BULGARIAN NATIONAL OLYMPIAD IN MATHEMATICS

Third round

1995

Problem 1. Let p and q be positive numbers such that the parabola $y = x^2 - 2px + q$ has no common point with the x-axis. Prove that there exist points A and B on the parabola such that the segment AB is parallel to the x-axis and $\angle AOB = 90^\circ$ (O is the coordinate origin) if and only if $p^2 < q \le \frac{1}{4}$. Find the values of p and q for which the points A and B are defined in an unique way.

Solution. Since the parabola has no common point with the x-axis, then the roots of the equation $x^2 - 2px + q = 0$ are not real and hence $p^2 < q$. Let the points $A(x_1, y_0)$ and $B(x_2, y_0)$ (Figure 1) be with the required properties. Then x_1 and x_2 are the roots of the equation $x^2 - 2px + q - y_0 = 0$ and $y_0 > q - p^2$, because the vertex of the parabola has coordinates $(p, q - p^2)$. On the other hand $OA^2 = x_1^2 + y_0^2$, $OB^2 = x_2^2 + y_0^2$, $AB^2 = (x_1 - x_2)^2$ and it follows from the Pythagorean theorem that $y_0^2 + x_1x_2 = 0$. But $x_1x_2 = q - y_0$ and thus $y_0^2 - y_0 + q = 0$. Consequently the existence of the points A and B is equivalent to the assertion that the equation $f(y) = y^2 - y + q = 0$ has a solution $y_0 > q - p^2$. (A and B are defined in an unique way if this is the only solution.) A necessary condition is that the discriminant of the equation is not negative, i.e. $q \le \frac{1}{4}$. The last condition is sufficient because $f(q - p^2) = (q - p^2) + p^2 > 0$ and $\frac{1}{2} > \frac{1}{4} \ge q - p^2$. The corresponding solution y_0 is unique iff $q = \frac{1}{4}$.

Figure 1.

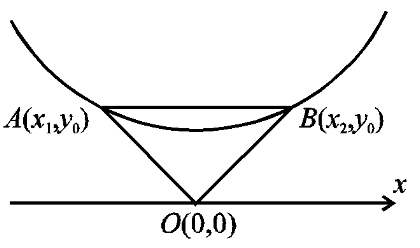
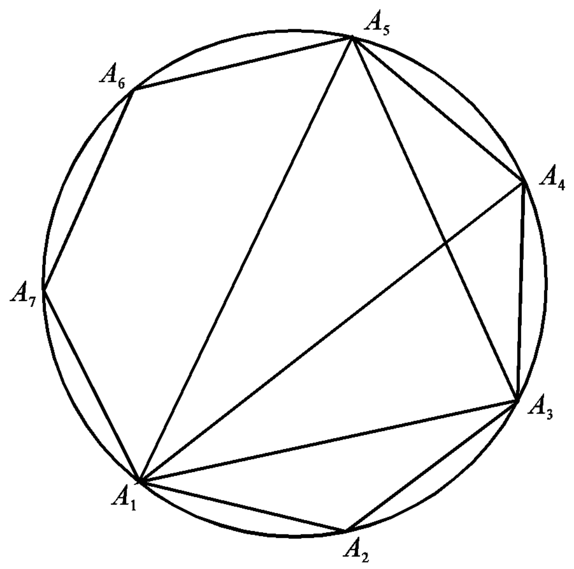


Figure 2.



Problem 2. Let $A_1A_2A_3A_4A_5A_6A_7$, $B_1B_2B_3B_4B_5B_6B_7$, $C_1C_2C_3C_4C_5C_6C_7$ be regular heptagons with areas S_A, S_B and S_C, respectively. Let $A_1A_2 = B_1B_3 = C_1C_4$. Prove that

$$\frac{1}{2} < \frac{S_B + S_C}{S_A} < 2 - \sqrt{2}.$$

Solution. Let $A_1A_2 = a$, $A_1A_3 = b$, $A_1A_4 = c$ (Figure 2). By the Ptolomeus theorem for the quadrangle $A_1A_3A_4A_5$ it follows that $ab+ac = bc$, i.e. $\frac{a}{b}+\frac{a}{c} = 1$. Since $\triangle A_1A_2A_3 \cong \triangle B_1B_2B_3$, then $\frac{B_1B_2}{B_1B_3} = \frac{a}{b}$ and hence $B_1B_2 = \frac{a^2}{b}$. Analogously $C_1C_2 = \frac{a^2}{c}$. Therefore $\frac{S_B + S_C}{S_A} = \frac{a^2}{b^2}+\frac{a^2}{c^2}$. Then $\frac{a^2}{b} + \frac{a^2}{c} > \frac{1}{2}(\frac{a}{b} + \frac{a}{c})^2 = \frac{1}{2}$ (equality is not possible because $\frac{a}{b} \neq \frac{a}{c}$). On the other hand

$$\frac{a^2}{b^2} + \frac{a^2}{c^2} = \left(\frac{a}{b} + \frac{a}{c}\right)^2 - \frac{2a^2}{bc} = 1 - \frac{2a^2}{bc}. \tag{1}$$

By the sine theorem we get $\frac{a^2}{bc} = \frac{\sin^2\frac{\pi}{7}}{\sin^2\frac{\pi}{7}\sin\frac{4\pi}{7}} = \frac{1}{4\cos\frac{2\pi}{7}(1+\cos\frac{2\pi}{7})}$. Since $\cos\frac{2\pi}{7} < \cos\frac{\pi}{4} = \frac{\sqrt{2}}{2}$, then $\frac{a^2}{bc} > \frac{1}{4\frac{\sqrt{2}}{2}(1+\frac{\sqrt{2}}{2})} = \sqrt{2}-1$. From here and from (1) we get the right hand side inequality of the problem.

Problem 3. Let $n > 1$ be an integer. Find the number of the permutations $(a_1, a_2, \ldots, a_n)$ of the numbers $1, 2, \ldots, n$ with the following property: there exists only one index $i \in \{1, 2, \ldots, n-1\}$ such that $a_i > a_{i+1}$.

Solution. Denote by p_n the number of the permutations with the given properties. Obviously, $p_1 = 0$ and $p_2 = 1$. Let $n \geq 2$. The number of the permutations with $a_n = n$ is equal to p_{n-1}. Consider all the permutations $(a_1, a_2, \ldots, a_n)$ with $a_i = n$, where $1 \leq i \leq n-1$ is fixed. Their number is $\binom{n-1}{i-1}$. Consequently

$$p_n = p_{n-1} + \sum_{i=1}^{n-1}\binom{n-1}{i-1} = p_{n-1} + 2^{n-1} - 1.$$

From here

$$\begin{aligned} p_n &= (2^{n-1}-1) + (2^{n-2}-1) + \cdots + (2-1) \\ &= 2^n - n - 1. \end{aligned}$$

Problem 4. Let $n \geq 2$ and $0 \leq x_i \leq 1$ for $i = 1, 2, \ldots, n$. Prove the inequality

$$(x_1 + x_2 + \cdots + x_n) - (x_1x_2 + x_2x_3 + \cdots + x_{n-1}x_n + x_nx_1) \leq \left[\frac{n}{2}\right].$$

When is there an equality?

Solution. Denote by $S(x_1, x_2, \ldots, x_n)$ the left hand side of the inequality. This function is linear with respect to each of the variables x_i. Particularly,

$$S(x_1, x_2, \ldots, x_n) \leq \max\left(S(0, x_2, \ldots, x_n), S(1, x_2, \ldots, x_n)\right).$$

From here it follows by induction that it is enough to prove the inequality when all x_i are equal to 0 or 1. On the other hand for arbitrary x_i we have

$$\begin{aligned} 2S(x_1, x_2, \ldots, x_n) &= n - (1-x_1)(1-x_2) - (1-x_2)(1-x_3) - \cdots - \\ &-(1-x_n)(1-x_1) - x_1x_2 - x_2x_3 - \cdots - x_nx_1 \end{aligned} \qquad (*)$$

i.e. $S(x_1, x_2, \ldots, x_n) \leq \dfrac{n}{2}$, when $x_i \in [0,1]$. In the case when x_i are equal to 0 or 1, the left hand side of the last inequality is an integer. Consequently $S(x_1, x_2, \ldots, x_n) \leq \left[\dfrac{n}{2}\right]$. It follows from $(*)$ that

- when n is even, the equality is satisfied iff $(x_1, x_2, \ldots, x_n) = (0, 1, 0, 1, \ldots, 0, 1)$;
- when n is odd, the equality is satisfied iff $(x_1, x_2, \ldots, x_n) = (x, 0, 1, 0, 1, \ldots, 0, 1)$, where $x \in [0,1]$ is arbitrary.

Problem 5. The points A_1, B_1, C_1 lie on the sides BC, CA, AB of the triangle ABC respectively and the lines AA_1, BB_1, CC_1 have a common point M. Prove that if the point M is center of gravity of $\triangle A_1B_1C_1$, then M is the center of gravity of $\triangle ABC$.

Solution. Let M be the center of gravity of $\triangle A_1B_1C_1$. Let A_2 be a point on $MA^{\rightharpoonup}$ such that $B_1A_1C_1A_2$ is a parallelogram. The points B_2 and C_2 are constructed analogously. Since $A_1C_1 \| A_1B_1 \| C_1B_2$, then the points A_2, C_1, B_2 are colinear and C_1 is the midpoint of A_2B_2. The same is true for the points A_2, B_1, C_2 and C_2, A_1, B_2. We shall prove that $A_2 = A$, $B_2 = B$ and $C_2 = C$, which will solve the problem.

Assume that $A_2 \neq A$ and let A be between A_2 and M. Then C_2 is between C and M, B is between B_2 and M and consequently A_2 is between A and M, which is a contradiction.

Problem 6. Find all pairs of positive integers (x, y) for which $\dfrac{x^2+y^2}{x-y}$ is an integer which is a divisor of 1995.

Solution. It is enough to find all pairs (x, y) for which $x > y$ and $x^2 + y^2 = k(x-y)$, where k divides $1995 = 3.5.7.19$. We shall use the following well-known fact: if p is a prime number of the type $4q + 3$ and if it divides $x^2 + y^2$, then p divides x and y. (For $p = 3, 7, 19$ this can be proved directly.) If k is divisible by 3 then x and y are divisible by 3 too. Simplifying by 9 we get an equality $x_1^2 + x_1^2 = k_1(x_1 - y_1)$, where k_1 divides 5.7.19. Considering 7 and 19 in an analogous way we get an equality $a^2 + b^2 = 5(a-b)$ (it is not possible to get an equality $a^2 + b^2 = (a-b)$), where $a > b$. From here $(2a-5)^2 + (2b-5)^2 = 50$, i.e. $a = 3$, $b = 1$ or $a = 2$, $b = 1$.

The above considerations imply that the pairs we are looking for are of the type $(3c, c)$, $(2c, c)$, $(c, 3c)$, $(c, 2c)$, where $c = 1, 3, 7, 19, 3.7, 3.19, 7.19, 3.7.19$.