BULGARIAN NATIONAL OLYMPIAD IN MATHEMATICS

Third round

1996

Problem 1. Prove that for all positive integers $n \geq 3$ there exist an odd positive integers x_n and y_n, such that

$$7x_n^2 + y_n^2 = 2^n.$$

Solution. If $n = 3$ we have $x_3 = y_3 = 1$.

Suppose that for an integer $n \geq 3$ there are odd positive integers x_n, y_n, such that $7x_n^2 + y_n^2 = 2^n$. We shall prove that for each pair

$$\left(X = \frac{x_n + y_n}{2}, Y = \frac{|7x_n - y_n|}{2}\right) \quad \text{and} \quad \left(X = \frac{|x_n - y_n|}{2}, Y = \frac{7x_n + y_n}{2}\right)$$

we have $7X^2 + Y^2 = 2^{n+1}$. Indeed,

$$7\left(\frac{x_n \pm y_n}{2}\right)^2 + \left(\frac{7x_n \mp y_n}{2}\right)^2 = 2\left(7x_n^2 + y_n^2\right) = 2 \cdot 2^n = 2^{n+1}.$$

Since x_n and y_n are odd, i.e. $x_n = 2k + 1$ and $y_n = 2l + 1$ (k, l are integers), then $\frac{x_n + y_n}{2} = k + l + 1$ and $\frac{|x_n - y_n|}{2} = |k - l|$, which shows that one of the numbers $\frac{x_n + y_n}{2}$ and $\frac{|x_n - y_n|}{2}$ is odd. Thus, for $n + 1$ there are odd natural numbers x_{n+1} and y_{n+1} with the required property.

Problem 2. The circles k_1 and k_2 with centers O_1 and O_2 respectively are externally tangent at the point C, while the circle k with center O is externally tangent to k_1 and k_2. Let ℓ be the common tangent of k_1 and k_2 at the point C and let AB be the diameter of k, which is perpendicular to ℓ, the points A and O_1 lie in one and the same semiplane with respect to the line ℓ. Prove that the lines AO_2, BO_1 and ℓ have a common point.

Figure 1.

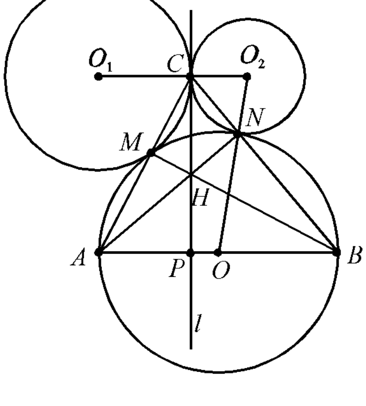
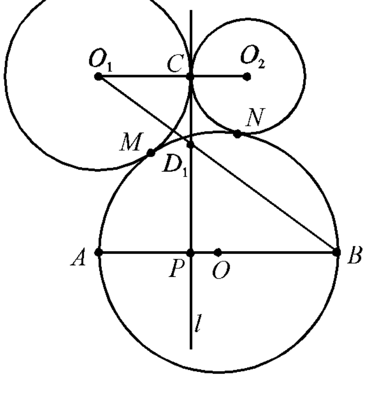


Figure 2.



Solution. Denote by r, r_1 and r_2 the radii of k, k_1 and k_2, by M and N the tangent points of k with k_1 and k_2, respectively and by P the common point of ℓ and AB (Figure 1).

It follows from $O_1O_2 \perp \ell$ and $AB \perp \ell$ that $\triangle BON \sim \triangle CO_2N$. Then $\angle CNO_2 = \angle ONB$, and consequently the points C, N and B are colinear. Also, $\dfrac{BN}{CN} = \dfrac{BO}{CO_2} = \dfrac{r}{r_2}$. Analogously, A, M and C are colinear and $\dfrac{AM}{MC} = \dfrac{r}{r_1}$.

The lines AN, BM and ℓ have a common point H, which is the altitude center of $\triangle ABC$. By Ceva's theorem we have: $\dfrac{AP}{PB} \cdot \dfrac{BN}{NC} \cdot \dfrac{CM}{MA} = \dfrac{AP}{PB} \cdot \dfrac{r}{r_2} \cdot \dfrac{r_1}{r} = 1$, from where:

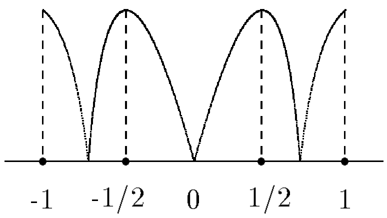
$$\frac{r_1}{PB} = \frac{r_2}{AP}. \tag{1}$$

Let now D_1 and D_2 be the common points of the line ℓ with the lines BO_1 and AO_2, respectively. Obviously, $\triangle O_1CD_1 \sim \triangle BPD_1$ (Figure 2), from where $\dfrac{CD_1}{D_1P} = \dfrac{r_1}{PB}$. Analogously, $\dfrac{CD_2}{D_2P} = \dfrac{r_2}{AP}$ and according to (1) we have $\dfrac{CD_1}{D_1P} = \dfrac{CD_2}{D_2P}$, which shows that $D_1 \equiv D_2$. Thus, the lines AO_2, BO_1 and ℓ have a common point.

Problem 3. a) Find the maximal value of the function $y = |4x^3 - 3x|$ in the interval $[-1, 1]$.

b)Let a, b and c be real numbers and M be the maximal value of the function $y = |4x^3 + ax^2 + bx + c|$ in the interval $[-1, 1]$. Prove that $M \geq 1$. For which a, b, c is the equality reached?

Figure 3.



Solution. a) Using that $\left(4x^3 - 3x\right)' = 12x^2 - 3 = 12\left(x - \dfrac{1}{2}\right)\left(x + \dfrac{1}{2}\right)$, we find that the function

$$y = \left|4x^3 - 3x\right|$$

has a local maximums when $x = \pm\dfrac{1}{2}$. Then its maximal value in the interval $[-1, 1]$ is the biggest among the numbers $y(-1)$, $y(1)$, $y\left(-\dfrac{1}{2}\right)$ and $y\left(\dfrac{1}{2}\right)$ (Figure 3).

But $y(-1) = y\left(-\dfrac{1}{2}\right) = y\left(\dfrac{1}{2}\right) = y(1) = 1$, thus the maximal value is equal to 1.

b) Let $f(x) = 4x^3 + ax^2 + bx + c$. Assume that there exist numbers a, b, c, for which the maximal value M of the function $y = |f(x)|$ in $[-1, 1]$ is less than 1, i.e. $M < 1$. Then $-1 < f(x) < 1$ for all $x \in [-1, 1]$.

Consider the function $g(x) = f(x) - (4x^3 - 3x) = ax^2 + (b+3)x + c$. We have $g(-1) > 0$, $g\left(-\dfrac{1}{2}\right) < 0$, $g\left(\dfrac{1}{2}\right) > 0$ and $g(1) < 0$. Consequently $g(x)$ changes its sign at least 3 times, which means that the quadratic equation $ax^2 + (b+3)x + c = 0$ has at least 3 different roots. This is possible only if $a = b + 3 = c = 0$, i.e. if $f(x) = 4x^3 - 3x$. According to a) the maximal value of $y = |4x^3 - 3x|$ is ≤ 1.

The equality $M = 1$ is reached only when $a = 0$, $b = -3$ and $c = 0$.

Problem 4. The real numbers $a_1, a_2, \ldots, a_n$ $(n \geq 3)$ form an arithmetic progression. There exists a permutation $a_{i_1}, a_{i_2}, \ldots, a_{i_n}$ of the same numbers, which is a geometric progression.

Find the numbers $a_1, a_2, \ldots, a_n$, if they are pairwise different and the biggest among them is equal to 1996.

Solution. Let $a_1 < a_2 < \cdots < a_n = 1996$ and q be the quotient of the geometric progression $a_{i_1}, a_{i_2}, \ldots, a_{i_n}$. We have $q \neq 0$ and $q \neq 1$. The numbers $a_{i_n}, a_{i_{n-1}}, \ldots, a_{i_1}$ form also a geometric progression which quotient is $\frac{1}{q}$. Thus, we can assume that $|q| > 1$, i.e. $q > 1$ or $q < -1$. Then $|a_{i_1}| < |a_{i_2}| < \cdots < |a_{i_n}|$, from where $a_i \neq 0$ for all i.

More exactly, either all numbers are positive ($q > 1$) and then $a_{i_1} < a_{i_2} < \cdots < a_{i_n}$, which together with $a_1 < a_2 < \cdots < a_n$ shows that $a_{i_k} = a_k$, i.e. the numbers $a_1, a_2, \ldots, a_n$ form an arithmetic as well as a geometric progression, or the numbers $a_{i_1}, a_{i_2}, \ldots, a_{i_n}$ change their signs alternatively ($q < -1$) and then the positive ones form an increasing geometric progression with quotient q^2, and the order is the same as in the arithmetic progression. (The numbers $a_1, a_2, \ldots, a_n$ could not be all negative, because $a_n = 1996 > 0$.)

Assume now that 3 among the numbers $a_1, a_2, \ldots, a_n$ are positive. Then $0 < a_{n-2} < a_{n-1} < a_n$ and they form a geometric as well as an arithmetic progression. Therefore $2a_{n-1} = a_{n-2} + a_n$ and $a_{n-1}^2 = a_{n-2}a_n$. From here $a_{n-2} = a_{n-1} = a_n$, which is a contradiction.

Thus at most two among the numbers are positive. Analogously, at most two among the numbers are negative. Consequently, $n \leq 4$.

Let $n = 4$. Then $a_1 < a_2 < 0 < a_3 < a_4$ and $2a_2 = a_1 + a_3$, $2a_3 = a_2 + a_4$. But $q < -1$ and the geometric progression is either a_3, a_2, a_4, a_1 or a_2, a_3, a_1, a_4. Let it be a_3, a_2, a_4, a_1. Then $a_2 = a_3q$, $a_4 = a_3q^2$ and $a_1 = a_3q^3$. Thus, $2a_3q = a_3q^3 + a_3$ and $2a_3 = a_3q + a_3q^2$. From here $q = 1$, which contradicts to $q < -1$.

So $n = 3$. There are two possibilities:

I. $a_1 < a_2 < 0 < a_3 = 1996$. Then the geometric progression is a_2, $a_3 = a_2q$, $a_1 = a_2q^2$. It follows from $2a_2 = a_1 + a_3$ that $2a_2 = a_2q^2 + a_2q$, i.e. $q^2 + q - 2 = 0$. Thus, $q = -2$, $-2a_2 = 1996$, $a_2 = -998$ and the numbers are $(-3992; -998; 1996)$.

II. $a_1 < 0 < a_2 < a_3 = 1996$. Now the geometric progression is a_2, $a_1 = a_2q$, $a_3 = a_2q^2$. From $2a_2 = a_1 + a_3$ we obtain $2a_2 = a_2q + a_2q^2$, i.e. again $q = -2$. Therefore, $a_3 = 4a_2 = 1996$ and $a_2 = 499$. The numbers are $(-998; 499; 1996)$.

Problem 5. A convex quadrilateral $ABCD$, for which $\angle ABC + \angle BCD < 180°$, is given. The common point of the lines AB and CD is E. Prove that $\angle ABC = \angle ADC$ if and only if

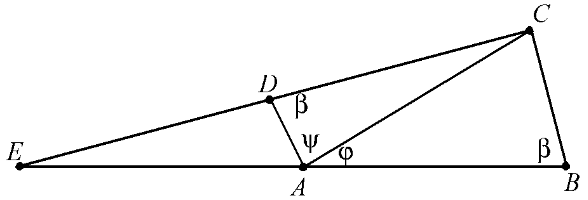
$$AC^2 = CD \cdot CE - AB \cdot AE.$$

Solution. Let $\angle ABC = \beta$, $\angle ADC = \delta$, $\angle BAC = \varphi$ and $\angle CAD = \psi$ (Figure 4). The point A is between E and B and the point D is between E and C. Also, $\angle AEC = \delta + \varphi + \psi - 180°$.

Applying the sine theorem to the triangles ACD, ACE, ABC and again to $\triangle ACE$, we obtain:

$$\begin{aligned}
\frac{CD}{AC} &= \frac{\sin\psi}{\sin\delta}, \\
\frac{CE}{AC} &= -\frac{\sin\varphi}{\sin(\delta+\varphi+\psi)}, \\
\frac{AB}{AC} &= \frac{\sin(\beta+\varphi)}{\sin\beta}, \\
\frac{AE}{AC} &= -\frac{\sin(\delta+\psi)}{\sin(\delta+\varphi+\psi)}.
\end{aligned}$$

Figure 4.



From here the equation $AC^2 = CD \cdot CE - AB \cdot AE$ is equivalent to the equations:

$$\frac{CD}{AC} \cdot \frac{CE}{AC} - \frac{AB}{AC} \cdot \frac{AE}{AC} - 1 = 0;$$

$$\sin(\beta+\varphi)\sin(\delta+\psi)\sin\delta - \sin\psi\sin\varphi\sin\beta \\ - \sin\delta\sin\beta\sin(\varphi+\psi+\delta) = 0;$$

$$(\cos(\beta+\varphi-\delta-\psi) - \cos(\beta+\varphi+\delta+\psi))\sin\delta \\ - (\cos(\varphi-\psi) - \cos(\varphi+\psi))\sin\beta \\ - (\cos(\beta-\delta) - \cos(\beta+\delta))\sin(\varphi+\psi+\delta) = 0$$

$$\sin(\beta+\varphi+\psi) - \sin(2\delta+\varphi+\psi-\beta) \\ + \sin(2\delta+\psi-\beta-\varphi) - \sin(\beta+\psi-\varphi) = 0;$$

$$\sin(\beta-\delta)\cos(\delta+\varphi+\psi) + \sin(\delta-\beta)\cos(\delta+\psi-\varphi) = 0;$$

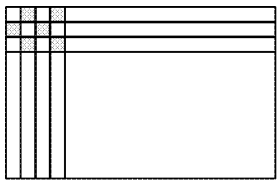
$$\sin(\beta-\delta)\sin(\delta+\psi)\sin\varphi = 0$$

But $\sin\varphi \neq 0$ and $\sin(\delta+\psi) \neq 0$. Consequently, $\sin(\beta-\delta) = 0$ and $\beta = \delta$.

Problem 6. A rectangle $m \times n$ ($m > 1$, $n > 1$) is divided into mn squares 1×1 with lines, parallel to its sides. In how many ways could two of the squares be canceled and the remaining part be covered with dominoes 2×1?

Solution. Denote by $F(m,n)$ the number we are looking for. Since every domino covers exactly two squares, then $F(m,n) = 0$ if m and n are odd.

Figure 5.



Let at least one of the numbers m and n be even. We color the squares in two colors — white and black in such a way that every two neighbor squares (with common side) are of different colors (Figure 5). The number S_0 of the white squares is equal to the number S_1 of the black ones and $S_0 = S_1 = \dfrac{mn}{2}$.

Each domino covers one white and one black square. If two white or two black squares are canceled, then it is impossible to cover by dominoes the remaining part of the rectangle. Now we shall show that if one white and one black squares are canceled then the remaining part can be covered by dominoes.

Since mn is even, then $mn = 2t$, where $t \geq 2$. We make induction with respect to t. The case $t = 2$ is obvious. Let $t_0 > 2$ and the proposition is true for all $2 \leq t \leq t_0$. Let $mn = 2(t_0+1)$. Denote by T_1 the rectangle consisted of the first two rows of the considered rectangle, and by T_2 — the rectangle consisted of the remaining rows. If the two canceled squares are in T_1 or in T_2, then we can cover each of the rectangles by dominoes and consequently we can cover the given rectangle.

Let one of the canceled squares be in T_1, and the other — in T_2. We place a domino in such a way that it covers one square from T_1 and one square from T_2. It is also possible that the canceled square from T_1 and the covered square from T_1 are of different colors. Thus, the canceled square from T_2 and the covered square from T_2 are of different colors. According to the inductive assumption the remaining part of T_1 and the remaining part of T_2 can be covered. Thus, the rectangle can be covered too.

Finally one white and one black squares can be chosen in $S_0 \cdot S_1 = \left(\dfrac{mn}{2}\right)^2$ ways. Therefore, in this case $F(m,n) = \dfrac{m^2n^2}{4}$.