BULGARIAN NATIONAL OLYMPIAD IN MATHEMATICS

Fourth round

1995

Problem 1. Find the number of all integers $n > 1$, for which the number $a^{25} - a$ is divisible by n for every integer a.

Solution. Let n be with the required property. Then p^2 (p prime) does not divide n since p^2 does not divide $p^{25} - p$. Hence n is a product of pairwise different prime numbers. On the other hand $2^{25} - 2 = 2.3^2.5.7.13.17.241$. But n is not divisible by 17 and 241 because $3^{25} \equiv -3 \pmod{17}$ and $3^{25} \equiv 32 \pmod{241}$. The Fermat theorem implies that $a^{25} \equiv a \pmod{p}$ when $p = 2, 3, 5, 7, 13$. Thus n should be equal to the divisor of $2 \cdot 3 \cdot 5 \cdot 7 \cdot 13$, different from 1. Therefore the number we are looking for is $2^5 - 1 = 31$.

Problem 2. A triangle ABC with semiperimeter p is given. Points E and F lie on the line AB and $CE = CF = p$. Prove that the excircle k_1 of $\triangle ABC$ to the side AB touches the circumcircle k of $\triangle EFC$.

Solution. Let P and Q be the tangent points of k_1 with the lines CA and CB, respectively. Since $CP = CQ = p$, then the points E, P, Q and F lie on the circle with center C and radius p. We denote by i the inversion defined by this circle. Since $i(P) = P$, $i(Q) = Q$, then $i(k_1) = k_1$. On the other hand $i(E) = E$ and $i(F) = F$. Hence $i(k)$ is the line AB. But k_1 touches AB and thus k touches k_1.

Problem 3. Two players **A** and **B** take stones one after the other from a heap with $n \geq 2$ stones. **A** begins the game and takes at least 1 stone but no more then $n - 1$ stones. Each player on his turn must take at least 1 stone but no more than the other player has taken before him. The player who takes the last stone is the winner. Find who of the players has a winning strategy.

Solution. Consider the pair (m, l), where m is the number of the stones in the heap and l is the maximal number of stones that could be taken by the player on turn. We must find for which n the position $(n, n-1)$ is winning (i.e. **A** wins) and for which n it is losing (**B** wins). We shall apply the following assertion several times: If (m, l) is a losing position and $l_1 < l$, then (m, l_1) is losing too.

Now we shall prove that $(n, n-1)$ is a losing position iff n is a power of 2.

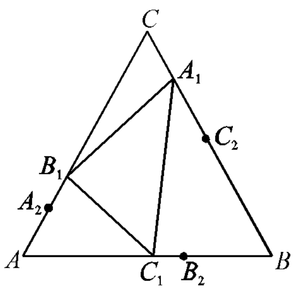
Sufficiency: Let $n = 2^k$, $k \geq 1$. If $k = 1$ then **B** wins on his first move. Assume that $(2^k, 2^k - 1)$ is a losing position and let consider the position $(2^{k+1}, 2^{k+1} - 1)$. If **A** takes at least 2^k stones on his first move, then **B** wins at ones. Let **A** take l stones, where $1 \leq l < 2^k$. By the inductive assumption **B** could play in such a way that he could win the game $(2^k, l)$ since $l \leq 2^k - 1$; the last move will be the move of **B**. After this move we get the position $(2^k, m)$ with $m \leq l$, which is losing for **A**, according to the inductive assumption.

Necessity: It is enough to prove that if n is not a power of 2, then $(n, n-1)$ is a winning position. Let $n = 2^k + r$, where $1 \leq r \leq 2^k - 1$. On his first move **A** takes r stones and **B** is faced to the position $(2^k, r)$, which is losing for **B**.

Problem 4. The points C_1, A_1 and B_1 lie on the sides AB, BC and CA of the equilateral triangle ABC respectively in such a way that the inradii of the triangles C_1AB_1, B_1CA_1, A_1BC_1 and $A_1B_1C_1$ are equal. Prove that A_1, B_1 and C_1 are the midpoints of the corresponding sides.

Solution. We shall prove that $BA_1 = CB_1 = AC_1$ (Figure 1). Assume the contrary and let $BA_1 \geq CB_1 > AC_1$. Let ρ be the rotation at 120^0 which center coincides with the in center of the incircle of $\triangle ABC$. This rotation transforms the incircles of the triangles C_1BA_1, A_1CB_1 and B_1AC_1 to the incircles of the triangles A_1CB_1, B_1AC_1 and C_1BA_1, respectively. Let $A_2 = \rho(A_1)$, $B_2 = \rho(B_1)$ and $C_2 = \rho(C_1)$. It follows that $BB_2 < BC_1$ and $BC_2 < BA_1$. But the incircles of the triangles BC_1A_1 and BC_2B_2 have equal radii (because $\rho(\triangle AC_1B_1) = \triangle BC_2B_2$), which is a contradiction.

Figure 1.



Let r be the radius of the incircles of the triangles C_1AB_1, B_1CA_1, A_1BC_1 and $A_1B_1C_1$. From the triangle B_1AC_1 we have $r = \dfrac{1 - B_1C_1}{2}.\dfrac{\sqrt{3}}{3}$, and from $\triangle A_1B_1C_1$ which is equilateral we have $r = B_1C_1.\dfrac{\sqrt{3}}{6}$. From here $B_1C_1 = \dfrac{1}{2}$ and consequently A_1, B_1, C_1 are midpoints of the corresponding sides.

Problem 5. Let $A = \{1, 2, \ldots, m + n\}$, where m and n are positive integers and let the function $f : A \to A$ be defined by the equations:

$$f(i) = i + 1 \quad \text{for} \quad i = 1, 2, \ldots, m - 1, m + 1, \ldots, m + n - 1$$

$$f(m) = 1 \quad \text{and} \quad f(m + n) = m + 1.$$

a) Prove that if m and n are odd then there exists a function $g : A \to A$ such that $g(g(a)) = f(a)$ for all $a \in A$.

b) Prove that if m is even then $m = n$ iff there exists a function $g : A \to A$ such that $g(g(a)) = f(a)$ for all $a \in A$.

Solution. a) Let $m = 2p+1$, $n = 2q+1$ and $g(i) = p+i+1$ for $i = 1, 2, \ldots, p$; $g(i) = q+i+1$ for $i = m + 1, m + 2, \ldots, m + q$; $g(2p + 1) = p + 1$; $g(p + 1) = 1$; $g(m + 2q + 1) = m + q + 1$; $g(m + q + 1) = m + 1$. It is easy to check that $g(g(a)) = f(a)$ for all $a \in A$.

b) Let $m = n$ and $g(i) = m + i$ for $i = 1, 2, \ldots, m$; $g(m + i) = i + 1$ for $i = 1, 2, \ldots, m - 1$; $g(2m) = 1$.

For the converse let $M = \{1, 2, \ldots, m\}$. It follows by the definition of f that the elements of M remain in M after applying the powers of f with respect to superposition. Moreover, these powers scoop out the whole M. The same is true for the set $A \setminus M$. The function f is bijective in A and if there exists g verifying the condition, then g is bijective too. We shall prove that $g(M) \bigcap M = \emptyset$. It follows from the contrary that there exists $i \in M$ such that $g(i) \in M$. Consider the sequence $i, g(i), g^2(i), \ldots$ and the subsequence $i, f(i), f^2(i), \ldots$. It is easy to see that $g(M) = M$. We deduce that there exists a permutation $a_1, a_2, \ldots, a_m$ of elements of M, such that $g(a_i) = a_{i+1}$ for $i = 1, 2, \ldots, m - 1$; $g(a_m) = a_1$ and $f(a_{2i-1}) = a_{2i+1}$ for $i = 1, 2, \ldots, s - 1$; $f(a_{2s-1}) = a_1$, where $m = 2s$. The last contradicts to the properties of f which were mentioned already. Thus $g(M) \bigcap M = \emptyset$. Analogously $g(A \setminus M) = A \setminus M$, if $g(i) \in A \setminus M$ for $i \in A \setminus M$. At last let us observe that when starting from an element of M and applying g we go to $A \setminus M$, but when applying g for a second time we go back to M. The same is true for the set $A \setminus M$.

From here and from the bijectivity of g it follows that M and $A \setminus M$ have one and the same number of elements, i.e. $n = m$.

Problem 6. Let x and y be different real numbers such that $\dfrac{x^n - y^n}{x - y}$ is an integer for some four consecutive positive integers n. Prove that $\dfrac{x^n - y^n}{x - y}$ is integer for all positive integers n.

Solution. Let $t_n = \dfrac{x^n - y^n}{x - y}$. Then $t_{n+2} + b.t_{n+1} + c.t_n = 0$ for $b = -(x + y)$, $c = xy$, where $t_0 = 0$, $t_1 = 1$. We shall show that $b, c \in \mathbb{Z}$. Let $t_n \in \mathbb{Z}$ for $n = m, m + 1, m + 2, m + 3$. Since $c^n = (xy)^n = t_{n+1}^2 - t_n.t_{n+2} \in \mathbb{Z}$ when $n = m, m + 1$, then $c^m, c^{m+1} \in \mathbb{Z}$. Therefore c is rational and from $c^{m+1} \in \mathbb{Z}$ it follows that $c \in \mathbb{Z}$. On the other hand

$$b = \frac{t_m t_{m+3} - t_{m+1} t_{m+2}}{c^m},$$

i.e. b is rational. From the recurrence equation it follows by induction that t_n could be represented in the following way $t_n = f_{n-1}(b)$, where $f_{n-1}(X)$ is a monic polynomial with integer coefficients and $\deg f_{n-1} = n - 1$. Since b is a root of the equation $f_m(X) = t_{m+1}$, then $b \in \mathbb{Z}$. Now from the recurrence equation it follows that $t_n \in \mathbb{Z}$ for all n.