 + \left(\frac{1}{3}\right)} = \dfrac{3}{4}(1 - \left(-\dfrac{1}{3}\right)^{n+1})$. Since $\alpha_0 = 0$ and $\lim\limits_{n\to\infty} \left(-\dfrac{1}{3}\right)^n = 0$ it follows that $\lim\limits_{n\to\infty} \alpha_n = \dfrac{3}{4}$. This shows that the sequence $\{x_n\}_{n=1}^{\infty}$ is convergent and $\lim\limits_{n\to\infty} x_n = a^{\frac{3}{4}}$.

Problem 11.2 Given a convex quadrilateral $ABCD$ where M is the intersecting point of its diagonals. It is known that $DB = 3DM, AM = MC$.

a) Express BC and CD by the sides of $\triangle ABD$.

b) Prove that if $2 \sphericalangle ADB - \sphericalangle ABD = 180^\circ$, then $\sphericalangle DBC = 2 \sphericalangle BDC$.

Solution: a) Denote $AB = c, BC = p, CD = q, DA = b$ and

$DB = a$. It follows from the condition of the problem that $DM = \frac{1}{3}a$, $MB = \frac{2}{3}a$. The formula for the median in a triangle applied for $\triangle ABC$ and $\triangle ACD$ gives

$$(1) \qquad \frac{16}{9}a^2 = 2p^2 + 2c^2 - 4AM^2 \frac{4}{9}a^2 = 2q^2 + 2b^2 - 4AM^2$$

Let $\measuredangle AMB = \alpha$. From the Law of Cosines for $\triangle AMB$ and $\triangle AMD$ we obtain

$$c^2 = AM^2 + \frac{4a^2}{9} - \frac{4}{3}AM.a\cos\alpha b^2 = AM^2 + \frac{a^2}{9} + \frac{2}{3}AM.a\cos\alpha$$

Thus $2b^2 + c^2 = 3AM^2 + \frac{2}{3}a^2$ and a substitution in (1) gives

$$(2) \qquad p^2 = \frac{4a^2 + 12b^2 - 3c^2}{9} q^2 = \frac{3b^2 + 6c^2 - 2a^2}{9}$$

b) It follows from the condition of the problem that $\measuredangle ADB > 90^\circ$ and so $c > a$. Let D_1 be a point on AB, such that $D_1B = DB$. Further $\measuredangle AD_1B = \measuredangle ABD + \frac{180^\circ - \measuredangle ABD}{2} = \frac{180^\circ + \measuredangle ABD}{2} = \measuredangle ADB$. Therefore $\triangle AD_1D \sim \triangle ADB$ and thus $b^2 = c(c-a)$. A substitution in (2) implies $p = \frac{3c - 2a}{3}$ and $q^2 = \frac{9c^2 - 3ac - 2a^2}{9}$. Hence $q^2 = p(p-a)$. Let B_1 be a point on BC (B lies between C and B_1), such that $BB_1 = BD = a$. Since $\frac{q}{p} = \frac{p+a}{q}$ it follows that $\triangle DBC \sim \triangle B_1DC$. Therefore $\measuredangle DBC = \measuredangle BB_1D + \measuredangle BDB_1 = 2\measuredangle BB_1D = 2\measuredangle BDC$.

Problem 11.3 See problem 10.3.

Spring Mathematical Tournament
Jambol, 24-26 March 2000

Problem 8.1. Let $f(x)$ be a linear function such that $f(0) = -5$ and $f(f(0)) = -15$. Find all values of m, for which the set of the solutions of the inequality $f(x).f(m-x) > 0$ is an interval of length 2.

Solution: Let $f(x) = ax+b$. It follows from $f(0) = -5$ that $b = -5$, and from $f(f(0)) = -15$ that $a = 2$. Therefore the function is $f(x) = 2x-5$. Consider the inequality $(2x-5)(2(m-x)-5) > 0 \iff (2x-5)(2m-5-2x) > 0$. The solution of the last inequality is an interval with end points $\frac{5}{2}$ and $\frac{2m-5}{2}$. Therefore $\left|\frac{5}{2} - \frac{2m-5}{2}\right| = 2$, so $|5-m| = 2$. Finally, we obtain that $m = 3$ and $m = 7$.

Problem 8.2 Given an isosceles right angle triangle ABC with $\sphericalangle ACB = 90°$. Point P lies on BC, M is the midpoint of AB and let L and N be points from the segment AP such that $CN \perp AP$ and $AL = CN$.

a) Find $\sphericalangle LMN$;

b) If the area of $\triangle ABC$ is 4 times greater than the area of $\triangle LMN$ find $\sphericalangle CAP$.

Solution: a) Let $\sphericalangle CAP = \alpha$. We find $\sphericalangle ACN = 90° - \alpha$ and $\sphericalangle MCN = \sphericalangle ACN - 45° = 45° - \alpha = \sphericalangle LAM$. Since $AM = CM$ and $AL = CN$ it follows that $\triangle AML \cong \triangle CMN$. Therefore $\sphericalangle AML = \sphericalangle CMN$ and so $\sphericalangle LMN = 90° - \sphericalangle AML + \sphericalangle CMN = 90°$.

b) It follows from $\triangle AML \cong \triangle CMN$ that $LM = MN$ and since

$\sphericalangle LMN = 90^\circ$ we get $S_{LMN} = \dfrac{MN^2}{2}$. Also $S_{ABC} = \dfrac{AC^2}{2}$. Now $\dfrac{4MN^2}{2} = \dfrac{AC^2}{2}$ applies that $MN = \dfrac{AC}{2}$. Denote by Q the midpoint of AC. Thus $QM = QN = \dfrac{AC}{2} = MN$. Therefore $\triangle QMN$ is equilateral and $\sphericalangle QNM = 60^\circ$, which implies $\sphericalangle QNA = 60^\circ - 45^\circ = 15^\circ$. But $AQ = QN$ and so $\sphericalangle CAP = \sphericalangle QNA = 15^\circ$.

Problem 8.3 There are 2000 white balls in a box. There are also sufficiently many white, green and red balls. The following operations are allowed:

1) Replacement of two white balls with a green ball;

2) Replacement of two red balls with a green ball;

3) Replacement of two green balls with a white bal and a red ball;

4) Replacement of a white ball and a green ball with a red ball;

5) Replacement of a green ball and a red ball with a white ball;

a) After finitely many of the above opperations there are three balls left in the box. Prove that at least one of them is a green ball.

b) Is it possible after finitely many opperations to have only one ball left in the box?

Solution: Consider a box with x white, y green and z red balls. Direct verification shows that after applying any of the allowed operations the sum $x + 2y + 3z$ does not change modulo 4. Since the initial values are $x = 2000, y = z = 0$, we obtain that this sum is congruent to 0 modulo 4.

a) There are 3 balls in the box and therefore $x + y + z = 3$. Moreover $x+2y+3z \equiv 0 (\mathrm{mod} 4)$. If a green ball is not in the box then $y = 0$ and so $x+z = 3$ and $x+3z \equiv x+3(3-x) \equiv 1-2x \equiv 0 (\mathrm{mod} 4)$, which is impossible.

b) Suppose that there is only one ball left in the box. Therefore $x+y+z=1$ and $x+2y+3z\equiv 0(\mathrm{mod}4)$, which is impossible.

Problem 9a.1 Find all values of m such that the equation

$$\left(\frac{1}{x+m}+\frac{m}{x-m}-\frac{2m}{m^2-x^2}\right)(|x-m|-m)=0$$

has exactly one nonnegative root.

Solution: The equation $\frac{1}{x+m}+\frac{m}{x-m}-\frac{2m}{m^2-x^2}=0$ is equivalent to (when $x\neq\pm m$) to $(m+1)x=-m(m+1)$. When $m=-1$ it has infinitely many roots – all numbers $x\neq\pm1$. If $m\neq-1$ we obtain $x=-m$. The equation $|x-m|-m=0$ has two roots: $x=0$ and $x=2m$ when $m>0$; an unique root: $x=0$ when $m=0$ and has no roots when $m<0$.

Let $m<0$ and $m\neq-1$. In this case the equation has an unique root $x=-m$ and it is nonnegative. Let $m=0$. Then the equation becomes $\frac{1}{x}|x|=0$, which obviously has no roots. Let $m>0$. The equation has three roots $x=-m, x=0, x=2m$ and two of them are nonnegative.

Thus, the desired values of m are $m<0$ and $m\neq-1$.

Problem 9a.2 Given an acute-angled triangle ABC and let α,β and γ be its angles respectively to A,B and C. For an arbitrary interior point M denote by A_1, B_1 and C_1 respectively the feet of the perpendiculars from M to BC, CA and AB. Find the locus of M for which the triangle $A_1B_1C_1$ is a right angle triangle.

Solution: Let M be a point such that $\sphericalangle A_1C_1B_1=90^\circ$. Since the quadrilateral AC_1MB_1 is inscribed we have $\sphericalangle B_1C_1M=\sphericalangle B_1AM=x$. Analogously $\sphericalangle A_1C_1M=\sphericalangle A_1BM=y$. Denote $z=\sphericalangle AMB$. It follows now from the quadrilateral $AMBC$ that $x+\gamma+y+(360^\circ-$

$z) = 360°$. Since $x + y = \sphericalangle A_1C_1B_1 = 90°$ we get $z = 90° + \gamma$. Therefore $\sphericalangle AMB = 90° + \gamma$. Thus, the locus of M such that $\sphericalangle A_1C_1B_1 = 90°$ is an arc G_1 in the interior of $\triangle ABC$ for which the segment AB is seen under angle $90° + \gamma$. Analogously one can prove that the locus of M such that $\sphericalangle C_1B_1A_1 = 90°$ is an arc G_2 in the interior of $\triangle ABC$, for which AC is seen under angle $90° + \beta$. Further, the locus of M such that $\sphericalangle B_1A_1C_1 = 90°$ is an arc G_3 in the interior of $\triangle ABC$ for which BC is seen under angle $90° + \alpha$. The desired locus is the union of the three arcs $G_1 \cup G_2 \cup G_3$.

Problem 9a.3 See Problem 8.3

Problem 9b.1 The real numbers x and y are such that $x^2+xy+y^2 = 1$. If $F = x^3y + xy^3$,

a) prove that $F \geq -2$;

b) find the greatest possible value of F.

Solution: a) It follows from the condition of the problem that $(x + y)^2 = 1 + xy$ and therefore $xy \geq -1$. From the other hand $x^2 + y^2 = 1 - xy$. Thus, $xy \leq 1$, so $z = xy \in [-1; 1]$. Hence $F = xy(x^2 + y^2) = z(1 - z)$ and the inequality $F \geq -2$ is equivalent to the inequality $z^2 - z - 2 \leq 0$. The latter one is true for any $z \in [-1; 2]$.

b) Assume that the greatest value of F exists and denote it by A. This implies that the system

$$(1) \qquad \left| \begin{array}{rcl} xy(x^2 + y^2) & = & A \\ xy + (x^2 + y^2) & = & 1 \end{array} \right.$$

has a solution. It is clear that $t_1 = xy$ and $t_2 = x^2 + y^2$ are roots of the quadratic equation $t^2 - t + A = 0$. But $t_2 - t_1 = \frac{1}{2}(x - y)^2 + \frac{1}{2}(x + y)^2 \geq 0$ and the equality is impossible since if so then $x = y = 0$ which is a contradiction to $x^2 + xy + y^2 = 1$. Therefore $t_2 > t_1$ and so $D = 1 - 4A > 0$, i.e. $A < \frac{1}{4}$. The system

(1) is equivalent to $\left|\begin{array}{rcl} xy & = & t_1 \\ x^2+y^2 & = & t_2, \end{array}\right.$ and the last is equivalent to $\left|\begin{array}{rcl} (x-y)^2 & = & t_2-2t_1 \\ (x+y)^2 & = & t_2+2t_1. \end{array}\right.$ The latter system has a solution iff the inequalities $t_2-2t_1 \geq 0$ and $t_2+2t_1 \geq 0$ hold true. It suffices to prove that when $A < \frac{1}{4}$ the roots $t_1 < t_2$ of the equation $g(t) = t^2 - t + A$ satisfy the inequalities $t_2 - 2t_1 \geq 0$ and $t_2 + 2t_1 \geq 0$. If $A = 0$ then $t_1 = 0$ and $t_2 = 1$ and the inequalities hold true. It is clear that we can consider only the case $A \in (0; \frac{1}{4})$. Now $t_1 > 0, t_2 > 0$ and the inequality $t_2 + 2t_1 \geq 0$ holds true. From the other hahd $t_1 + t_2 = 1$ and the inequality $t_2 - 2t_1 \geq 0$ is equivalent to $t_2 \geq \frac{2}{3}$. The same inequality is equivalent to $t_1 \leq \frac{1}{3}$. For the roots t_1 and t_2 we obtain $t_1 \leq \frac{1}{3}$ $t_2 \geq \frac{2}{3}$, which is equivalent to $\left|\begin{array}{rcl} g(\frac{1}{3}) & \leq & 0 \\ g(\frac{2}{3}) & \leq & 0. \end{array}\right.$ Therefore $A \leq \frac{2}{9}$. Conversely, if $A = \frac{2}{9}$ then $x = y = \pm\frac{\sqrt{3}}{3}$ satisfy the condition of the problem.

Problem 9b.2 A line l is drown through the orthocenter of an acute-angled triangle ABC. Prove that the lines symmetric to l with respect to the sides of the triangle intersect in a point.

Solution: Let the intersecting points of l with the sides AC and BC be Q and P respectively and let the intersecting point of l with the extention of AB be R. Without loss of generality A lies between R and B. Denote the symmetric points of H with respect to AC, BC and AB by B_1, A_1 and C_1 respectively. It is well known that A_1, B_1 and C_1 lie on the circumcircle k of $\triangle ABC$. It is clear also that the symmetric lines of l are B_1Q, A_1P and C_1R. Let $B_1Q \cap A_1P = S$ and $B_1Q \cap C_1R = T$. We obtain $\sphericalangle CB_1Q + \sphericalangle CA_1P = \sphericalangle CHQ + \sphericalangle CHP$ $= 180°$ and therefore SA_1CB_1 is inscribed. Thus, $S = k \cap B_1Q$. From the other hand $\sphericalangle RC_1A = \sphericalangle RHA = \sphericalangle AB_1T$ and therefore B_1ATC_1 is inscribed. Hence, $T = k \cap B_1Q$. This implies that $T \equiv S$ which shows that the three lines intersect in a point.

Problem 9b.3 See Problem 8.3.

Problem 10.1 Solve the equation $\sqrt{x} + \sqrt[3]{x+7} = \sqrt[4]{x+80}$.

Solution: It is clear that $x \geq 0$. Note that $x = 1$ is a root of the equation. We shall prove that there are no other roots. Rasing the equation to the forth power we get $x^2 + 4(\sqrt{x})^3\sqrt[3]{x+7} + 6(\sqrt{x})^2(\sqrt[3]{x+7})^2 + 4(\sqrt{x})(\sqrt[3]{x+7})^3 + (\sqrt[3]{x+7})^4$
$= x + 80$.

Let $f(x) = 4(\sqrt{x})^3\sqrt[3]{x+7} + 6(\sqrt{x})^2(\sqrt[3]{x+7})^2$
$+4(\sqrt{x})(\sqrt[3]{x+7})^3 + (\sqrt[3]{x+7})^4$. Obviously $f(x)$ is an increasing function. If $x > 1$ then it follows from the inequalities $x^2 > x, f(x) > f(1) = 80$ that $x^2 + f(x) > x + 80$, a contradiction. If $x < 1$ then it follows from the inequalities $x^2 < x, f(x) < f(1) = 80$ that $x^2 + f(x) < x + 80$, a contradiction. This completes the prove.

Problem 10.2 The incircle of an isosceles $\triangle ABC$ touches the legs AC and BC at points M and N respectively. A tangent t is drawn to the smaller of the arcs $\overset{\frown}{MN}$ and let t intersects NC and MC at points P and Q respectively. Let T be the intersecting point of the lines AP and BQ.

a) Prove that T lies on the segment MN;

b) Prove that the sum of the areas of triangles ATQ and BTP is the smallest possible when t is parallel to AB.

Solution: a) Let the incircle touches AB and PQ at points R and S respectively. Let MN and SR intersect QB in points T_1 and T_2 respectively. Since $\sphericalangle T_1MQ = \sphericalangle T_1NC = 180° - \sphericalangle T_1NB$ and $\sphericalangle MT_1D = \sphericalangle BT_1N$ it follows from the Law of Sines for $\triangle MT_1D$ and

$\triangle BT_1N$ that $\dfrac{DT_1}{MD} = \dfrac{BT_1}{BN}$, so $\dfrac{QT_1}{BT_1} = \dfrac{MQ}{BN}$. By analogy $\dfrac{QT_2}{BT_2} = \dfrac{SQ}{BR}$. It follows now from $MQ = SQ$ and $BN = BR$ that $\dfrac{QT_1}{BT_1} = \dfrac{QT_2}{BT_2}$, so $T_1 = T_2$. In the same manner one can prove that AP passes through the intersecting point of MN and SR. This implies that AP, BQ, MN and SR intersect in T.

b) We have $S_{ATQ} + S_{BPT} = S_{ABQ} + S_{ABP} - 2S_{ABT}$. Since $\triangle ABC$ is isosceles we get $MN \parallel AB$ and therefore S_{ABT} is constant. Thus, $S_{ATQ} + S_{BPT}$ is minimal exactly when $S_{ABQ} + S_{ABP}$ is minimal. The latter sum equals to $\dfrac{AQ.AB\sin\alpha + BP.AB\sin\alpha}{2} =$ $\frac{(AQ+BP)AB\sin\alpha}{2}$, where α is the angle to the base of the triangle. Therefore it suffices to find the minimum of $AQ + BP$. It is easily seen that $AQ + BP = AM + BN + PQ$ and therefore we have to find when PQ is minimal. Let r be the inradius and O be the center of the incircle of $\triangle ABC$. Using that $PQ = r(\operatorname{cotg}\phi + \operatorname{cotg}\psi)$ where $\phi = \sphericalangle OQP, \psi = \sphericalangle OPQ$ we obtain from $ABPQ$ that $2\phi + 2\psi + 2\alpha = 360°$. Therefore $\phi + \psi = 180° - \alpha$. Thus $PQ = \dfrac{2r\sin\alpha}{\cos(\phi - \psi) + \cos\alpha}$. It is clear now that PQ is minimal exactly when $\cos(\phi - \psi) = 1$, so $\phi = \psi \Longrightarrow PQ \parallel AB$.

Problem 10.3 There are $n \geq 4$ points in the plane such that the distance between any two of them is an integer. Prove that at least $\dfrac{1}{6}$ from the distances between them are divisible by 3.

Solution: We show first that the assertion from the problem is true for $n = 4$ i.e. for 4 points with integer distances between them at least one distance is divisible by 3. Denote the points by A, B, C and D (it is easy to be seen that WLOG $\sphericalangle BAD = \sphericalangle BAC + \sphericalangle CAD$). By the Law of Cosines for $\triangle ABC, \triangle ACD, \triangle ABD$ we obtain

$$BC^2 = AB^2 + AC^2 - 2.AB.AC\cos\alpha$$

$$CD^2 = AD^2 + AC^2 - 2.AD.AC\cos\beta$$
$$BD^2 = AB^2 + AD^2 - 2.AB.AD\cos\gamma$$
where $\alpha = \measuredangle BAC, \beta = \measuredangle CAD, \gamma = \measuredangle BAD = \alpha + \beta$.

Suppose that all distances are integers not divisible by 3. Therefore $AB^2 \equiv AC^2 \equiv AD^2 \equiv BC^2 \equiv CD^2 \equiv BD^2 \equiv 1 \pmod 3$ and so $2.AB.AC\cos\alpha \equiv 2.AD.AC\cos\beta \equiv 2.AB.AD\cos\gamma \equiv 1 \pmod 3$. Thus, $2.AB.AC\cos\alpha.2.AD.AC\cos\beta \equiv 4.AC^2.AB.AD.\cos\alpha\cos\beta$ $\equiv AC^2.AB.AD.\cos\alpha\cos\beta \equiv 1 \pmod 3$.

Note that $\cos\alpha, \cos\beta$ and $\cos\gamma$ are rational numbers. Moreover, if $\cos\alpha = \dfrac{p}{q}, \cos\beta = \dfrac{r}{s}$, where p, q and r, s are relatively prime then p, q, r and s are not divisible by 3. Hence $p^2 \equiv q^2 \equiv r^2 \equiv s^2 \equiv 1 \pmod 3$ and $\cos\gamma = \cos\alpha\cos\beta - \sin\alpha\sin\beta = \cos\alpha\cos\beta - \dfrac{\sqrt{q^2-p^2}}{q}\dfrac{\sqrt{s^2-r^2}}{s}$. Therefore $2.AC^2.AB.AD.\sin\alpha\sin\beta$ is divisible by 3 and after multiplying $2.AB.AD\cos\gamma \equiv 1 \pmod 3$ by AC^2 we obtain $2.AC^2.AB.AD.\cos\alpha\cos\beta \equiv 1 \pmod 3$, a contradiction to $AC^2.AB.AD.\cos\alpha\cos\beta \equiv 1 \pmod 3$. Therefore at least one of the distances is divisible by 3.

Let $n \geq 4$. Since there exist $\dbinom{n}{4}$ sets with four elements each there exist at least $\dbinom{n}{4}$ distances (counted more than once) divisible by 3. Each such distance is counted exactly $\dbinom{n-2}{2}$ times and we get that the desired number is at least $\dfrac{\binom{n}{4}}{\binom{n-2}{2}} = \dfrac{1}{6}\dbinom{n}{2}$.

Problem 11.1 Let $f(x) = \dfrac{x^2+4x+3}{x^2+7x+14}$.

a) Find the greatest value of $f(x)$;

b) Find the greatest value of the function $\left(\dfrac{x^2-5x+10}{x^2+5x+20}\right)^{f(x)}$.

Solution: a) We shall prove that the greatest value of $f(x)$ equals 2. Since $x^2+7x+14>0, \forall x$, we obtain $f(x)\leq 2 \Longrightarrow (x+5)^2\geq 0$, and an equality occurs only when $x=-5$.

b) Let $g(x)=\dfrac{x^2-5x+10}{x^2+5x+20}$. Since $x^2+5x+20>0, \forall x$, we obtain $g(x)\leq 3 \Longrightarrow (x+5)^2\geq 0$, and equality occurs only when $x=-5$. Since $x^2-5x+10>0, \forall x$, we have that the function $h(x)=g(x)^{f(x)}$ is correctly defined. Further, when $f(x)>0$, i.e. $x\in[-3,-1]$, then $h(x)\leq 3^2=9$. From the other hand $g(x)\geq 1 \Longrightarrow x\leq -1$ and so $h(x)\leq 1$ when $f(x)\leq 0$. Therefore the greatest value of $h(x)$ equals 9 and $h(x)=9$ when $x=-5$.

Problem 11.2 A point A_1 is chosen on the side BC of a triangle ABC such that the inradii of $\triangle ABA_1$ and $\triangle ACA_1$ are equal. Denote the diameters of the incircles of $\triangle ABA_1$ and $\triangle ACA_1$ by d_a. In the same manner define d_b d_c. If $BC=a, CA=b, AB=c, p=\dfrac{a+b+c}{2}$ and h_a, h_b, h_c are the altitudes of the triangle ABC and d is the diameter of the incircle of $\triangle ABC$ prove that:

a) $d_a+\dfrac{\sqrt{p(p-a)}}{a}d=h_a$;

b) $d_a+d_b+d_c+p\geq h_a+h_b+h_c$.

Solution: a) Let O_1 and O_2 be the incenters of $\triangle ABA_1$ and $\triangle ACA_1$ and p_1 and p_2 be semiperimeters of the same triangles. We have $S_{ABC}=S_{ABA_1}+S_{ACA_1}=r_a.p_1+r_a.p_2=r_c(p_1+p_2)=r_c.(p+CC_1)$, where $r_a=\dfrac{d_a}{2}$. Therefore $r_a(p+AA_1)=S_{ABC}$. If P and Q are the touching points of the incircles of $\triangle ABA_1$ and $\triangle ACA_1$ with

BC then the quadrilateral O_1PQO_2 is rectangle and so $O_1O_2 = PQ = PA_1 + QA_1 = p_1 - AB + p_2 - AC = p_1 + p_2 - AC - BC = p + AA_1 - AC - BC$. It follows from the similarity of $\triangle IO_1O_2$ and $\triangle IBC$, where I is the incenter of $\triangle ABC$ that $\dfrac{O_1O_2}{BC} = \dfrac{r - r_a}{r} \Longrightarrow \dfrac{AA_1 + p - AC - BC}{BC} = \dfrac{r - r_c}{r}$. We obtain the system

$$\left| \begin{array}{ccc} r_a(p + AA_1) & = & S \\ \dfrac{AA_1 + AB - p}{c} & = & \dfrac{r - r_a}{r} \end{array} \right.$$

Thus, $AA_1 = \sqrt{p(p-a)}$ and $r_a = \dfrac{rp}{p + \sqrt{p(p-a)}} = \dfrac{r\sqrt{p}(\sqrt{p} - \sqrt{p-a})}{a} = \dfrac{h_a}{2} - \dfrac{r\sqrt{p(p-a)}}{a}$, which implies a).

b) By a) the inequality is equivalent to $\dfrac{\sqrt{p-a}}{a} + \dfrac{\sqrt{p-b}}{b} + \dfrac{\sqrt{p-c}}{c} \leq \dfrac{\sqrt{p}}{d}$. Further $\dfrac{\sqrt{p-a}}{a} + \dfrac{\sqrt{p-b}}{b} + \dfrac{\sqrt{p-c}}{c} = \dfrac{\sqrt{p-a}}{(p-b)+(p-c)} + \dfrac{\sqrt{p-b}}{(p-a)+(p-c)} + \dfrac{\sqrt{p-c}}{(p-a)+(p-b)} \leq \dfrac{\sqrt{p-a}}{2\sqrt{p-b}\sqrt{p-c}} + \dfrac{\sqrt{p-b}}{2\sqrt{p-a}\sqrt{p-c}} + \dfrac{\sqrt{p-c}}{2\sqrt{p-a}\sqrt{p-b}} = \dfrac{p}{\sqrt{(p-a)(p-b)(p-c)}} = \dfrac{\sqrt{p}}{d}$.

Problem 11.3 See problem 10.3.

$XLIX$ National Mathematical Olympiad
Third Round, 15-16 April 2000

Problem 1. Find all value of the real parameter a such that the equation

$$9^t - 4a3^t + 4 - a^2 = 0$$

has an unique root in the interval $(0, 1)$.

Solution: After the substitutions $x = 3^t$ and $f(x) = x^2 - 4ax + 4 - a^2$ the problem is equivalent to: find all values of a such that the equation $f(x) = 0$ has an unique root in the interval $(1, 3)$. It is easily seen that if $f(1) = 0$ or $f(3) = 0$, then $a = 1, a = -5$ or $a = -13$ are not solutions of the problem. Therefore the equation $f(x) = 0$ has an unique root in the interval $(1, 3)$ when $f(1).f(3) < 0$ or $D_f = 0, 2a \in (1, 3)$. It follows now that $f(1).f(3) < 0 \iff (a-1)^2(a+5)(a+13) < 0 \iff a \in (-13, -5)$ or $5a^2 - 4 = 0, 2a \in (1, 3)$, and so $a = \dfrac{2\sqrt{5}}{5}$.

Problem 2. In $\triangle ABC, CH (H \in AB)$ is altitude and CM and $CN (M, N \in AB)$ are bisectors respectively of $\sphericalangle ACH$ and $\sphericalangle BCH$. The circumcenter of $\triangle CMN$ coincides with the incenter of $\triangle ABC$. Prove that $S_{\triangle ABC} = \dfrac{AN.BM}{2}$.

Solution: Let I be the incenter of ABC. Denote by P, Q and R the common points of the incircle of ABC respectively with AB, BC and CA. It follows from $IP = IQ = IR$ and $IC = IM = IN$ that $\triangle IMP, \triangle INP, \triangle ICQ$ and $\triangle ICR$ are congruent. Let $\sphericalangle MIP = \sphericalangle NIP = \sphericalangle QIC = \sphericalangle CIR = \delta$. We have that $\sphericalangle MCN = \dfrac{1}{2} \sphericalangle MIN = \delta$. Since CM and CN are bisectors we get $\sphericalangle ACB = 2 \sphericalangle MCN = 2\delta$. It follows now from the quadrilateral $IQCR$ that $\sphericalangle QIR+$

$\sphericalangle QCR = 180^\circ \iff 4\delta = 180^\circ \iff \delta = 45^\circ$. Therefore $\triangle ABC$ is a right angle triangle which implies $\sphericalangle BCM = \sphericalangle BMC = \sphericalangle BAC + \frac{\sphericalangle ABC}{2}$ and $\sphericalangle ANC = \sphericalangle ACN = \sphericalangle ABC + \frac{\sphericalangle BAC}{2}$. Therefore $BC = BM$ and $AC = AN$ and so $S_{ABC} = \frac{AC.BC}{2} = \frac{AN.BM}{2}$.

Problem 3. Let $\{a_n\}_{n=1}^{\infty}$ be a sequence such that $a_1 = 43, a_2 = 142, a_{n+1} = 3a_n + a_{n-1}$ for $n \geq 2$. Prove that:

a) a_n and a_{n+1} are relatively prime for all n;

b) for every natural number m there exist infinitely many natural numbers n, such that $a_n - 1$ and $a_{n+1} - 1$ both are divisible by m.

Solution: a) Suppose that there exist natural numbers n and $m > 1$, such that m divides both a_n and a_{n+1}. It follows from $a_{n-1} = a_{n+1} - 3a_n$ that m divides a_{n-1}. By induction m divides both a_1 and a_2, which is impossible since a_1 and a_2 are relatively prime.

b) Consider the sequence $\{a_n\}$ defined by $a_{n-1} = a_{n+1} - 3a_n$ for negative indices. Compute $a_0 = a_2 - 3a_1 = 13; a_{-1} = a_1 - 3a_0 = 4; a_{-2} = a_0 - 3a_1 = 1; a_{-3} = a_{-1} - 3a_{-2} = 1$ and so on. We find a sequences $\{a_n\}_{-\infty}^{\infty}$ of integers such that $a_{n+1} = 3a_n + a_{n-1}$ for every n. Since the pairs $(p(\text{mod } m), q(\text{mod } m))$ are finitely many we get that there exist integers r and $s > r$ such that $a_r \equiv a_s (\text{mod } m)$ and $a_{r+1} \equiv a_{s+1} (\text{mod } m)$. It is clear now that $a_{r+i} \equiv a_{s+i} (\text{mod } m)$ for every i. Therefore the sequence $\{a_n\}_{-\infty}^{\infty}$ is periodic. Since $a_{-3} \equiv a_{-2} \equiv 1 (\text{mod } m)$, there exist infinitely many natural numbers n such that both $a_n - 1$ and $a_{n+1} - 1$ are divisible by m.

Problem 4. Given a convex quadrilateral $ABCD$, such that $\sphericalangle BCD = \sphericalangle CDA$. The bisector of $\sphericalangle ABC$ intersects the segment CD in point E. Prove that $\sphericalangle AEB = 90^\circ$ if and only if $AB = AD + BC$.

Solution: Let $\sphericalangle AEB = 90^\circ$. Since $\sphericalangle CEB < 90^\circ$ there exists a point F on the side AB such that $\sphericalangle BEF = \sphericalangle BEC$. Thus $\triangle BCE \cong \triangle BFE$, which implies $BF = BC$ and $\sphericalangle BFE = \sphericalangle BCE$. From the other hand

$$(*) \qquad \frac{AE}{\sin \sphericalangle AFE} = \frac{AF}{\sin \sphericalangle AEF}, \frac{AE}{\sin \sphericalangle ADE} = \frac{AD}{\sin \sphericalangle AED}.$$

Since $\sphericalangle AED = \sphericalangle AEF$ and $\sphericalangle AFE + \sphericalangle ADE = 180^\circ$, we obtain $AF = AD$ and so $AB = AD + BC$.

Conversely, let $AB = AD + BC$. There exists a point F on the segment AB such that $AF = AD$ and $BF = BC$. Therefore $\triangle BFE \cong \triangle BCE$, so $\sphericalangle BFE = \sphericalangle BCE$ and $\sphericalangle BEF = \sphericalangle BEC$. It follows from $(*)$ and $AF = AD$ that $\sin \sphericalangle AED = \sin \sphericalangle AEF$. Since $\sphericalangle AED + \sphericalangle AEF < 180^\circ$, we have $\sphericalangle AED = \sphericalangle AEF$ and therefore $\sphericalangle AEB = 90^\circ$.

Problem 5. Prove that for any two real numbers a and b there exists a real number $c \in (0,1)$, such that

$$\left| ac + b + \frac{1}{c+1} \right| > \frac{1}{24}.$$

Solution: Consider $f(x) = ax + \dfrac{1}{x+1}$ and let m and M be respectively the minimum and the maximum value of the function f in the interval $[0,1]$. Since $f(0) = 1, f(1) = a + \frac{1}{2}$ and $f'(x) = a - \dfrac{1}{(x+1)^2}$ there are four cases for the difference $M - m$.

1. $a \leq \dfrac{1}{4}$. Thus $f'(x) \leq 0$ for $x \in [0,1]$ and so $M - m = f(0) - f(1) = \dfrac{1}{2} - a \geq \dfrac{1}{4}$.

2. $a \geq 1$. Thus $f'(x) \geq 0$ for $x \in [0,1]$ and so $M - m = f(1) - f(0) = a - \frac{1}{2} \geq \frac{1}{2}$.

3. $\frac{1}{4} \leq a \leq 1$. Thus $d = \frac{1}{\sqrt{a}} - 1 \in [0,1]$, $f'(x) \leq 0$ for $x \in [0,d]$ and $f'(x) \geq 0$ for $x \in [d,1]$.

3.1 $\frac{1}{4} \leq a \leq \frac{1}{2}$. Since $f(0) \geq f(1)$, we have $M - m = f(0) - f(d) = (1-\sqrt{a})^2 \geq (1 - \frac{\sqrt{2}}{2})^2$.

3.2 $\frac{1}{2} \leq a \leq 1$. Since $f(1) \geq f(0)$, we have $M - m = f(1) - f(d) = \frac{1}{2}(2\sqrt{a}-1)^2 \geq \frac{1}{2}(\sqrt{2}-1)^2$.

In all four cases $M - m > \frac{1}{12}$, which implies $M + b > \frac{1}{24}$ or $m + b < \frac{1}{24}$. The assertion of the problem follows now by continuity.

Problem 6. Find all sets S of four points in the plane such that: for any two circles k_1 and k_2, having diameters with endpoints - points from S there exists a point $A \in S \cap k_1 \cap k_2$.

Solution: Let $S = \{A, B, C, D\}$. Consider the circles k_1 and k_2 with diameters respectively AB and CD. It follows that at least one of the angles ACB, ADB, CAD and CBD is right angle. Without loss of generality $\sphericalangle ACB = 90^\circ$. Let k_3 and k_4 be circles with diameters respectively AC and BD. Since $\sphericalangle ABC < 90^\circ$, we obtain that the common point of k_3 and k_4 is one of the points A, C or D.

1. Let $A \in k_3 \cap k_4$. Now $\sphericalangle BAD = 90^\circ$ and therefore $D \in l_1$, $l_1 \perp AB, A \in l_1$. It is easily seen that $\sphericalangle BDC < 90^\circ$ and since $\sphericalangle BAC < 90^\circ, \sphericalangle ABD < 90^\circ$, it follows that the common point of

the circles with diameters AD and BC is the point C and $\sphericalangle ACD = 90^\circ$. Therefore $D = l_1 \cap BC$.

2. Let $C \in k_3 \cap k_4$. Now $\sphericalangle BCD = 90^\circ$ and therefore D lies on the line AC. Since $\sphericalangle BDC < 90^\circ, \sphericalangle BAC < 90^\circ$ and $\sphericalangle ACD < 90^\circ$, we obtain that the common point of the circles with diameters AD and BC is B and $\sphericalangle ABD = 90^\circ$. Therefore $D = l_2 \cap AC$, where $l_2 \perp AB, B \in l_2$.

3. Let $D \in k_3 \cap k_4$. Now $\sphericalangle ADC = 90^\circ$ and therefore $D \in k_3$. Since $\sphericalangle BAC < 90^\circ$ and $\sphericalangle ACD < 90^\circ$, we obtain that the common point of the circles with diameters AD and BC is B or D. In the first case $\sphericalangle ABD = 90^\circ$ and thus $D \in l_2$; from the other hand $l_2 \cap k_3 = \emptyset$, which is a contradiction. In the second case $\sphericalangle BDC = 90^\circ$, and we get that D is the orthogonal projection of C on AB.

In conclusion, all sets S, satisfying the condition of the problem are those consisting of the vertices of a right triangle and the foot of the altitude to the hypothenusis.

$XLIX$ National Mathematical Olympiad
Fourth Round, 16-17 May 2000

Problem 1. In an orthogonal coordinate system xOy a set consisiting of 2000 points $M_i(x_i, y_i)$, is called "good" if $0 \leq x_i \leq 83, 0 \leq y_i \leq 1 \quad i = 1, 2, \ldots, 2000$ and $x_i \neq x_j$ for $i \neq j$. Find all natural numbers n with the following properties: : a) For any "good" set some n of its points lie in a square of side length 1.

b) There exists a "good" set such that no $n + 1$ of its points lie in a square of side length 1.

(A point on a side of a square lies in the square).

Solution: We shall prove that $n = 25$ is the only solution of the problem. We show first that for a "good" set some 25 points lie in a square of side length 1. All points lie in the rectangle $1 \leq x \leq 83, 0 \leq y \leq 1$. Divide this rectangle to 83 squares of side length 1. If some 25 points lie in one of these rectangles then we are done. Conversely, in every square there are at least 26 or at most 24 points. We prove now that there exists a square with at least 26 points and there exists a square with at most 24 points. Indeed, if a square with 26 points does not exist then the points are at most $83.24 = 1992 < 2000$. If there is no square with less than 25 points then the points are $83.26 - 82 > 2000$. Further, move the square with more than 25 points towards the square with less that 25 points. Since the number of points in this square changes at most by one we get the assertion of the problem.

To prove b) let $x_1 = 0, x_i = x_{i-1} + \dfrac{83}{1999}$ and $y_i = 0$ for $i = 0, 2, 4, \ldots, 2000$ while $y_i = 1$ for $i = 1, 3, \ldots, 1999$. Let $XYZT$ be an unit square. WLOG we assume that it intersects the lines

$y = 0$ and $y = 1$ in the points P, Q and R, S respectively. Then

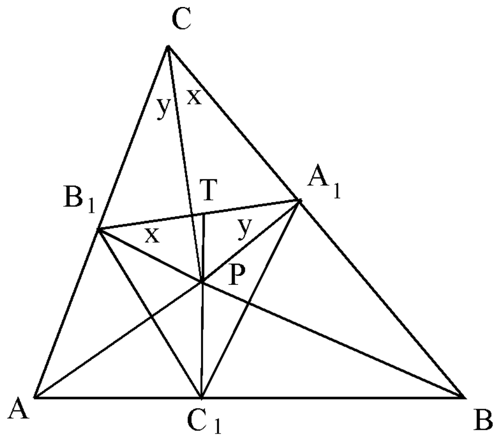
$$PQ+QR+RS+SP < PY+YQ+QZ+ZR+RT+TS+SX+XP = 4.$$

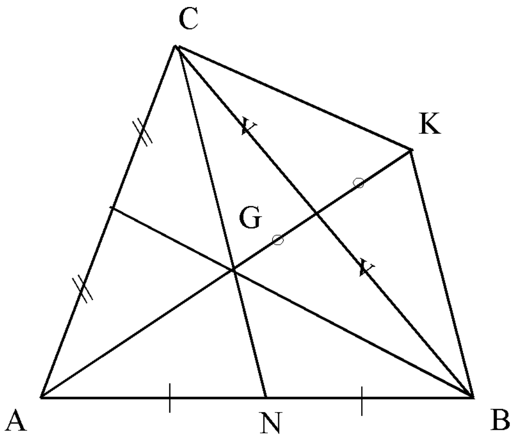
But $SP, QR \geq 1$ and we get $PQ + RS < 2$. This implies that in the square $XYZT$ there are no more than 25 points.

We found that $n = 25$ is a solution of the problem. It is unique since $24 < \dfrac{2000}{83} < 25$.

Problem 2. Given an acute $\triangle ABC$. Prove that there exist unique points A_1, B_1 and C_1 on BC, AC and AB respectively with the following property: Each of the points is the midpoint of the segment with ends the orthogonal projections of the other two points on the corresponding side. Prove that $\triangle A_1B_1C_1$ is similar to the triangle formed by the medians of $\triangle ABC$.

Solution: It is easy to be seen that the perpendicular through C_1 to AB intersects A_1B_1 at its midpoint. The same is applied for A_1 and B_1. Therefore the tree perpendiculars intersect at the medcenter of $\triangle A_1B_1C_1-$ the point P.





Let T be the midpoint of A_1B_1, $\sphericalangle PB_1A_1 = x = \sphericalangle PCA_1$; $\sphericalangle PA_1B_1 = y = \sphericalangle PCB_1$. The Law of Sines for $\triangle B_1TP$ and $\triangle A_1TP$ gives:

$$\frac{\sin\alpha}{\sin x} = \frac{B_1T}{TP} = \frac{A_1T}{AP} = \frac{\sin\beta}{\sin y},$$

where $\dfrac{\sin x}{\sin y} = \dfrac{\sin\alpha}{\sin\beta}$. Further, if CN is the median in $\triangle ABC$, similar arguments imply that $\dfrac{\sin\sphericalangle ACG}{\sin\sphericalangle BCG} = \dfrac{\sin\alpha}{\sin\beta}$. Since $x + y = \sphericalangle ACG + \sphericalangle BCG = \gamma$ and γ is acute it follows that $x = \sphericalangle ACG, y = \sphericalangle BCG$. It is clear now that CP is symmetric to the median in $\triangle ABC$ through C with respect to the bisector of angle γ. The same is true for AP and BP. Therefore the point P is unique (and therefore A_1, B_1 and C_1 are unique). Further, $\sphericalangle B_1C_1A_1 = \sphericalangle B_1C_1P + \sphericalangle A_1C_1P = \sphericalangle B_1AP + \sphericalangle A_1BP = \sphericalangle BAG + \sphericalangle ABG = \sphericalangle BGK$, where K is the symmetric point of G with respect to the midpoint of BC. By analogy $\sphericalangle C_1A_1B_1 = \sphericalangle GBK$ and $\sphericalangle A_1B_1C_1 = \sphericalangle GKB$. Therefore $\triangle A_1B_1C_1$ is similar to the triangle formed by the medians of $\triangle A_1B_1C_1$.

Problem 3. Let $p \geq 3$ be a prime number and $a_1, a_2, \dots, a_{p-2}$ be a sequence of natural numbers such that p does not divide both a_k and $a_k^k - 1$ for all $k = 1, 2, \dots, p-2$. Prove that the product of some elements of the sequence is congruent to 2 modulo p.

Solution: Consider the sequence $1, a_1, a_2, \dots, a_{p-2}$. We shall prove by induction that for any $i = 2, 3, \dots, p-1$ there exist integers $b_1, b_2, \dots, b_i$ each of which is a product of some elements from the above sequence and $b_m \not\equiv b_n (\text{mod } p)$ for $m \neq n$. Indeed, for $i = 2$ we can choose $b_1 = 1, b_2 = a_1 (a_1 \not\equiv 1 (\text{mod } p))$. Suppose we have chosen $b_1, b_2, \dots, b_i$ such that $b_m \not\equiv b_n (\text{mod } p)$ for $m \neq n$. Consider $b_1 a_i, b_2 a_i, \dots, b_i a_i$. It is easily seen that any two of them are not congruent modulo p. Further, if $b_j a_i$ for any j is congruent to b_l for some l we get that $b_1 a_i, b_2 a_i, \dots, b_i a_i$ modulo p is a permutation of $b_1, b_2, \dots, b_i$ modulo p. Thus,

$$(b_1 a_i).(b_2 a_i).\dots.(b_i a_i) \equiv b_1.b_2.\dots.b_i (\text{mod } p)$$

and therefore $a_i^i \equiv 1 (\text{mod } p)-$ a contradiction.

It follows now that for any $s = 2, 3, \dots, p-1$ there exist few elements of the sequence from the condition of the problem such that their product is congruent to s modulo p.

Problem 4. Find all polynomials $P(x)$ with real coeficients such that

$$P(x).P(x+1) = P(x^2)$$

for any real x.

Solution: We show first that for any natural number n there exists at most one polynomial $P(x)$ of degree n such that $P(x).P(x+1) =$

$P(x^2)$. Indeed, if

$$P(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0,$$

comparing the coefficients in front of the same degrees of x in $P(x).P(x+1)$ and $P(x^2)$, we get $a_n = 1$ and $a_{n-1} = -\frac{n}{2}$. It is easily seen that each of the subsequent coefficients is a solution of an equation of first degree of the form $2x + b = 0$, where b is a function of already chosen coefficients. Therefore, if a polynomial of degree n with the desired property exists it is unique one.

We prove now that a polynomial $P(x)$ of odd degree such that $P(x).P(x+1) = P(x^2)$ does not exist. Let $P(x) = x^m(x-1)^n.Q(x)$ (m or n could be zero), where $Q(x)$ is a polynomial such that $Q(0) \neq 0$ and $Q(1) \neq 0$. A substitution gives

$$(x+1)^m.x^n.Q(x).Q(x+1) = x^m.(x+1)^n.Q(x^2).$$

If $m \neq n$ the substitution $x = 0$ leeds to a contradiction. Therefore $m = n$ and so $Q(x).Q(x+1) = Q(x^2)$, where $Q(x)$ is of odd degree. Therefore there exists a real number $x_0 \neq 0, 1$, such that $Q(x_0) = 0$. It is clear that $x_0 = -1$ since otherwise the substitution $x = -1$ shows that $Q(1) = 0$ which is impossible. A substitution $x = x_0$ gives that $Q(x_0^2) = 0$, thus $Q(x_0^4) = 0$ and so on. Since $x_0 \neq \pm 1, 0$ it follows that there are infinitely many distinct terms in the sequence $x_0, x_0^2, \ldots, x_0^{2^k}$, which is impossible.

Direct verification shows that for any even natural number $n = 2k$, the polynomial $P(x) = x^k(x-1)^k$ is of degree n and is a solution of the problem.

Therefore all polynomials such that $P(x).P(x+1) = P(x^2)$ are $P(x) = x^k(x-1)^k$.

Problem 5. Let D be the midpoint of the base AB of an isosceles acute $\triangle ABC$. A point E is chosen on AB and O is the circumcenter of $\triangle ACE$. Prove that the line through D perpendicular to DO, the line through E, perpendicular to BC and the line through B, parallel to AC intersect in a point.

Solution: Let $\sphericalangle ABC = \sphericalangle BAC = \alpha$ and let G be the circumcenter of $\triangle ABC$. Consider points F' and F'' on the line through B parallel to AC such that $OD \perp DF'$ and $BC \perp EF''$. Denote by H' and H'' respectively the projection points of F' and F'' on the line AB. Since O is inner point for $\sphericalangle ADC$ and $\sphericalangle ACB < 90°$, we have that F' and F'' lie in the interior of $\sphericalangle BAC$. It suffices to prove that $F' \equiv F''$, i.e. $F'H' = F''H''$. Let O' and G' be respectively the projection points of O on AB and of G on OO'. Since $\triangle DH'F' \sim \triangle OO'D$, it follows that $\dfrac{DH'}{F'H'} = \dfrac{OO'}{DO'}$. Also, $\sphericalangle GOG' = \alpha$ and so $\dfrac{B'H'}{F'H'} = \dfrac{OG'}{GG'}$. It follows from $GG' = DO', O'G' = DG$ and $\sphericalangle DBG = 2\alpha - 90°$ that $F'H' = \dfrac{BD.O'D'}{GD} = -\text{tg}2\alpha.O'D$. Denote $BC \cap EF'' = I$. Since $\sphericalangle CBF'' = 180° - 2\alpha$ and $BE = 2O'D$, it is clear that $F''H'' = BF'' \sin\alpha = \dfrac{BI \sin\alpha}{\cos(180° - 2\alpha)} = -BE\dfrac{\sin\alpha\cos\alpha}{\cos 2\alpha} = -O'D\text{tg}2\alpha = F'H'$. This completes the prove.

Problem 6. Let $\mathcal{A}$ be the set of all binary sequences of length n and let $\mathbf{0} \in \mathcal{A}$ be the sequence with zero elements. The sequence $c = c_1, c_2, \ldots, c_n$ is called sum of $a = a_1, a_2, \ldots, a_n$ and $b = b_1, b_2, \ldots, b_n$ if $c_i = 0$ when $a_i = b_i$ and $c_i = 1$ when $a_i \neq b_i$. Let $f : \mathcal{A} \to \mathcal{A}$ be a function such that $f(\mathbf{0}) = \mathbf{0}$ and if the sequences a and b differ in exactly k terms then the sequences $f(a)$ and $f(b)$ differ also exactly in k terms. Prove that if a, b and c are sequences from $\mathcal{A}$ such that $a + b + c = \mathbf{0}$, then $f(a) + f(b) + f(c) = \mathbf{0}$.

Solution: Consider the sequence $e_1 = 1, 0, \ldots, 0, 0$; $e_2 = 0, 1, \ldots, 0$; $\ldots, e_{n-1} = 0, 0, \ldots, 1, 0$; $e_n = 0, 0, \ldots, 0, 1$. It follows from the condition of the problem that (since $f(\mathbf{0}) = \mathbf{0}$) for all $p, 1 \leq p \leq n$ there exists $q, 1 \leq q \leq n$, such that $f(e_p) = e_q$.

It is clear also that $f(e_p) \neq f(e_q)$ for $1 \leq p, q \leq n; p \neq q$. Therefore

$$(1) \qquad \{f(e_1), f(e_2), \dots, f(e_n)\} = \{e_1, e_2, \dots, e_n\}.$$

Consider an arbitrary sequence $a = a_1, a_2, \dots, a_n$ with t ones. If $f(e_p) = e_q$, and $a_p = 1$, then the $q-$th term of the sequence $f(a)$ is also 1 (otherwise e_p and a differ at $t-1$ terms whereas $f(e_p) = e_q$ and $f(a)$ differ at $t+1$ terms). By analogy if $a_p = 0$, then the $q-$th term of $f(a)$ is also 0.

Finaly, consider the sequences $a = a_1, a_2, \dots, a_n; b = b_1, b_2, \dots, b_n$ and $c = c_1, c_2, \dots, c_n$ such that $a + b + c = \mathbf{0}$. This means that for every $i, 1 \leq i \leq n$ the sum $a_i + b_i + c_i$ is even number. Fix $i, 1 \leq i \leq n$ and let $f(e_i) = e_j$. It follows now that the $j-$th terms of the sequences $f(a), f(b), f(c)$ coincide with a_i, b_i, c_i respectively and using (1) we obtain that $f(a) + f(b) + f(c) = \mathbf{0}$.