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Problem 9.1. a) Draw all points in the plane with coordinates $(x;y)$ such that

$$(|3x-y|-3)(|3x+y|-3)=0;$$

b) Find all x and y for which

$$\left|\begin{array}{rcl} (|3x-y|-3)(|3x+y|-3) & = & 0 \\ y-\{4x\} & = & 0 \\ -1\le x\le 1 & & \end{array}\right.$$

(For a real number x we denote the unique number in the interval $[0,1)$ for which $x-\{x\}$ is an integer by $\{x\}$).

Solution: a) It is easy to see that these are the points on the four lines $l_1: 3x-y=3$, $l_2: 3x-y=-3$, $l_3: 3x+y=3$ and $l_4: 3x+y=-3$.

b) There are two types of solutions:

1. Solutions of $\left|\begin{array}{ccc} y & = & -3x+3 \\ y & = & \{4x\} \\ -1 & \le x \le & 1 \end{array}\right.$

2. Solutions of $\left|\begin{array}{ccc} y & = & 3x+3 \\ y & = & \{4x\} \\ -1 & \le x \le & 1 \end{array}\right.$

Denote the integer part of $4x$ by $[4x]$, i.e. $4x=[4x]+\{4x\}$. In the first case we have $-3x+3=\{4x\}=4x-[4x]$, i.e $7x=3+[4x]$.

Since the right hand side is an integer all posiible values of x are $0, \pm\frac{1}{7}, \pm\frac{2}{7}, \ldots, \pm 1$. Direct verification shows that only $x = \frac{5}{7}, y = \frac{6}{7}$; $x = \frac{6}{7}, y = \frac{3}{7}$; $x = 1, y = 0$ are solutions.

In the second case $3x + 3 = \{4x\} = 4x - [4x]$, i.e. $x = 3 + [4x]$. It follows that x is an integer, i.e. $x = -1, 0$ or 1. Direct verification shows that solution is only $x = -1, y = 0$.

Thus, there are four solutions:

$$x = \frac{5}{7}, y = \frac{6}{7}; x = \frac{6}{7}, y = \frac{3}{7}; x = 1, y = 0; x = -1, y = 0.$$

Problem 9.2. Points A_1, B_1 and C_1 are chosen on the sides BC, CA and AB of a triangle ABC. Point G is the centroid of $\triangle ABC$, and G_a, G_b and G_c are centroids of $\triangle AB_1C_1, \triangle BA_1C_1$ and $\triangle CA_1B_1$ respectively. The centroids of $\triangle A_1B_1C_1$ and $\triangle G_aG_bG_c$ are denoted by G_1 and G_2 respectively. Prove that:

a) the points G, G_1 and G_2 lie on a straight line;

b) lines AG_a, BG_b and CG_c intersect in a point if and only if AA_1, BB_1 and CC_1 intersect in a point.

Solution: a) Let O be an arbitrary point and

$$\vec{OA_1} = \alpha\vec{OB} + (1-\alpha)\vec{OC}; \vec{OB_1} = \beta\vec{OA} + (1-\beta)\vec{OC};$$
$$\vec{OC_1} = \gamma\vec{OA} + (1-\gamma)\vec{OB},$$

where $\alpha, \beta, \gamma \in (0,1)$. The existence of α, β and γ follows from the fact that A_1, B_1 and C_1 lie on the sides BC, CA and AB of $\triangle ABC$. Then we have

$$\vec{OG} = \frac{1}{3}\left(\vec{OA} + \vec{OB} + \vec{OC}\right) = \frac{1}{9}\left(3\vec{OA} + 3\vec{OB} + 3\vec{OC}\right);$$
$$\vec{OG_1} = \tfrac{1}{3}\left(\vec{OA_1} + \vec{OB_1} + \vec{OC_1}\right) =$$

$\frac{1}{3}\left[(\beta+\gamma)\vec{OA}+(\alpha-\gamma+1)\vec{OB}+(2-\alpha-\beta)\vec{OC}\right].$

By analogy $\vec{OG_a}=\frac{1}{3}\left[(1+\beta+\gamma)\vec{OA}+(1-\gamma)\vec{OB}+(1-\beta)\vec{OC}\right];$

$\vec{OG_b}=\frac{1}{3}\left[\gamma\vec{OA}+(2+\alpha-\gamma)\vec{OB}+(1-\alpha)\vec{OC}\right];$

$\vec{OG_c}=\frac{1}{3}\left[\beta\vec{OA}+\alpha\vec{OB}+(3-\alpha-\beta)\vec{OC}\right].$

Thus, $\vec{OG_2}=\frac{1}{3}\left(\vec{OA_1}+\vec{OB_1}+\vec{OC_1}\right)=$

$\frac{1}{9}\left[(1+2\beta+2\gamma)\vec{OA}+(3+2\alpha-2\gamma)\vec{OB}+(5-2\alpha-2\beta)\vec{OC}\right].$

Since $\vec{GG_1}=\vec{OG_1}-\vec{OG}$ we have

$$\vec{GG_1}=\frac{1}{3}\left[(\beta+\gamma-1)\vec{OA}+(\alpha-\gamma)\vec{OB}+(1-\alpha-\beta)\vec{OC}\right].$$

Using the same arguments we obtain $\vec{GG_2}=\vec{OG_2}-\vec{OG}=$ $\frac{1}{9}\left[(2\beta+2\gamma-2)\vec{OA}+(2\alpha-2\gamma)\vec{OB}+(2-2\alpha-2\beta)\vec{OC}\right]$. It follows from the last two equalities that $\vec{GG_1}=\frac{3}{2}\vec{GG_2}$ and we are done.

b) Since the lines AA_1, BB_1 and CC_1 intersect in a point Ceva's theorem gives $\frac{AC_1}{C_1B}\cdot\frac{BA_1}{A_1C}\cdot\frac{CB_1}{B_1A}=1.$

Denote the intersecting points of AG_a, BG_b and CG_c with the sides BC, CA and AB by A_2, B_2 and C_2 respectively. A necessary and sufficient condition for lines AG_a, BG_b and CG_c to intersect in a point is $\frac{CA_2}{A_2B}\cdot\frac{BC_2}{C_2A}\cdot\frac{AB_2}{B_2C}=1.$

Denote the midpoint of B_1C_1 by A_3. Let h_1 and h_2 be the altitudes of $\triangle AA_2C$ and AA_3B_1 from A_2 and A_3. We have

$$\frac{S_{AA_2C}}{S_{AA_3B_1}} = \frac{h_1.AC}{h_2.AB_1} = \frac{AA_2.AC}{AA_3.AB_1}.$$

By analogy

$$\frac{S_{AA_2B}}{S_{AA_3C_1}} = \frac{AA_2.AB}{AA_3.AC_1}.$$

Dividing the above equalities and using that $S_{AA_3C_1} = S_{AA_3B_1}$ and $\dfrac{S_{AA_2C}}{S_{AA_2B}} = \dfrac{CA_2}{BA_2}$ we obtain

$$\frac{CA_2}{A_2B} = \frac{AC}{AB} \cdot \frac{AC_1}{AB_1}.$$

By analogy

$$\frac{BC_2}{C_2A} = \frac{CB}{CA} \cdot \frac{CB_1}{CA_1}, \frac{AB_2}{B_2C} = \frac{BA}{BC} \cdot \frac{BA_1}{BC_1}.$$

Multiplying the above equalities gives

$$\frac{CA_2}{A_2B}\frac{BC_2}{C_2A}\frac{AB_2}{B_2C} = \frac{AC}{AB}\frac{AC_1}{AB_1} \cdot \frac{BC}{AC}\frac{CB_1}{CA_1} \cdot \frac{BA}{BC}\frac{BA_1}{BC_1} = \frac{AC_1}{C_1B}\frac{BA_1}{A_1C}\frac{CB_1}{B_1A}.$$

Therefore the lines AG_a, BG_b and CG_c intersect in a point iff the lines AA_1, BB_1 and CC_1 interect in a point.

Problem 9.3. Let A_n be the number of sequences from 0's and 1's of length n, such that no four consequtive elements equal 0101. Find the parity of A_{2001}.

Solution: Denote the number of sequences of length n, the last three terms of which are ijk, where $i, j, k \in \{0, 1\}$ and no four consequtive elements equal to 0101 by a^n_{ijk}.

Obviously $a_{ijk}^{n+1} = a_{0ij}^n + a_{1ij}^n$ when $ijk \neq 101$ and $a_{101}^{n+1} = a_{110}^n$. Adding the above equalities for all values of ijk we obtain
$A_{n+1} = 2A_n - a_{010}^n$.

Therefore $A_{n+1} \equiv a_{010}^n \equiv a_{001}^{n-1} + a_{101}^{n-1} \equiv a_{000}^{n-2} + a_{100}^{n-2} + a_{110}^{n-2} \equiv a_{000}^{n-3} + a_{100}^{n-3} \equiv a_{000}^{n-4} + a_{100}^{n-4} + a_{010}^{n-4} + a_{110}^{n-4} \equiv a_{000}^{n-5} + a_{001}^{n-5} + a_{010}^{n-5} + a_{100}^{n-5} + a_{111}^{n-5} + a_{110}^{n-5} + a_{101}^{n-5} + a_{011}^{n-5} + \equiv A_{n-5} \pmod 2$.

Thus, $A_{k+6m} \equiv A_k \pmod 2$ for all k and m.

Therefore $A_{2001} \equiv A_3 = 8 \equiv 0 \pmod 2$.

Problem 10a.1. Find all pairs $(a; b)$ of integers such that the system

$$\left| \begin{array}{rcl} x^2 + 2ax - 3a - 1 & = & 0 \\ y^2 - 2by + x & = & 0 \end{array} \right.$$

has exactly three real solutions.

Solution: Let $(a; b)$ satisfy the condition of the problem. Since there are exactly three real solutions the first equation has two distinct roots $x_1 < x_2$, which implies $D_1 = 4(a^2 + 3a + 1) > 0$. The discriminant of the second equation in respect to y equals $D_2 = 4(b^2 - x)$ and therefore there are exactly three real solutions iff $x_1 < x_2 = b^2$.

These conditions are satisfied iff a and b are integers such that $a^2 + 3a + 1 > 0$ and $b^2 = -a + \sqrt{a^2 + 3a + 1}$. It follows from the latter equality that $a^2 + 3a + 1 = c^2$, where c is a positive integer. Therefore the discriminant of $a^2 + 3a + 1 - c^2 = 0$ is a perfect square, i.e. $9 - 4(1 - c^2) = d^2$, where d is nonnegative integer. The last equality can be written in the form $(d - 2c)(d + 2c) = 5$ which implies that $d - 2c = 1, d + 2c = 5$, i.e. $d = 3, c = 1$. Hence, $a^2 + 3a + 1 = 1$ with roots $a = 0$ and $a = -3$. Respectively $b = \pm 1$ or $b = \pm 2$. Direct verification shows that all pairs $(a, b) = (0, 1), (0, -1), (-3, 2)$ and $(-3, -2)$ satisfy the condition of the problem.

Problem 10a.2. The tangential point of a circle k through the vertex C of a $\triangle ABC$ and the line AB is the vertex B. The circle k intersects for a second time the side AC snd the median of $\triangle ABC$ through C at points D and E respectively. Prove that if the intersecting point of the tangents to k through C and E lies on the line BD then $\sphericalangle ABC = 90^\circ$.

Solution: Let F be the intersecting point of the tangents t_C and t_E to k at C and $E, G = BD \cap CE$ and $ACBH$ is parallelogram. We have that $\sphericalangle ABC = 90^\circ \iff t_C \parallel AB \iff \frac{FD}{FB} = \frac{CD}{CA}$. From the other hand $\frac{CD}{CA} = \frac{CD}{BH} = \frac{GD}{GB}$, i.e. it suffices to show that $\frac{FD}{FB} = \frac{GD}{GB}(*)$. Since $\triangle FBE \sim \triangle FED$ and $\triangle FBC \sim \triangle FCD$, we obtain $\frac{FB}{FE} = \frac{BE}{ED}$ and $\frac{FB}{FC} = \frac{BC}{CD}$. Further, $FC^2 = FE^2 = FB.FD$ and so $\frac{FD}{FB} = \frac{CD.ED}{CB.EB}$. Now the equality from $(*)$ follows from $\frac{CD.ED}{CB.EB} = \frac{S_{CED}}{S_{CEB}} = \frac{GD}{GB}$, which completes the proof.

Problem 10a.3. Ivan and Peter alternatively write down 0 or 1 until each of them has written 2001 digits. Peter is a winner if the number, which binary representation has been obtained, cannot be expressed as a sum of two perfect squares. Prove that Peter has a winning strategy.

Solution: First we prove that if the binary representation of a positive integer ends with two ones and even number of zeroes then this integer cannot be represented as sum of two squares. Indeed, such a number is of the form $4^k(4s + 3)$ and if we suppose that $x^2 + y^2 = 4^k(4s + 3)$, then using the fact that $x^2 + y^2$ equals 0 modulo 4 iff x and y are even we get that there exist integers p, q for which $p^2 + q^2 = 4s + 3$, which is a contradiction.

The winning strategy of Peter could be:

If one of Ivan's digits is 1 then Peter simply repeats all digits written by Ivan. The final number is of the form $4^k(4s+3)$ and cannot be written as x^2+y^2. If all Ivan's digits are zeroes then the first three digits of Peter are $1,1,1$ after which he writes only zeroes. The final number is $(0101010\ldots0)_2 = 21.4^{1998}$, and cannot be represented as p^2+q^2 since 21 cannot be written in that form.

Problem 10b.1. Find all values of the real parameter a such that the equation

$$\log_x (x^2+x+a)^2 = 4$$

has unique solution.

Solution: The equation is equivalent to $(x^2+x+a)^2 = x^4, x > 0, x \neq 1$. Further, $(x^2+x+a)^2 = x^4 \iff (x+a)(2x^2+x+a) = 0$ with roots $x_1 = -a$ and $x_{2/3} = \dfrac{-1 \pm \sqrt{1-8a}}{4}$, provided $1-8a \geq 0$. Since $\dfrac{-1-\sqrt{1-8a}}{4} < 0$, we obtain that only $x_1 = -a, x_2 = \dfrac{-1+\sqrt{1-8a}}{4}$ can be roots of the equation from the problem. If $a \geq 0$ then $x_1 \leq 0$ and $x_2 \leq 0$, which implies that the equation has no roots. If $a < 0$ then $x_1 > 0$ and $x_2 > 0$. Note that $x_1 = x_2$ implies $a = 0$ which is a contradiction. Therefore if the equation has unique solution then it is necessary to have $a < 0$ and one of the two roots equals 1. Thus, $x_1 = 1 \Rightarrow a = -1$ and $x_2 = 1 \Rightarrow a = -3$.

Problem 10b.2. On each side of a right isosceles triangle with legs of length 1 is chosen a point such that the triangle formed from these three points is a right triangle. What is the least value of the hypotenuse of this triangle?

Solution: Consider a right isosceles triangle ABC with right angle at C. Let $A_1 \in BC, B_1 \in CA$ and $C_1 \in AB$ be such that $\triangle A_1B_1C_1$ is right triangle.

1. Suppose $\sphericalangle B_1A_1C_1 = 90°$. Assume that the circle k with diameter B_1C_1 intersects BC at point $X \neq A_1$. It is easy to be seen that X is an interior point for the line segment BC. Draw a tangent l to k which is parallel to BC and the tangential point Y belongs to the smaller of the arcs $\overset{\frown}{A_1X}$. Denote the intersecting points of l with AB and AC by C_2 and B_2 respectively. Consider a homothety of center A such that the image of $B_2(C_2)$ is $C(B)$. It is clear that the image of $\triangle B_1C_1Y$ is inscribed in $\triangle ABC$ and its hypotenuse is less than B_1C_1. Therefore wlog we may assume that BC is tangent to k. Thus, if $\sphericalangle C_1B_1A_1 = \alpha$, then $\sphericalangle C_1A_1B = \alpha$. From the Sine Low for $\triangle BC_1A_1$ we obtain $BA_1 = \dfrac{B_1C_1 \sin\alpha \sin(\alpha + 45°)}{\sin 45°} =$ $B_1C_1 \sin\alpha(\sin\alpha + \cos\alpha)$. Hence

(1) $1 = BA_1 + A_1C = B_1C_1 \cos\alpha \sin\alpha + B_1C_1 \sin\alpha(\sin\alpha + \cos\alpha)$

Therefore $1 = B_1C_1(\sin^2\alpha + \sin 2\alpha) \iff B_1C_1(1 + 2\sin 2\alpha - \cos 2\alpha) = 2$. From the other hand $2\sin 2\alpha - \cos 2\alpha =$ $\sqrt{5}\left(\dfrac{2}{\sqrt{5}}\sin 2\alpha - \dfrac{1}{\sqrt{5}}\cos 2\alpha\right) = \sqrt{5}\sin(2\alpha - \delta) \leq \sqrt{5}$, where δ is such that $\sin\delta = \dfrac{1}{\sqrt{5}}$ and $\cos\delta = \dfrac{2}{\sqrt{5}}$. Note that the equality holds when $2\alpha - \delta = 90°$ and since $\delta < 90°$ we have that $\alpha < 90°$.

Further $B_1C_1(1 + \sqrt{5}) \geq B_1C_1(1 + 2\sin 2\alpha - \cos 2\alpha) = 2$ and so $B_1C_1 \geq \dfrac{\sqrt{5} - 1}{2}$. It is easily seen that if $B_1C_1 = \dfrac{\sqrt{5} - 1}{2}$ and $2\alpha - \delta = 90°$ then (1) holds true and therefore there exists triangle with $B_1C_1 = \dfrac{\sqrt{5} - 1}{2}$.

2. Suppose $\sphericalangle A_1C_1B_1 = 90°$. Denote the projection of A_1 and

B_1 on AB by A_2 and B_2 respectively. Obviously $AB_2 = B_1B_2$ and $BA_2 = A_1A_2$. Let P be the midpoint of A_1B_1 and Q – the midpoint of A_2B_2. Thus, $A_1B_1 = 2PC_1 \geq 2PQ = A_1A_2 + B_1B_2$ and $A_1B_1 \geq A_2B_2 = AB - AB_2 - BA_2 = AB - B_1B_2 - A_1A_2$. Adding the above equalities gives $2A_1B_1 \geq AB = \sqrt{2}$ and so $A_1B_1 \geq \dfrac{\sqrt{2}}{2}$.

Since $\dfrac{\sqrt{2}}{2} > \dfrac{\sqrt{5}-1}{2}$ we obtain that the least possible value equals $\dfrac{\sqrt{5}-1}{2}$.

Problem 10b.3. An element x is chosen from the set $A = \{1, 2, \ldots, 2^n\}, n \geq 3$. Questions of the type: Does x belong to $B \subset A$ where the sum of the elements of B equals $2^{n-2}(2^n + 1)$ are allowed? Prove that one can find x with exactly n questions stated in advance.

Solution:

Lemma There exist n sets each with 2^{n-1} elements, the sum of the elements of each set equals $2^{n-2}(2^n + 1)$ with the following property: the elements from the set $\{1, 2, \ldots, 2^n\}$ get as answers distinct $n-$tuples from "yes" and "no".

Proof: Induction by $n \geq 3$. For $n = 3$ the sum of the elements of B is 18 and we use the sets $B_1 = \{1, 2, 7, 8\}$, $B_2 = \{1, 3, 6, 8\}$ and $B_3 = \{1, 4, 6, 7\}$. The table shows that the elements from A get as answers distinct triples.

	1	2	3	4	5	6	7	8
$B_1 = \{1, 2, 7, 8\}$	+	+	-	-	-	-	+	+
$B_2 = \{1, 3, 6, 8\}$	+	-	+	-	-	+	-	+
$B_3 = \{1, 4, 6, 7\}$	+	-	-	+	-	+	+	-

$+$ means "yes", $-$ means "no".

The number of elements in each set is $2^2 = 2^{3-1}$.

Suppose the assertion is true for some m. Therefore there exist sets $B_1, B_2, \ldots, B_m$ each with 2^{m-1} elements, the sum of the elements of each set equals $2^{m-2}(2^m+1)$ and every element from $\{1, 2, \ldots, 2^m\}$ gets distinct $m-$tuple.

Consider the set $\{1, 2, \ldots, 2^{m+1}\}$. For any $i, 1 \leq i \leq m$ if $B_i = \{a_{1i}, a_{2i}, \ldots, a_{2^{m-1}i}\}$, let

$$D_i = \{a_{1i}, a_{2i}, \ldots, a_{2^{m-1}i}, a_{1i} + 2^m, a_{2i} + 2^m, \ldots, a_{2^{m-1}i} + 2^m\}.$$

It is clear that each set $D_i, 1 \leq i \leq m$ has exactly 2^m elements and the sum of the elements of D_i is equal to: $2(a_{1i} + a_{2i} + \ldots + a_{2^{m-1}i}) + 2^{m-1}.2^m = 2.2^{m-2}(2^m+1) + 2^{2m-1} = 2^{2m-1} + 2^{m-1} + 2^{2m-1} = 2^{m-1}(2^{m+1}+1)$. It is easily seen that only the elements t and $t + 2^m$ for $1 \leq t \leq 2^m$ get equal $m-$tuples. Let P and Q be nonintersecting sets such that $|P| = |Q| = 2^{m-1}$ and $P \cup Q = \{1, 2, \ldots, 2^m\}$. Consider a set $\bar{Q}$ obtained from Q by adding 2^m to each of its elements. The set $D_{m+1} = P \cup \bar{Q}$ has 2^m elements and the sum of its elements is equal to $\sum_{s \in P} s + \sum_{s \in Q} s + 2^{m-1}.2^m = 2^{m-1}(2^m + 1) + 2^{2m-1} = 2^{m-1}(2^{m+1} + 1)$. It is clear that exactly one element of each pair $(t, t + 2^m)$ for $1 \leq t \leq 2^m$ belongs to D_{m+1}. Hence, we have found the desired sets $D_1, D_2, \ldots, D_{m+1}$.

It is obvious now that the sets, given by the Lemma solve the problem.

Problem 11.1. A sequence $a_1, a_2, \ldots, a_n, \ldots$ is defined by

$$a_1 = k; a_2 = 5k - 2 \text{ and } a_{n+2} = 3a_{n+1} - 2a_n, n \geq 1,$$

where k is a real number.

a) Find all values of k, such that the sequence $\{a_n\}_{n=1}^{\infty}$ is convergent.

b) Prove that if $k = 1$ then

$$a_{n+2} = \left[\frac{7a_{n+1}^2 - 8a_n a_{n+1}}{1 + a_n + a_{n+1}}\right], n \geq 1,$$

where $[x]$ denotes the integer part of x.

Solution: a) Write the given reccurent relation in the form $a_{n+2} - a_{n+1} = 2(a_{n+1} - a_n)$ and consider the sequence $c_n = a_{n+1} - a_n$. It is obvious that if the sequence $\{a_n\}_{n=1}^{\infty}$ is convergent then the sequence $\{c_n\}_{n=1}^{\infty}$ is also convergent. From the other hand it follows from $c_1 = a_2 - a_1 = 4k - 2$ and $c_{n+1} = 2c_n$ that the sequence $\{c_n\}_{n=1}^{\infty}$ is an arithmetic progression and $c_n = (4k - 2).2^{n-1}$ for any positive integer n. Thus, if $4k - 2 \neq 0$ then the sequence $\{c_n\}_{n=1}^{\infty}$ is unbounded and therefore is not convergent one. Therefore $4k - 2 = 0$, i.e. $k = \frac{1}{2}$. For this value of k all terms of the sequence $\{a_n\}_{n=1}^{\infty}$ are equal to $\frac{1}{2}$ and therefore the sequence is convergent.

b) It easily follows by induction that $a_n = 2^n - 1$. From the other hand the equality from the condition of the problem is satisfied for $n = 1, 2$. Suppose it is true for $n = m$ and $n = m + 1$. Then

$$\left[\frac{7a_{m+1}^2 - 8a_m a_{m+1}}{1 + a_m + a_{m+1}}\right] = \left[\frac{7(2^{m+1} - 1)^2 - 8(2^m - 1)(2^{m+1} - 1)}{1 + 2^m - 1 + 2^{m+1} - 1}\right] =$$

$$\left[\frac{3.2^{2m+2} - 2^{m+2}}{3.2^m - 1} + \frac{-1}{3.2^m - 1}\right] = 2^{m+2} - 1 = a_{m+1},$$

which completes the proof.

Problem 11.2. On each side of a triangle with angles $30°, 60°$ and $90°$ and hypotenuse 1 is chosen a point such that the triangle formed from these three points is a right triangle. What is the least value of the hypotenuse of this triangle?

Solution: Consider triangle ABC with angles α, β and γ. Let $A_1 \in BC, B_1 \in CA$ and $C_1 \in AB$ be such that $\triangle A_1B_1C_1$ is right with $\sphericalangle A_1C_1B_1 = 90°$. Suppose that the circle k of diameter A_1B_1 intersects AB at point $X \neq C_1$. It is easy to see that if $\alpha \leq 90°$ and $\beta \leq 90°$ then X is an interior point for line segment AB. Draw a tangent l to k which is parallel to AB and the tangential point Y belongs to the smaller of the arcs $\overset{\frown}{C_1X}$. Denote the intersecting points of l and lines AC and BC by B_2 and A_2 respectively. Consider homothety of center C for which the image of $B_2(A_2)$ is the point $A(B)$. It is obvious that the image of $\triangle B_1A_1Y$ is inscribed in $\triangle ABC$ and its hypotenuse is less than B_1A_1. Therefore wlog we may suppose that AB is tangent to k. In this case if $\sphericalangle B_1A_1C_1 = \delta$ then $\sphericalangle B_1C_1A = \delta$.

The Sine Low for $\triangle AC_1B_1$ and $\triangle BC_1A_1$ gives

$$AC_1 = \frac{B_1A_1 \sin\delta \sin(\alpha+\delta)}{\sin\alpha}; BC_1 = \frac{B_1A_1 \cos\delta \sin(90° - \delta + \beta)}{\sin\beta}.$$

Since $AB = AC_1 + C_1B$ we obtain

$$(1) \qquad AB = \frac{B_1A_1 \sin\delta \sin(\alpha+\delta)}{\sin\alpha} + \frac{B_1A_1 \cos\delta \sin(90° - \delta + \beta)}{\sin\beta}$$

This equality is equivalent to

$$2AB = A_1B_1(\text{cotg}\alpha + \text{cotg}\beta + 2\sin 2\delta - (\text{cotg}\alpha - \text{cotg}\beta)\cos 2\delta).$$

From the other hand

$$2\sin 2\delta - (\text{cotg}\alpha - \text{cotg}\beta)\cos 2\delta =$$

$$\sqrt{4 + (\text{cotg}\alpha - \text{cotg}\beta)^2}\sin(2\delta - \phi) \leq \sqrt{4 + (\text{cotg}\alpha - \text{cotg}\beta)^2},$$

where

$$\cos\phi = \frac{2}{\sqrt{4 + (\text{cotg}\alpha - \text{cotg}\beta)^2}}, \sin\phi = \frac{\text{cotg}\alpha - \text{cotg}\beta}{\sqrt{4 + (\text{cotg}\alpha - \text{cotg}\beta)^2}}.$$

It is easily seen that such an angle δ always exists.

Finally, the least value is

$$B_1A_1 = \frac{2AB}{\mathrm{cotg}\alpha + \mathrm{cotg}\beta + \sqrt{4 + (\mathrm{cotg}\alpha - \mathrm{cotg}\beta)^2}}.$$

It is clear that if $B_1A_1 = \dfrac{2AB}{\mathrm{cotg}\alpha + \mathrm{cotg}\beta + \sqrt{4 + (\mathrm{cotg}\alpha - \mathrm{cotg}\beta)^2}}$ and $2\delta - \phi = 90^\circ$ then (1) holds true and therefore there exists a triangle with $B_1A_1 = \dfrac{2AB}{\mathrm{cotg}\alpha + \mathrm{cotg}\beta + \sqrt{4 + (\mathrm{cotg}\alpha - \mathrm{cotg}\beta)^2}}$.

It remains to compute the above expression for the three possible ways to inscribe a right triangle in triangle of angles $30^\circ, 60^\circ$ and 90° and hypotenuse 1. We obtain $\dfrac{\sqrt{39} - \sqrt{3}}{12}, \dfrac{\sqrt{21} - 3}{4}, \dfrac{\sqrt{3}}{4}$. The least value is $\dfrac{\sqrt{39} - \sqrt{3}}{12}$ which is the answer of the problem.

Problem 11.3. The plane is divided into unit squares by lines parallel to coordinate axes of an orthogonal coordinate system. Find the number of paths of length n from the point with coordinates $(0; 0)$ to the point with coordinates $(a; b)$ moving along the sides of the unit squares.

Solution: Divide the path into unit paths, i.e. paths between two neighbouring points of integer coordinates. Denote the number of moves up by x_1, down by y_1, right by x_2 and left by y_2. The condition of the problem gives $x_1 + x_2 + y_1 + y_2 = n, x_1 - y_1 = b, x_2 - y_2 = a$. Thus, $y_1 = x_1 - b; x_2 = \dfrac{n + a + b}{2} - x_1; y_2 = \dfrac{n - a + b}{2} - x_1$. Since $|a| + |b| = |x_2 - y_2| + |x_1 - y_1| \leq x_1 + x_2 + y_1 + y_2 = n$ we obtain that it is necessary to have $a + b \leq n$ and $a + b \equiv n (\mathrm{mod}\ 2)$. Let us first fix the moves up and rigth. This can be done by $\binom{n}{\frac{n+a+b}{2}}$ ways. After that in these already fixed $\dfrac{n+a+b}{2}$ positions fix the moves up and finally, in the remaining $\dfrac{n-a-b}{2}$ positions fix the moves left. We obtain

$$\binom{n}{\frac{n+a+b}{2}} \sum_{i=b}^{\frac{n+a-b}{2}} \binom{\frac{n+a+b}{2}}{i} \binom{\frac{n-a-b}{2}}{\frac{n-a+b}{2}-i}.$$

The sum $\displaystyle\sum_{i=b}^{\frac{n+a-b}{2}} \binom{\frac{n+a+b}{2}}{i} \binom{\frac{n-a-b}{2}}{\frac{n-a+b}{2}-i}$ equals to the coefficient in front of $x^{\frac{n-a+b}{2}}$ in the expansion of $(1+x)^{\frac{n+a+b}{2}}(1+x)^{\frac{n-a-b}{2}} = (1+x)^n$. Therefore the sum equals $\binom{n}{\frac{n-a+b}{2}}$. The number of paths is:

$$\binom{n}{\frac{n+a+b}{2}} \binom{n}{\frac{n-a+b}{2}}.$$

Answer: If $|a|+|b|>n$ or $a+b \not\equiv n \pmod{2}$, the number of paths is 0, otherwise the number of paths is $\binom{n}{\frac{n+a+b}{2}} \binom{n}{\frac{n-a+b}{2}}$.

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Problem 8.1. Let a be a real parameter such that $0 \leq a \leq 1$. Prove that the solutions of the inequality

$$|x| + |ax + \frac{1}{2}| \leq 1$$

form an interval of length greater or equal to 1.

Solution: If $a = 0$ then all numbers from the interval $\left[\frac{1}{2}, -\frac{1}{2}\right]$ of length 1 are solutions of the problem. If $a = 1$ then the inequality becomes $|x| + |x + \frac{1}{2}| \leq 1$. The solutions of this inequality are all numbers from the interval $\left[-\frac{3}{4}, \frac{1}{4}\right]$, which is of length 1.

Further, assume that $0 < a < 1$ and let $x \geq 0$. In this case $ax + \frac{1}{2} > 0$ and therefore the solutions are $x \leq \frac{1}{2(1+a)}$. Let $x < 0$. If $ax + \frac{1}{2} \geq 0$, i.e. $x \geq -\frac{1}{2}$, the solutions satisfy $x \geq -\frac{1}{2(1-a)}$. If $ax + \frac{1}{2} < 0$, i.e. $x < -\frac{1}{2a}$, the solutions satisfy $x \geq -\frac{1}{3(1+a)}$.

Let $a \leq \frac{1}{2}$. Then $-\frac{1}{2(1-a)} \geq -\frac{1}{2a}$ and so all numbers $x \geq -\frac{1}{2(1-a)}$ are solutions of the inequality. Since $-\frac{1}{3(1+a)} > -\frac{1}{2a}$ it has no other solutions. Therefore all solutions form an interval $\left[-\frac{1}{2(1-a)}, \frac{1}{2(1+a)}\right]$, of length $\frac{1}{2(1+a)} + \frac{1}{2(1-a)} = \frac{1}{1-a^2} > 1$.

Let $a > \frac{1}{2}$. Then $-\frac{3}{2(1+a)} < -\frac{1}{2a}$ and the solutions are $x \geq$

$-\dfrac{3}{2(1+a)}$ $\left(x \geq -\dfrac{1}{2a} \text{ or } x \in \left[-\dfrac{3}{2(1+a)}, -\dfrac{1}{2a}\right]\right)$. Therefore all solutions form an interval $\left[-\dfrac{3}{2(1+a)}, \dfrac{1}{2(1+a)}\right]$, of length $\dfrac{2}{1+a} \geq 1$.

Problem 8.2. Given a square $ABCD$ of side length 1. Point $M \in BC$ and point $N \in CD$ are such that the perimeter of $\triangle MCN$ is 2.

a) Find $\sphericalangle MAN$;

b) If P is the foot of the perpendicular from A to MN, find the locus of the point P.

Solution: a) Let K be a point on the extension of CB such that $BK = DN$. Then $MK = MB + BK = MB + DN = 1 - CM + 1 - CN = 2 - (CM + CN) = MN$ (using that $CM + CN + MN = 2$). Since $\triangle ABK \cong \triangle ADN$ we have that $AK = AN$. Therefore $\triangle AMN \cong \triangle AMK$ and it follows that $\sphericalangle MAN = \sphericalangle MAK$, i.e. $\sphericalangle MAN = \dfrac{1}{2} \sphericalangle KAN$. Furthermore $\sphericalangle KAB = \sphericalangle NAD$, which implies that $\sphericalangle KAN = \sphericalangle BAD = 90^\circ$. Therefore $\sphericalangle MAN = 45^\circ$.

b) It follows from $\triangle AMN \cong \triangle AMK$ that $\sphericalangle AMN = \sphericalangle AMK$. Thus, $\triangle APM \cong \triangle ABM$ and $AP = AB = 1$. Therefore P lies on a circle of center A and radius 1. Finally, the locus of P is an arc of a circle with center A and radius 1 excluding points B and D.

Problem 8.3. a) Prove that

$$1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

holds true for any positive integer n.

b) Find the least integer $n, n > 1$ for which

$$\frac{1^2 + 2^2 + \cdots + n^2}{n}$$

is a perfect square.

Solution:

a) Direct verification shows that

$$\frac{n(n+1)(2n+1)}{6} + (n+1)^2 = \frac{(n+1)(n+2)(2n+3)}{6}$$

for any integer n. Using this equality the proof is easily done by induction.

b) It follows from a) that $n(n+1)(2n+1) = 6m^2$. Since $2n+1$ is odd then $n+1$ is even, i.e. n is odd. Let $n = 2k-1$. Then $k(4k-1) = 3m^2$. Therefore $3/k$ or $3/4k-1$ i.e. $3/k$ or $3/k-1$. Let $k = 3l$. Then $l(12l-1) = m^2$. Since $(l, 12l-1) = 1$, we have $l = n^2$ and $12l-1 = v^2$. The latter equality is impossible (contradiction both modulo 4 and modulo 3). Let $k = 3l+1$. Then $(3l+1)(4l+1) = m^2$. Since $(3l+1, 4l+1) = 1$ we get that $3l+1 = u^2, 4l+1 = v^2, u > 1, v > 1$. Verifying consequently for $v = 3, 5, 7, 9, 11, 13, \ldots$ we obtain $l = 2, 6, 12, 20, 30, 42, 56, \ldots$. Thus, $3l+1 = 7, 19, 37, 61, 91, 127, 169$ (first perfect square). So, the least n is $2k-1$, where $k = 3l+1 = 3.56+1 = 169$ and therefore $n = 337$.

Problem 9.1. Let $f(x) = x^2 + 6ax - a$ where a is a real parameter.

a) Find all values of a for which the equation $f(x) = 0$ has at least one real root.

b) If x_1 and x_2 are the real roots of $f(x) = 0$ (not necessarily distinct) find the least value of the expression

$$A = \frac{9a - 4a^2}{(1+x_1)(1+x_2)} - \frac{70a^3 + 1}{(1-6a-x_1)(1-6a-x_2)}.$$

Solution: a) The discriminant of $f(x)$ is $D = 4(9a^2 + a)$. Therefore the equation $f(x) = 0$ has at least one real root iff $9a^2 + a \geq 0$, so giving $a \in \left(-\infty, -\frac{1}{9}\right] \cup [0, +\infty)$.

b) Since $f(x) = (x - x_1)(x - x_2)$ we obtain $f(-1) = (-1 - x_1)(-1 - x_2) = (1 + x_1)(1 + x_2) = (-1)^2 + 6a(-1) - a = 1 - 7a$ and $f(1-6a) = (1-6a-x_1)(1-6a-x_2) = (1-6a)^2 + 6a(1-6a) - a = 1 - 7a$. Therefore the denominators of the two fractions of A are equal to $1 - 7a$ and

$$A = \frac{-70a^3 - 4a^2 + 9a - 1}{1 - 7a} = \frac{(1 - 7a)(10a^2 + 2a - 1)}{1 - 7a} = 10a^2+2a-1.$$

The quadratic function $g(a) = 10a^2 + 2a - 1$ attains its minimal value for $a = -\frac{1}{10} \notin \left(-\infty, -\frac{1}{9}\right] \cup [0, +\infty)$. Therefore the minimal value of A equals to the smallest of the numbers $g(0)$ and $g\left(-\frac{1}{9}\right)$, i.e. this value is $g\left(-\frac{1}{9}\right) = -\frac{89}{81}$.

Problem 9.2. Given a convex quadrilateral $ABCD$ such that $OA = \frac{OB.OD}{OC + OD}$, where O is the intersecting point of its diagonals. The circumcircle of $\triangle ABC$ intersects the line BD at point Q. Prove that CQ is the bisector of $\measuredangle DCA$.

Solution: Let $CQ_1, Q_1 \in BD$ be the bisector of $\measuredangle DCO$. Therefore

$$\frac{DQ_1}{Q_1O} = \frac{DC}{CO}$$

This equality, combined with the condition of the problem gives

$$OA(OC + OD) = OB.OD \iff OA\frac{Q_1O + DQ_1}{Q_1O}CD = OB.OD$$

$$\iff OA.CO.\frac{DO}{Q_1O} = OB.OD \iff OA.CO = Q_1O.OB.$$

Therefore the quadrilateral $ABCQ_1$ is cyclic. Thus, $Q_1 \equiv Q$.

Problem 9.3. Prove that there exist eight consecutive positive integers such that non of them can be written in the form $|7x^2 + 9xy - 5y^2|$, where x and y are integers.

Solution: Denote $f(x, y) = 7x^2 + 9xy - 5y^2$. Since $f(0, 0) = 0$, $f(0, 1) = 5$, $f(1, 0) = 7$, $f(1, 1) = 11$ and $f(0, 2) = 20$, first possible sequence of eight positive integers is $12, 13, \ldots, 19$. We shall prove that non of these integers can be written in the form $|7x^2+9xy-5y^2|$, where x and y are integers.

Let $f(x, y) = \pm k$, where x and y are integers. It suffices to prove that $f(x, y) = \pm k$ has no solutions for $k \in \{12, 13, \ldots, 19\}$.

Suppose k is even. Then x and y are also even. If $x = 2x_1$ and $y = 2y_1$ we get the equality $4f(x_1, y_1) = \pm k$ which implies that k is divisible by 4. Thus, $k \neq 14$ and $k \neq 18$. Let $k = 16$ and consider the equation $4f(x_1, y_1) = \pm 16$ which is equivalent to $f(x_1, y_1) = \pm 4$. As above we conclude that x_1 and y_1 are both even and let $x_1 = 2x_2$ and $y_1 = 2y_2$. Therefore $f(x, y) = \pm 1$. By analogy if $k = 12$ we get the equation $f(x, y) = \pm 3$.

Multiply the equation $f(x, y) = \pm k$ by 28 and write it in the form

$$(14x + 9y)^2 - 221y^2 = \pm 28k.$$

Since $221 = 13.17$ it is appropriate to consider modules 13 and 17.

Denote $t = 14x + 9y$ and consider all possibilities for k, i.e. $k \in \{1, 3, 13, 15, 17, 19\}$.

1) If $k = 13$ then $t^2 \equiv \pm 28.13 \pmod{17}$ and so $t^2 \equiv \pm 7 \pmod{17}$. Raising this congruence to 8-th power gives $t^{16} \equiv (\pm 7)^8 \equiv -1$

(mod 17) which is a contradiction to Fermat's theorem. Therefore $k \neq 13$.

2) If $k = 15$ then $t^2 \equiv \pm 28.15 \equiv \mp 5 \pmod{17}$. Raising in 8-th power gives $t^{16} \equiv (\mp 5)^8 \equiv -1 \pmod{17}$, a contradiction.

The cases $k = 17, 19, 1, 3$ are treated similarly.

Problem 10a.1. Let a and b be positive numbers such that both of the equations $(a+b-x)^2 = a-b$ and $(ab+1-x)^2 = ab-1$ have two distinct real roots. Prove that if the two bigger roots are equal then the two smaller roots are also equal.

Solution: Let a and b satisfy the condition of the problem. Both equations have two distinct roots iff $a > b$ and $ab > 1$. Since $a > 0$ it follows that $a^2 > ab > 1$, i.e. $a > 1$. Therefore $a > b > \frac{1}{a}$ and $a > 1$. It follows from the condition of the problem that $a + b + \sqrt{a-b} = ab + 1 + \sqrt{ab-1}$, i.e. $\sqrt{a-b} = (a-1)(b-1) + \sqrt{ab-1}$. If $a > b \geq 1$ then $\sqrt{a-b} \geq \sqrt{ab-1} \iff (a+1)(b-1) \leq 0 \iff b \leq 1$. Therefore $b = 1$. Conversely if $\frac{1}{a} < b \leq 1$ then $\sqrt{a-b} \leq \sqrt{ab-1} \iff (a+1)(b-1) \geq 0 \iff b \geq 1$, i.e. $b = 1$. Thus, the two bigger roots are equal iff $b = 1$ and $a > 1$. In this case the two smaller roots are also equal.

Problem 10a.2. Let A_1 and B_1 be points respectively on the sides BC and AC of $\triangle ABC$, $D = AA_1 \cap BB_1$ and $E = A_1B_1 \cap CD$. Prove that if $\sphericalangle A_1EC = 90^\circ$ and the points A, B, A_1, E lie on a circle, then $AA_1 = BA_1$.

Solution: Let $F = AE \cap BC$. We prove that EA_1 is the bisector of $\sphericalangle BEF$, which solves the problem. Indeed, then we have $\sphericalangle BAA_1 = \sphericalangle BEA_1 = \sphericalangle FEA_1 = \sphericalangle ABA_1$, i.e. $AA_1 = BA_1$.

Using Ceva's and Menelaus's theorems for $\triangle AA_1C$ we obtain $\frac{AD}{A_1D}\cdot\frac{A_1F}{CF}\cdot\frac{CB_1}{AB_1}=1$, $\frac{AD}{A_1D}\cdot\frac{A_1B}{CB}\cdot\frac{CB_1}{AB_1}=1$ and so

$$\frac{A_1F}{A_1B}=\frac{CF}{CB} \tag{1}$$

Let B' be a point on the ray $A_1B^{\rightarrow}$ such that $\sphericalangle B'EA_1=\sphericalangle A_1EF$. Then EA_1 is the bisector of $\sphericalangle B'EF$, and since $\sphericalangle A_1EC=90^\circ$ it follows that EC is the external bisector of the same angle. Therefore $\frac{A_1F}{A_1B'}=\frac{CF}{CB'}$ and it follows from (1) that $\frac{A_1B}{CB}=\frac{A_1B'}{CB'}$, i.e. $B=B'$ which completes the proof.

Problem 10a.3. Find all positive integers x and y such that

$$\frac{x^3+y^3-x^2y^2}{(x+y)^2}$$

is a nonnegative integer.

Solution: Let x and y be positive integers such that

$$z=\frac{x^3+y^3-x^2y^2}{(x+y)^2}$$

is a nonnegative integer. Substitute $a=x+y$ and $b=xy$ and write the expression in the form $b^2+3ab-a^2(a-z)=0$. The discriminant of this quadratic equation $a^2(4a+9-4z)$ is a perfect square, so $(4a+9-4z)=(2t+1)^2$. Thus, $a=t^2+t+z-2$ and from the equation for b we obtain that $b=a(t-1)$. Since $t\geq 2$ we have $(x-y)^2=a^2-4a(t-1)<(a-2(t-1))^2$. From the other hand $a\geq t^2$ and therefore $a^2-4a(t-1)\geq(a-2(t-1)-2)^2$. Since $a^2-4a(t-1)\neq(a-2(t-1)-1)^2$ (the two numbers are of different parity) it follows that $(x-y)^2=a^2-4a(t-1)=(a-2(t-1)-2)^2$.

Thus, $a = t^2$ i.e. $t + z = 2$ and so $t = 2, z = 0$, which implies $a = b = 4$. Therefore $x = y = 2$ and these are the only positive integers satisfying the condition of the problem.

Problem 10b.1. Solve the equation:

$$3^{\log_3(\cos x+\sin x)+\frac{1}{2}} - 2^{\log_2(\cos x-\sin x)} = \sqrt{2}.$$

Solution: Note that the admissible values are those x for which $\cos x + \sin x > 0, \cos x - \sin x > 0$. After simple calculations the equation becomes

$$(\sqrt{3} - 1)\cos x + (\sqrt{3} + 1)\sin x = \sqrt{2}.$$

This equation is equivalent to

$$\frac{\sqrt{6} - \sqrt{2}}{4}\cos x + \frac{\sqrt{6} + \sqrt{2}}{4}\sin x = \frac{1}{2}.$$

Since $\cos 15° = \frac{\sqrt{6}+\sqrt{2}}{4}$ and $\sin 15° = \frac{\sqrt{6}-\sqrt{2}}{4}$ we have

$$\sin(x + 15°) = \frac{1}{2}.$$

From the roots of the later equation only $x = 15° \pm k360°$ are admissible. Therefore the roots are $x = 15° \pm k360°$.

Problem 10b.2. Given a triangle ABC. Let M be such an interior point of $\sphericalangle BAC$ that $\sphericalangle MAB = \sphericalangle MCA$ and $\sphericalangle MAC = \sphericalangle MBA$. Analogously, let N be such an interior point of $\sphericalangle ABC$ that $\sphericalangle NBA = \sphericalangle NCB$ and $\sphericalangle NBC = \sphericalangle NAB$, and let P be such an interior point of $\sphericalangle ACB$ that $\sphericalangle PCA = \sphericalangle PBC$ and $\sphericalangle PCB = \sphericalangle PAC$. Prove that lines AM, BN and CP intersect in a point on circumcircle of $\triangle MNP$.

Solution: Suppose $\sphericalangle C < 90^\circ$. The case $\sphericalangle C > 90^\circ$ is treated similarly. Denote the intersecting point of line CP and line segment AB by C_1. Since PC_1 is the bisector of $\sphericalangle APB(\sphericalangle APC_1 = \sphericalangle BPC_1 = \gamma = \sphericalangle ACB)$ we get $\dfrac{AC_1}{BC_1} = \dfrac{AP}{BP}$. Triangles APC and BPC are similar which gives $\dfrac{AP}{CP} = \dfrac{AC}{BC}$ and $\dfrac{BP}{CP} = \dfrac{BC}{AC} \Longrightarrow \dfrac{AP}{BP} = \dfrac{AC^2}{BC^2} = \dfrac{b^2}{a^2}$. By analogy, if A_1 and B_1 are the intersecting points of AM and BN with BC and AC respectively then $\dfrac{BA_1}{A_1C} = \dfrac{c^2}{b^2}$ and $\dfrac{CB_1}{B_1A} = \dfrac{a^2}{c^2}$. Thus, $\dfrac{AC_1.BA_1.CA_1}{C_1B.A_1C.B_1A} = 1$, i.e. according to Ceva's theorem lines AM, BN and CP intersect in a point. Dnote this point by K. Since $\sphericalangle APB = \sphericalangle AOB = 2\gamma$, where O is the circumcenter of $\triangle ABC$ we get that A, P, O and B lie on a circle and PC_1 intersects the arc $\overset{\frown}{AB}$ at its midpoint C_1. Therefore OC_2 is a diameter of this circle and $\sphericalangle OPC_2 = \sphericalangle OPC = 90^\circ$. It easily follows now that M, N and P lie on a circle of diameter OK.

Problem 10b.3. Consider a set P of six four-letter words over an alphabet of two letters a and b. Denote by Q_P the set of all words over the same alphabet which do not contain as subwords the words from P. Prove that:

a) if Q_P is finite then it does not contain words of length ≥ 11;

b) there exists a set P such that Q_P is finite and it contains word of length 10.

Solution: a) Suppose that Q_P contains a word of length 11. We will show that Q_P contains words of any length. Let ω be a word of length 11. There are 9 subwords of ω of length 3. Since there are 8 distinct words of length 3 it follows that there exists a word α that appears as subword of ω twise. Let $\alpha = \alpha_1\alpha_2\alpha_3$ where $\alpha_i \in \{a, b\}$.

Consider the subword of ω obtained after the second appearance of α, i.e. consider the word

$$\dots \alpha_1\alpha_2\alpha_3\gamma_1 \dots \alpha_1\alpha_2\alpha_3$$

Write γ_1 after α_3. Obviously the new word does not contain subwords from P.

$$\dots \alpha_2\alpha_3\gamma_1\gamma_2 \dots \alpha_2\alpha_3\gamma_1$$

By analogy write γ_2 and so on. Thus, we find words of any length without subwords from P. This contradiction shows that there are no words of length ≥ 11 in Q_P.

b) A direct verification shows that the set

$$P = \{0000, 1000, 1001, 1010, 1101, 1111, \}$$

is such that Q_P is finite and $0001011100 \in Q_P$.

Problem 11.1. Prove that there exist unique numbers α and β such that $\cos\alpha = \alpha^2$, $\beta \mathrm{tg}\beta = 1$ and $0 < \alpha < \beta < 1$.

Solution: The function $f(x) = \cos x - x^2$ is continuous in the interval $(0;1), f(0) = 1 > 0$ and $f(1) = \cos 1 - 1 < 0$. Therefore there exists α such that $\cos\alpha = \alpha^2$. In the interval $(0;1)$ the function $\cos x$ is decreasing one, whereas the function x^2 is increasing. Therefore there exists unique $\alpha \in (0;1)$ such that $\cos\alpha = \alpha^2$.

We show now that in the interval $[0;1]$ there exists a unique β such that $g(\beta) = 0$ where $g(x) = x\mathrm{tgx} - 1$. The function $g(x)$ is increasing one because $x\mathrm{tgx}$ is increasing as product of two increasing functions. Moreover $\mathrm{tg}\left(\frac{\pi}{4}\right) = \frac{\pi}{4} - 1 < 0, \mathrm{g}(1) = \mathrm{tg}1 - 1 = \mathrm{tg}1 - \mathrm{tg}\frac{\pi}{4} > 0$ and the uniquennes of β follows. Since $\sin x < x$ for positive

x it follows that $g(\alpha) = \alpha \mathrm{tg}\alpha - 1 = \dfrac{\sin\alpha}{\alpha} - 1 < 0$. Further, $g(x)$ is increasing in the interval $[0;1]$ and so $\alpha < \beta$.

Problem 11.2. Let AA_1 and BB_1 be the altituds of obtuse non-isosceles $\triangle ABC$, and O and O_1 are circumcenters of $\triangle ABC$ and $\triangle A_1B_1C$ respectively. A line through C intersects the line segments AB and A_1B_1 at points D and D_1 respectively and E is point on the line OO_1 such that $\sphericalangle ECD = 90°$. Prove that $\dfrac{EO_1}{EO} = \dfrac{CD_1}{CD}$.

Solution: Let F and F_1 be the feet of the perpendiculars from O and O_1 to CD. Since $CO_1 \perp AB$ we have

(1)
$$\frac{EO_1}{EO} = \frac{CF_1}{CF} = \frac{CO_1 \cos \sphericalangle O_1CF_1}{CO \cos \sphericalangle OCF} = \frac{CO_1 \sin \sphericalangle BDC}{CO \sin(\sphericalangle DCB + \sphericalangle BAC)}.$$

From the other hand the Low of Sine's for $\triangle A_1B_1C, \triangle ABC$, $\triangle A_1D_1C$ and $\triangle BDC$ gives

(2) $$CO_1 = \frac{A_1C}{2\sin \sphericalangle A_1B_1C} = \frac{A_1C}{2\sin \sphericalangle ABC}$$

(3) $$CO = \frac{BC}{2\sin \sphericalangle BAC}$$

(4)
$$CD_1 = CA_1 \frac{\sin \sphericalangle B_1A_1C}{\sin(\sphericalangle B_1A_1C + \sphericalangle D_1CA_1)} = \frac{CA_1 \sin \sphericalangle BAC}{\sin(\sphericalangle BAC + \sphericalangle DCB)}$$

(5) $$CD = \frac{BC \sin \sphericalangle ABC}{\sin \sphericalangle BDC}$$

It follows now from (1),(2),(3),(4) and (5) that

$$\frac{EO_1}{EO} = \frac{A_1C}{BC} \frac{\sin \measuredangle BAC \sin \measuredangle BDC}{\sin \measuredangle ABC \sin(\measuredangle BAC + \measuredangle DCB)} = \frac{CD_1}{CD},$$

which completes the proof.

Problem 11.3. There are 2001 towns in a country every one of which is connected with at least 1600 towns by direct bus line. Find the largest n for which there exist n towns any two of which are connected by direct bus line.

Solution: Let S_1 snd S_1 be two towns connected by direct bus line. If k is the number of towns connected to both S_1 and S_2 by bus line then $(1599-k)+(1599-k)+k \leq 1999$ which implies that $k \geq 1198$. Therefore there exists town S_3 connected to both S_1 and S_2. Further, let k be the number of towns connected to all S_1, S_2 and S_3. Therefore $(1197-k)+(1598-k)+k \leq 1998$, so giving $k \geq 797$. Therefore there exists town S_4 connected to all S_1, S_2 and S_3. By analogy let k be the number of towns connected to all S_1, S_2, S_3 and S_4. We have $(796-k)+(1597-k)+k \leq 1997$, which implies $k \geq 396$. Therefore there exists S_5 connected to all S_1, S_2, S_3 and S_4. For the number n from the condition of the problem we obtain $n \geq 5$. We show that $n = 5$. For, number the towns by $S_1, S_2, \dots, S_{2001}$ and connect S_k and S_m with direct line for all k and m for which $k \not\equiv m (mod 5)$. Since $\left[\frac{2001}{5}\right] = 400$, we have that each town is connested to 1600 or 1601 other towns, i.e. the condition of the problem is satisfied.

For arbitrary 6 towns the numbers of at least two are equal modulo 5 and therefore they are not connected to each other. Therefore $n < 6$ and so $n = 5$.

L National Mathematics Olympiad
3rd round, 28-29 April 2001

Problem 1. For which values of the real parameter a the equation

$\lg(4x^2-(8a-1)x+5a^2)+x^2+(1-2a)x+2a^2 = \lg(x^2-2(a+1)x-a^2)$

has exactly one root?

Solution: Write the equation in the form:

$$\lg(4x^2-(8a-1)x+5a^2)+\frac{4x^2-(8a-1)x+5a^2}{3} =$$
$$\lg(x^2-2(a+1)x-a^2)+\frac{x^2-2(a+1)x-a^2}{3}.$$

Since $\lg t + \dfrac{t}{3}$ is an increasing function this equality is equivalent to

$$\left|\begin{array}{l} 4x^2-(8a-1)x+5a^2 = x^2-2(a+1)x-a^2 \\ x^2-2(a+1)x-a^2 > 0 \end{array}\right.$$

The equation in the above system is equivalent to $f(x)=0$ where $f(x) = x^2-(2a-1)x+2a^2$. Substitution $x^2 = 2ax-x-2a^2$ in the inequality gives $x < -a^2$. Therefore we have to find those a for which the equation $f(x) = x^2-(2a-1)x+2a^2 = 0$ has exactly one root such that $x < -a^2$. There are three cases to be considered:

1. $f(x)=0$ has two roots and $-a^2$ is between the roots. In this case $f(-a^2)<0 \Rightarrow a^2(a+1)^2 < 0$, which is impossible.

2. $f(x) = 0$ has two roots one of which equals $-a^2$ and the second one is less than $-a^2$. in this case $f(-a^2)=0$ and so $a=-1$

or $a = 0$. When $a = -1$ we obtain $x_1 = -2, x_2 = -1$, which implies that $a = -1$ is a solution. If $a = 0$ we get $x_1 = -1, x_2 = 0$, which implies that $a = 0$ is also a solution.

3. $f(x) = 0$ has one root which is less than $-a^2$. Since $D = 0$ we get that $-4a^2 - 4a + 1 = 0$, and so $a_{1/2} = \dfrac{-1 \pm \sqrt{2}}{2}$. It is easy to check that for both values of a the corresponding root is less than $-a^2$.

Therefore the solutions are $a = -1, a = 0, a = \dfrac{-1 \pm \sqrt{2}}{2}$.

Problem 2. Diagonals AC and BD of a cyclic quadrilateral $ABCD$ intersect in a point E. Prove that if $\sphericalangle BAD = 60^\circ$ and $AE = 3CE$ then the sum of two of the sides of the quadrilateral equals the sum of the other two.

Solution: Set $\sphericalangle ABD = x$ and $\sphericalangle CBD = y$. From the Low of Sine's we obtain

$$\frac{AE}{AB} = \frac{\sin x}{\sin(120^\circ + y - x)}, \frac{CE}{BC} = \frac{\sin y}{\sin(60^\circ + x - y)}$$

and $\dfrac{AB}{BC} = \dfrac{\sin(120^\circ - x)}{\sin(60^\circ - y)}$. Therefore $3 = \dfrac{AE}{CE} = \dfrac{\sin x . \sin(120^\circ - x)}{\sin y . \sin(60^\circ - y)}$.

Hence $3(\cos(2y - 60^\circ) - \cos 60^\circ) = \cos(2x - 120^\circ) - \cos 120^\circ$, i.e. $1 - \cos(2x - 120^\circ) = 3(1 - \cos(2y - 60^\circ))$ and so $\sin^2(x - 60^\circ) = 3\sin^2(y - 30^\circ)$. Therefore $\sin(x - 60^\circ)\cos 60^\circ = \pm \cos 30^\circ \sin(y - 30^\circ)$, i.e. $\sin x - \sin(120^\circ - x) = \pm(\sin y - \sin(60^\circ - y))$. Again fom the Low of Sine's we get $AD - AB = \pm(CD - BC)$, i.e.

$$AD + BC = AB + CD \text{ or } AD + CD = AB + BC.$$

Problem 3. Find the least positive integer n such that there exists a group of n people such that:

1. There is no group of four every two of which are friends;

2. For any choice of $k \geq 1$ people among which there are no friends there exists a group of three among the remaining $n - k$ every two of which are friends.

Solution: Consider a group of 7 people $A_1, A_2, \ldots, A_7$, such that $A_i, i = 1, 2 \ldots, 7$ is not friend only with A_{i+1} and A_{i-1} (we set that $A_8 = A_1$ and $A_0 = A_7$). It is easily seen that there are no four every two of which are friends. Also, for any choice of $k \geq 1$ people (in this case k is 1 or 2) every two of which are not friends there exists a group of three among remaining $7 - k$ people every two of which are friends. Therefore $k \leq 7$.

We prove that for any group of 6 which satisfy the condition 1) it is possible to choose a group of $k \geq 1$ every two of which are not friends such that among remaining $6 - k$ a group of three every two of which are friends does not exist.

Denote the people by $A_1, A_2, \ldots A_6$. If some of them is friend with the other 5 (wlog suppose this is A_1) it is clear that there are no three among $A_2, A_3, \ldots A_5$ every two of which are friends. Therefore the choice of A_1 solves the problem.

Suppose one of then is friend with exactly four others (wlog assume A_1 is friend with A_2, A_3, A_4 and A_5) Then the choice of A_1 and A_6 solves the problem.

Therefore each person has $0, 1, 2$ or 3 friends. It is obviuos that there exists a group of three any two of which are friends (otherwise the problem is trivial). Assume that the group A_1, A_2 and A_3 has this property. Wlog A_1 and A_4 are not friends. Therefore among A_2, A_3, A_5 and A_6 there is a group of three any two of which are friends. If this is A_5 and A_6 together with one of A_2 or A_3 then there exists a person who is friend with at least four others, a contradiction. Therefore wlog suppose that this group is A_2, A_3 and A_5. Since A_1 and A_5 are not friends and since among the others there is a group of three friends we obtain that either A_2 or A_3 has at least four friends which is a contradiction.

Therefore $k = 7$.

Problem 4. Given a right triangle ABC with hypotenuse AB. A point D distinct from A and C is chosen on the ray $AC^{\rightarrow}$ such that the line through incenter of $\triangle ABC$ parallel to the bisector of $\sphericalangle ADB$ is tangent to the incircle of $\triangle BCD$. Prove that $AD = BD$.

Solution: First we show that C lies on the line segment AD. For, suppose the contrary, i.e. $\sphericalangle ADB > 90°$. Denote the tangential point of incircle $k(I, r)$ of $\triangle BCD$ and the side BD by P and the tangential point of k and the line through the center J parallel to bisector l of $\sphericalangle ADB$ by T. Since $l \perp DI$ we get $T \in DI$. Thus, $\sphericalangle IJT = \dfrac{\sphericalangle CBD}{2} = \sphericalangle IBP$ and since $IT = r = IP$ we have that $\triangle IJT$ and $\triangle IBP$ are congruent. In particular $IJ = IB$ which implies that $\sphericalangle BJC < 90°$, a contradiction. Let A' be such point on the ray $DA^{\rightarrow}$ that $DA' = DB$ and E is the midpoint of $A'B$. Denote the tangential point of incircle of $\triangle A'BC$ by F and its incenter by J'. Then we have

$$\begin{aligned} &EF = |A'E - A'F| = \frac{1}{2}|A'B - (A'B + A'C - BC)| = \\ &= \frac{1}{2}|BC - A'C| = \frac{1}{2}|BC - (A'D - CD)| = \\ &\tfrac{1}{2}|BC + CD - BD| = r. \end{aligned}$$

Therefore the line $J'F$ parallel to l is tangent to k. This implies that $J' \in JF$. From the other hand $J' \in JC$ and since JC is not parallel to $JF(\sphericalangle ACJ = 45^\circ > \frac{1}{2} \sphericalangle ADB = \sphericalangle ADI)$, we get $J = J'$. Thus, $\sphericalangle ABJ = \sphericalangle CBJ = \sphericalangle A'BJ$, i.e. $A = A'$ which completes the proof.

Problem 5. Find all triples of positive integers (a, b, c) such that $a^3 + b^3 + c^3$ is divisible by a^2b, b^2c and c^2a.

Solution: If $d = gcd(a, b, c)$ it is easy to see that $\left(\frac{a}{d}, \frac{b}{d}, \frac{c}{d}\right)$ is also a solution of the problem. Thus, it suffices to find all triples (a, b, c) such that $gcd(a, b, c) = 1$. Let $gcd(a, b) = s$ and suppose $s > 1$. If p is a prime divisor of s then p divides a; p divides b and p divides $a^3 + b^3 + c^3$. Hence p divides c^3 and so p divides c. This is a contradiction to $gcd(a, b, c) = 1$. Therefore $gcd(a, b) = 1$ and by analogy $gcd(a, c) = gcd(b, c) = 1$. It follows from $a^2/(a^3 + b^3 + c^3), b^2/(a^3 + b^3 + c^3), c^2/(a^3 + b^3 + c^3)$ and $gcd(a^2, b^2) = gcd(a^2, c^2) = gcd(b^2, c^2) = 1$ that $a^3 + b^3 + c^3$ is divisible by $a^2b^2c^2$. In particular $a^3 + b^3 + c^3 \geq a^2b^2c^2$. Wlog we may assume that $a \leq b \leq c$. Then $3c^3 \geq a^3 + b^3 + c^3 \geq a^2b^2c^2 \Rightarrow c \geq \frac{a^2b^2}{3}$. Suppose $a > 1$. Then $\frac{a^2b^2}{3} > b^2 \geq b^2 + a(a - b) = a^2 - ab + b^2$ and the inequality (1), $c > a^2 - ab + b^2$ holds. From the other hand since $b \geq a \geq 2$ we have that $b^2 \geq 2b \geq a + b$ and since $c > b^2$ we obtain (2), $c > a + b$. Now (1) and (2) give $\frac{(a^2 - ab + b^2)(a + b)}{c^2} < 1 \iff \frac{a^3 + b^3}{c^2} < 1$, which is impossible since $\frac{a^3+b^3}{c^2}$ is an integer. Thus, $a = 1$. In this case we have that $1 + b^3 + c^3$ is divisible by b^2c^2. Consider the following cases:

1) if $b = c$ it is easy to see that $b = c = 1$. Indeed, $(1, 1, 1)$ is a solution of the problem.

2) if $b = 1$ we obtain the same solution.

3) if $b = 2$ we obtain $c = 3$. Triple $(1, 2, 3)$ is a solution of the

problem.

Supose now that $c > b \geq 3$. Since $1 + b^3 + c^3 \geq b^2c^2$, it follows that $2c^3 > 1+b^3+c^3$ and so $2c > b^2$ or $c > \frac{b^2}{2}$. It follows from $2c > b^2$ that $2c > b^2 - b + 1$, i.e. (3), $\frac{b^2 - b + 1}{c} < 2$. From the other hand when $b \geq 5$ the inequalities $\frac{c}{2} > \frac{b^2}{4} > b + 1 \Rightarrow (4) \frac{b+1}{c} < \frac{1}{2}$ hold. Multiplying (3) and (4) gives $\frac{b^3+1}{c^2} < 1$, which is impossible since this number is an integer. Direct verificstion shows that when $b = 3$ or $b = 4$ we get no new solutions. Therefore all solutions of the problem are triples (k, k, k) and $(k, 2k, 3k)$ (and its permutations) for arbitrary positive integer k.

Problem 6. Given a pack of 52 cards. The following opperations are allowed:

1. Swap the first two cards;
2. Put the first card on the last place.

Prove that using these opperations one can order the cards in arbitrary manner.

Solution: First we show that it is possible to change any two neighbouring cards. Indeed, using 2) we can move the two cards on the first two positions. After that apply 1) and again 2) to put all remaining cards on their initial positions.

Consider two arbitrary cards a_i and a_{i+j}. We can change these two cards by changing a_i and a_{i+1} then a_i and a_{i+2} and so on, up to a_i and a_{i+j}. After that we change a_{i+j} and a_{i+j-1} and so on, up to a_{i+j} and a_{i+1}.

Since we can change any two cards we can order the cards in arbitrary manner.

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Problem 1. Consider the sequence $\{a_n\}$ such that $a_0 = 4$, $a_1 = 22$ and $a_n - 6a_{n-1} + a_{n-2} = 0$ for $n \geq 2$. Prove that there exist sequences $\{x_n\}$ and $\{y_n\}$ of positive integers such that $a_n = \dfrac{y_n^2 + 7}{x_n - y_n}$ for any $n \geq 0$.

Solution: Let $x_n = \dfrac{a_n + a_{n-1}}{2}$, $x_0 = 3$ and $y_n = \dfrac{a_n - a_{n-1}}{2}$, $y_0 = 1$. Then $x_n = 3x_{n-1} + 4y_{n-1}$ and $y_n = 2x_{n-1} + 3y_{n-1}$. Since $a_n = x_n + y_n$, it suffices to prove that $x_n + y_n = \dfrac{y_n^2 + 7}{x_n - y_n}$, i.e. $x_n^2 = 2y_n^2 + 7$. We prove this by induction. The assertion is obvious for $n = 0$. Suppose that $x_{n-1}^2 = 2y_{n-1}^2 + 7$. Writing this equality in the form $(3x_{n-1} + 4y_{n-1})^2 = 2(3y_{n-1} + 2x_{n-1})^2 + 7$ gives $x_n^2 = 2y_n^2 + 7$, which completes the proof.

Problem 2. Given nonisosceles triangle ABC. Denote the tangential points of the inscribed circle k of center O with the sides AB, BC and CA by C_1, A_1 and B_1 respectively. Let $AA_1 \cap k = A_2$, $BB_1 \cap k = B_2$ and let A_1A_3, B_1B_3 be bisectors in triangle $A_1B_1C_1$ ($A_3 \in B_1C_1, B_3 \in A_1C_1$). Prove that:

a) A_2A_3 is bisector of $\sphericalangle B_1A_2C_1$;

b) if P and Q are the intersecting points of circumcircles of triangle $A_1A_2A_3$ and triangle $B_1B_2B_3$ then the point O lies on the line PQ.

Solution: a) From the Low of Sine's we obtain

$$\frac{AB_1}{AA_1} = \frac{\sin \sphericalangle B_1A_1A_2}{\sin \gamma_1}, \quad \frac{AC_1}{AA_1} = \frac{\sin \sphericalangle C_1A_1A_2}{\sin \beta_1},$$

where $\beta_1 = \sphericalangle A_1B_1C_1$, $\gamma_1 = A_1C_1B_1$. Since $AB_1 = AC_1$ we get

$$\frac{A_2B_1}{A_2C_1} = \frac{\sin \sphericalangle B_1A_1A_2}{\sin \sphericalangle C_1A_1A_2} = \frac{\sin \gamma_1}{\sin \beta_1} = \frac{A_1B_1}{A_1C_1} = \frac{A_3B_1}{A_3C_1},$$

which implies that A_2A_3 is the bisector of $\sphericalangle B_1A_2C_1$.

b) Let $M = AA_1 \cap BB_1$. It follows from $MA_1 \,.\, MA_2 = MB_1 \,.\, MB_2$ that M lies on the line PQ. Therefore it suffices to prove that $OM \perp O_1O_2$, where O_1 and O_2 are circumcenters of $\triangle A_1A_2A_3$ and $\triangle B_1B_2B_3$. It follows from a) that the diametrically opposite point of A_3 in k_1 – the circumcircle of $\triangle A_1A_2A_3$, lies on the line B_1C_1. Therefore $O_1 \in B_1C_1$. Moreover $\sphericalangle B_1B_3A_1 = \sphericalangle CA_1A_3 = \gamma + \dfrac{\alpha_1}{2}$. It easily follows now that O_1 coinsides with the intersecting point of B_1C_1 and BC. Let $OO_1 \cap A_1A_2 = N$ and $OO_2 \cap B_1B_2 = K$. It follows from $\triangle OA_1O_1$ that $ON \,.\, OO_1 = OA_1^2 = r^2$ and by analogy $OK \,.\, OO_2 = r^2$ where r is the radius of k. Since O, N, M and K lie on the circle k_3 of diameter OM we have that the line O_1O_2 is the image of k_3 by inversion of center O and degree r^2, i.e. $OO_1 \perp OM$.

Problem 3. For a permutation $a_1, a_2, \dots a_n$ of the numbers $1, 2 \dots n$ it is allowed to change the places of any two consecutive blocks, i.e. from

$$a_1, \dots, a_i, \underbrace{a_{i+1}, a_{i+2} \dots, a_{i+p}}_{A}, \underbrace{a_{i+p+1}, a_{i+p+2} \dots, a_{i+q}}_{B}, a_{i+q+1}, \dots, a_n$$

by replacing A and B one can obtain

$$a_1, \dots, a_i, \underbrace{a_{i+p+1}, a_{i+p+2} \dots, a_{i+q}}_{B}, \underbrace{a_{i+1}, a_{i+2} \dots, a_{i+p}}_{A}, a_{i+q+1}, \dots, a_n.$$

Find the least number of such changes after which from $n, n-1, \dots 1$ one can obtain $1, 2, \dots n$.

Solution: Call the change of two blocks a move. We shall prove that the least number of moves such that from $n, n-1, \dots 1$ one can obtain $1, 2, \ldots n$ is $\left\lceil \frac{n+1}{2} \right\rceil$.

Consider the number of pairs a_i, a_{i+1} such that $a_i < a_{i+1}$. This number is 0 in the initial permutation $n, n-1, \ldots 2, 1$ and is $n-1$ in the final permutation $1, 2, \ldots, n-1, n$.

First we show that a move changes the number of pairs a_i, a_{i+1} such that $a_i < a_{i+1}$ at most by two. For, consider a move of the blocks $a \ldots, b$ and $c, \ldots d$ of the permutation

$$\ldots p, a \ldots, b, c, \ldots d, q \ldots$$

As a result we obtain

$$\ldots p, c \ldots, d, a, \ldots b, q \ldots$$

It is clear that at most three pairs can change the ordering. Suppose the elements in all three pairs change the ordering, i.e. from $p > a$, $b > c$ and $d > q$ we get $p < c$, $d < a$ and $b < q$. Adding the first three inequalities gives $p + b + d > a + c + q$ and adding the last three implies $p + b + d < a + c + q$, a contradiction. Therefore a move changes the number of pairs a_i, a_{i+1} for which $a_i < a_{i+1}$ at most by two. It is easily seen that the first and the last moves change this number by one. Therefore if x is desired number we have $2 + 2(x-2) \geq n-1$ which implies $x \geq \left\lceil \frac{n+1}{2} \right\rceil$. It remaines to find a sequence of $\left\lceil \frac{n+1}{2} \right\rceil$ moves such that from $n, n-1, \ldots 2, 1$ we get $1, 2, \ldots, n-1, n$. Let n be even number, i.e. $n = 2k$. Number the positions from right to left by $1, 2 \ldots n$. First change the places of the blocks from positions $1, 2 \ldots, k-1$ and $k, k+1$. Next, change the places of the blocks from positions $2, 3 \ldots, k$ and $k+1, k+2$. Third, change the blocks $3, 4 \ldots, k+1$ and $k+2, k+3$ and so on. On the k-th step change the blocks from positions $k, k+1 \ldots, 2k-2$ and $2k-1, 2k$. As a result we obtain $k+1, k+2, \ldots, 2k, 1, 2, \ldots, k$. The last change is $1, 2, \ldots, k$ and $k+1, k+2 \ldots, 2k$. When n is odd, i.e. $n = 2k+1$ the first change is of the blocks from positions $1, 2, \ldots, k$ and $k+1, k+2$ after that $2, 3, \ldots, k+1$ and $k+2, k+3$ and so on. The last $k+1$ change is $1, 2 \ldots k$ and $k+1, \ldots, 2k$ and we obtain $1, 2, \ldots, 2k+1$.

Problem 4. Let $n \geq 2$ be fixed integer. At any point with integer coordinates $(i,\, j)$ we write $i + j$ modulo n. Find all pairs $(a; b)$ of positive integers such that the rectangle with vertices $(0,\, 0)$, $(a,\, 0)$, $(a,\, b)$, $(0,\, b)$ has the following properties:

1) the remainders $0,\, 1,\, \ldots, n-1$ written in its interior points appear equal number of times;

2) the remainders $0,\, 1,\, \ldots, n-1$ written on its boundary appear equal number of times.

Solution: Let p_i and s_i for $i = 0,\, 1,\, \ldots,\, n-1$ be the number of residues i on the sides and in the interior of the rectangle respectively. If $a' = a + kn$ for an integer k then the corresponding numbers for $(a',\, b)$ are $p'_i = p_i + 2k$ and $s'_i = s_i + k(b-1)$.

Therefore if $(a,\, b)$ is a solution then $(a + kn,\, b)$ and by analogy $(a,\, b + ln)$ are also solutions. Thus, wlog we may suppose that $1 \leq a,\, b \leq n$.

If $n = 2$ then all possible values of $(a,\, b)$ are

$$(a,\, b) = (1,\, 1),\ (1,\, 2),\ (2,\, 1),\ (2,\, 2).$$

Pair $(2,\, 2)$ is not a solution because there is unique interior point $(s_0 = 1,\, s_1 = 0)$ for the rectangle. The remaining three cases give solutions.

Therefore for $n = 2$ solutions are all pairs $(a; b)$ where either a or b is an odd number.

Let $n > 2$ and suppose $(a,\, b)$ is a solution of the problem for which $1 \leq a, b < n$. In the vertices $(0, 0)$ and (a, b) are written 0 and $a + b$ modulo n respectively, whereas all residues $1, 2, \ldots, a + b - 1$ appear on the boundary even number of times. (once on the boundary $(1, 0), \ldots, (a, 0), (a, 1), \ldots, (a, b-1)$ and once on the boundary $(0, 1), \ldots, (0, b), (1, b), \ldots, (a-1, b)$). If n does not divide $a + b$ then 0 and the residue of $a + b$ appear odd number of times whereas at least one of the remaining residues ($n > 2$) appears even number of times. Therefore n divides $a + b$ and so $a + b = n$ or $a + b = 2n$. When $a + b = 2n$ we have that $a = b = n$. This pair is not a solution since the number of interior points is $(n-1)^2$ and is not divisible by n. It remains to consider the case $a + b = n$. If $a > 1$ and $b > 1$ then the rectangle has interior points. For any such point (i, j) we have $0 < i < a$, $0 < j < b$, $0 < i + j < a + b = n$ and therefore the residue modulo n is not 0. Therefore $a = 1$, $b = n - 1$ (or $a = n - 1$, $b = 1$) which is a solution. Therefore for $n > 2$ all solutions are $a = 1 + kn$, $b = n - 1 + ln$ and $a = n - 1 + kn$, $b = 1 + ln$, where $k, l = 0, 1, 2, \ldots$.

Problem 5. Find all real numbers t for which there exist real numbers x, y, z such that

$$3x^2 + 3xz + z^2 = 1,\ 3y^2 + 3yz + z^2 = 4,\ x^2 - xy + y^2 = t.$$

Solution. We shall prove that the answer is $t \in \left(\dfrac{3-\sqrt{5}}{2}, 1\right)$.

Let x, y, t, α satisfy the conditions of the problem. Consider four points A, B, C, O in the plane such that $AO = x, BO = y, CO = \dfrac{z}{\sqrt{3}}$ and $\sphericalangle AOB = 60°, \sphericalangle BOC = \sphericalangle COA = 150°$. By the cosine theorem it follows that $t > 0, AB = \sqrt{t}, BC = \dfrac{2}{\sqrt{3}}, CA = \dfrac{1}{\sqrt{3}}$. Since $\sphericalangle ACB < \sphericalangle AOB = 60°$ and $\sphericalangle BAC < \sphericalangle BOC = 150°$, we have

$$(1) \qquad \frac{\left(\frac{1}{\sqrt{3}}\right)^2+\left(\frac{2}{\sqrt{3}}\right)^2-(\sqrt{t})^2}{2\frac{1}{\sqrt{3}}\frac{2}{\sqrt{3}}}=\cos\sphericalangle ACB>\frac{1}{2},$$

$$(2) \qquad \frac{(\sqrt{t})^2+\left(\frac{1}{\sqrt{3}}\right)^2-\left(\frac{2}{\sqrt{3}}\right)^2}{2\sqrt{\alpha}\frac{2}{\sqrt{3}}}=\cos\sphericalangle BAC>-\frac{\sqrt{3}}{2}.$$

These inequalities are equivalent to $t<1$ and $t+\sqrt{t}-1>0$, i.e. $t\in\left(\dfrac{3-\sqrt{5}}{2},1\right)$.

Conversely, let $t\in\left(\dfrac{3-\sqrt{5}}{2},1\right)$. Then we can construct a triangle ABC such that $AB=\sqrt{t}, BC=\dfrac{2}{\sqrt{3}}, CA=\dfrac{1}{\sqrt{3}}$. By (1) and (2) it follows that $\sphericalangle ACB<60^\circ$ and $\sphericalangle ABC<\sphericalangle BAC<150^\circ$. Let k_A be the circle through B and C such that the arc BC, lying in the half-plane (with respect to BC) containing the point A, is equal to 60°. Denote by k_B the analogous circle trough A and C. Then it is easy to see that the second common point O of k_A and k_B lies in the interior of $\triangle ABC$. (Indeed, assume the contrary. If O lies in $\sphericalangle ACB$, then $\sphericalangle AOB=\sphericalangle AOC+\sphericalangle BOC=150^\circ+150^\circ=300^\circ$ which is impossible. If O lies in the opposite angle of $\sphericalangle ACB$, then $\sphericalangle AOB=\sphericalangle AOC+\sphericalangle BOC=30^\circ+30^\circ=60^\circ$ which contardicts to the inequalities $\sphericalangle AOB<\sphericalangle ACB<60^\circ$. The other cases for the position of O can be rejected in the same manner.) Hence $\sphericalangle AOB=360^\circ-\sphericalangle AOC-\sphericalangle BOC=360^\circ-150^\circ-150^\circ=60^\circ$. Set $x=AO, y=BO, z=CO\sqrt{3}$. Applying the cosine theorem for triangles AOB, BOC and COA it follows that x,y,z,t satisfy the conditions of the problem.

Problem 6. Given the equation

$$(p+2)x^2-(p+1)y^2+px+(p+2)y=1,$$

where p is fixed prime number of the form $4k+3$. Prove that:

a) If (x_0, y_0) is a solution of the equation where x_0 and y_0 are positive integers, then p divides x_0;

b) The given equation has infinitely many solutions (x_0, y_0) where x_0 and y_0 are positive integers.

Solution: a) Set $y-1=z$ and write the equation in the form

$$x^2=(z-x)((p+1)(z+x)+p). \tag{3}$$

If $z-x$ and $(p+1)(z+x)+p$ are relatively prime then they are perfect squares which is impossible since the second one is of the form $4k+3$. Let q be a common divisor of these numbers. It follows from (3) that q/x and so q/z. Since $q/(p+1)(z+x)+p$ we have q/p giving $q=p$ which completes the proof.

b) It suffices to prove that (3) has infinitely many solutions in positive integers. Let $x=px_1$ and $z=pz_1$. Then $x_1^2=(z_1-x_1)((p+1)(z_1+x_1)+1)$ and therefore there exist positive integers a and b such that $z_1-x_1=a^2$, $x_1=ab$ and $(p+1)(z_1+x_1)+1=b^2$. It follows now that

$$(p+2)b^2-(p+1)(a+b)^2=1. \tag{4}$$

Let $\left(\sqrt{p+2}+\sqrt{p+1}\right)^{2k+1}=m_k\sqrt{p+2}+n_k\sqrt{p+1}$ for any $k=0,1,\dots$, where m_k and n_k are positive integers. It is obvious that

$$(\sqrt{p+2}-\sqrt{p+1})^{2k+1}=m_k\sqrt{p+2}-n_k\sqrt{p+1}$$

and after multiplying we obtain that $(p+2)m_k^2-(p+1)n_k^2=1$, i.e. $b=m_k$ and $a+b=n_k$ are solutions of (4). Hence, $x=pm_k(n_k-m_k)$ and

$z = pn_k(n_k - m_k)$ are solutions of (3). The assertion of b) follows from the fact that both sequences $m_1, m_2, \ldots$ and $n_1, n_2, \ldots$ are strictly increasing.