BULGARIAN NATIONAL OLYMPIAD IN MATHEMATICS

Fourth round

1996

Problem 1. Find all primary numbers p and q, for which $\dfrac{(5^p-2^p)(5^q-2^q)}{pq}$ is an integer.

Solution. Let p be prime number and $p|\,(5^p-2^p)$. It follows by the Fermat theorem that $5^p-2^p\equiv 3 \pmod p$. Consequently $p=3$.

Let now p and q be such prime numbers that $\dfrac{(5^p-2^p)(5^q-2^q)}{pq}$ is an integer. If $p|\,(5^p-2^p)$, then $p=3$. Since $5^3-2^3=3\cdot 3\cdot 13$, then either $q|\,(5^q-2^q)$, i.e. $q=3$, or $q=13$. Therefore the pairs (3,3), (3,13), (13,3) satisfy the problem condition. It remains the case when $p\neq 3, q\neq 3$. Now $p|\,(5^q-2^q)$ and $q|\,(5^p-2^p)$. We can assume that $p>q$. It is clear that $(p,q-1)=1$ and consequently, there are positive integers a and b, for which $ap-b(q-1)=1$ (Bezou theorem). Since $(q,5)=(q,2)=1$, it follows by the Fermat theorem that $5^{q-1}\equiv 2^{q-1} \pmod q$. From $5^p\equiv 2^p(\pmod q)$ we deduce that $5^{ap}\equiv 2^{ap} \pmod q$ and therefore, $5^{b(q-1)+1}\equiv 2^{b(q-1)+1} \pmod q$. But $5^{b(q-1)+1}\equiv 5 \pmod q$ and $2^{b(q-1)+1}\equiv 2 \pmod q$. Thus, $q=3$, which is a contradiction. Finally, $(p,q)=(3,3),(3,13),(13,3)$.

Problem 2. Find the side length of the smallest equilateral triangle in which three disks with radii 2, 3 and 4 without common inner points can be placed.

Solution. Let in a equilateral $\triangle ABC$ two disks with radii 3 and 4 without common inner points be placed. It is clear that a line ℓ exists, which separates them, i.e. the disks are in different semiplanes with respect to ℓ (Figure 1).

Figure 1.

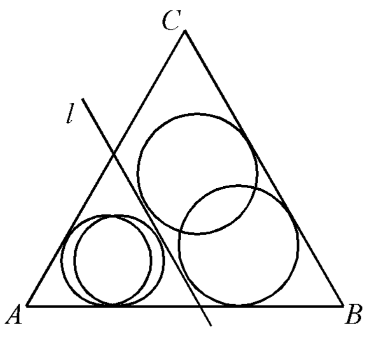
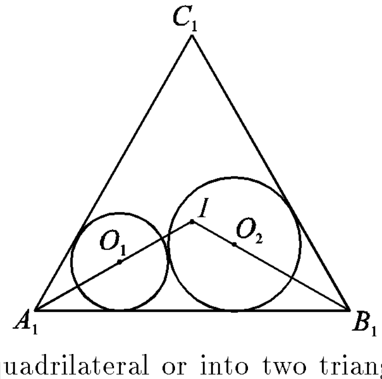
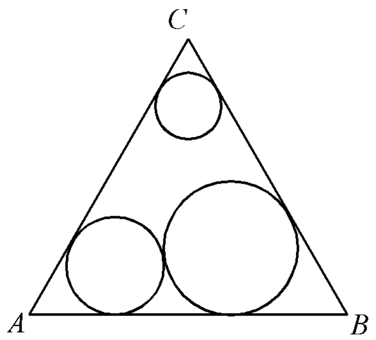


Figure 2.



This line divides the triangle into a triangle and a quadrilateral or into two triangles. In both cases the disks can be replaced in the figure in a way that each of them is tangent to two of the sides of $\triangle ABC$. It is clear that the new disks have no common inner point. Let the disks be inscribed in $\angle A$ and $\angle B$ of $\triangle ABC$, respectively. We translate the side BC parallelly to itself towards the point A, till the disk which is inscribed in $\angle B$ touches the disk which is inscribed in $\angle A$ (Figure 2). Thus, we get an equilateral $\triangle A_1B_1C_1$ with a smaller sides, in which two disks with radii 3 and 4 and without common inner points are placed.

Figure 3.



Let $A_1B_1 = x$, I be the incenter of $\triangle A_1B_1C_1$, while O_1 and O_2 be the centers of the two disks. Then $A_1I = B_1I = \dfrac{x}{\sqrt{3}}, A_1O_1 = 6, B_1O_2 = 8$. Since the disk with radius 4 is inside the $\triangle A_1B_1C_1$, then $O_2 \in IB_1$. Thus, $B_1O_2 \leq B_1I$, i.e. $x \geq 8\sqrt{3}$. On the other hand $O_1I = \dfrac{x}{\sqrt{3}} - 6, O_2I = \dfrac{x}{\sqrt{3}} - 8, O_1O_2 = 7$ and by the cosine theorem for $\triangle O_1O_2I$ we find that $\left(\dfrac{x}{\sqrt{3}} - 6\right)^2 + \left(\dfrac{x}{\sqrt{3}} - 8\right)^2 + \left(\dfrac{x}{\sqrt{3}} - 6\right)\left(\dfrac{x}{\sqrt{3}} - 8\right) = 49$. But $x \geq 8\sqrt{3}$, and from here $x = 11\sqrt{3}$. Consequently, $AB \geq 11\sqrt{3}$. On the other hand in the equilateral $\triangle ABC$ with side length $11\sqrt{3}$ three disks with radii 2, 3 and 4 (without common inner points) can be placed inscribing circles with these radii in the angles of the triangle (Figure 3).

Note that the disks wit radii 3 and 4 are tangent to each other. It follows from the above considerations that the solution of the problem is $11\sqrt{3}$.

Problem 3. The quadratic functions $f(x)$ and $g(x)$ are with real coefficients and have the following property: if the number $g(x)$ is integer for a positive x, then the number $f(x)$ is integer too. Prove that there are such integers m and n, that $f(x) = mg(x) + n$ for all real x.

Solution. Let $g(x) = px^2+qx+r$. We can assume that $p > 0$. Since $g(x) = p(x+\dfrac{q}{2p})^2+r-\dfrac{q^2}{4p}$ after the variable change of x by $x + \dfrac{q}{2p}$ we reduce the problem for the following quadratic functions $f(x) = ax^2+bx+c$ and $g(x) = px^2+s, p > 0$. Let k be such an integer that $k > s$ and $\sqrt{\dfrac{k-s}{p}} > \dfrac{q}{2p}$. Since $g\left(\sqrt{\dfrac{k-s}{p}}\right) = k$ is integer, then $f\left(\sqrt{\dfrac{k-s}{p}}\right) = \dfrac{a(k-s)}{p} + b\sqrt{\dfrac{k-s}{p}} + c$ is an integer too. Consequently, the number

$$f\left(\sqrt{\frac{k+1-s}{p}}\right) - f\left(\sqrt{\frac{k-s}{p}}\right) = \frac{b}{\sqrt{p}}.\frac{1}{\sqrt{k+1-s}+\sqrt{k-s}} + \frac{a}{p} \tag{1}$$

is an integer for all k which are sufficiently big. It follows from here that $\dfrac{a}{p}$ is an integer. Indeed, suppose that $\dfrac{a}{p}$ is not an integer. If $b > 0$, we chose k in a way that

$$\frac{b}{\sqrt{p}}.\frac{1}{\sqrt{k+1-s}+\sqrt{k-1}} < \left[\frac{a}{p}\right] + 1 - \frac{a}{p},$$

and if $b < 0$, we chose k in a way that

$$\frac{b}{\sqrt{p}}.\frac{1}{\sqrt{k+1-s}+\sqrt{k-1}} > \left[\frac{a}{p}\right] - \frac{a}{p}.$$

In both cases there is a contradiction with the fact that (1) is an integer. Now, it follows that

$$\frac{b}{\sqrt{p}}.\frac{1}{\sqrt{k+1-s}+\sqrt{k-s}}$$

for all k, which are sufficiently big. This is possible only when $b = 0$.

Let $\frac{a}{p} = m$. Then, $f\left(\sqrt{\frac{k-s}{p}}\right) = m(k-s)+c$ is an integer (when k is sufficiently big), i.e. $c - ms$ is an integer. Let $n = c - ms$. Now it is clear that $f(x) = mg(x) + n$ for all x.

Problem 4. The sequence $\{a_n\}_{n=1}^{\infty}$ is defined by

$$a_1 = 1, \quad a_{n+1} = \frac{a_n}{n} + \frac{n}{a_n}, \quad n \geq 1.$$

Prove that $\lfloor a_n^2 \rfloor = n$ when $n \geq 4$ (it is denoted by $\lfloor x \rfloor$ the integer part of the number x).

Solution. Let $f(x) = \frac{x}{n} + \frac{n}{x}$. Since $f(a) - f(b) = \frac{(a-b)(ab-n^2)}{abn}$, it follows that the function $f(x)$ is decreasing in the interval $(0, n)$.

Firstly, by induction we shall prove that $\sqrt{n} \leq a_n \leq \frac{n}{\sqrt{n-1}}$ when $n \geq 3$. We have $a_1 = 1, a_2 = 2$ and $a_3 = 2$, i.e. $\sqrt{3} \leq a_3 \leq \frac{3}{\sqrt{2}}$. Let $\sqrt{n} \leq a_n \leq \frac{n}{\sqrt{n-1}}$ for an integer $n \geq 3$. Then, $a_{n+1} = f(a_n) \leq f(\sqrt{n}) = \frac{n+1}{\sqrt{n}}$ and $a_{n+1} = f(a_n) \geq f\left(\frac{n}{\sqrt{n-1}}\right) = \frac{n}{\sqrt{n-1}} > \sqrt{n+1}$ and thus the induction finishes.

Since $a_n \geq \sqrt{n}$, it remains to prove, that $a_n < \sqrt{n+1}$. We have $a_{n+1} = f(a_n) \geq f\left(\frac{n}{\sqrt{n-1}}\right) = \frac{n}{\sqrt{n-1}}$ when $n \geq 3$. Consequently, $a_n \geq \frac{n-1}{\sqrt{n-2}}$ when $n \geq 4$. Then,

$$a_{n+1} = f(a_n) < f\left(\frac{n-1}{\sqrt{n-2}}\right) = \frac{(n-1)^2 + n^2(n-2)}{(n-1)n\sqrt{n-2}} < \sqrt{n+2}$$

when $n \geq 4$. (The last inequality is equivalent to $2n^2(n-3) + 4n - 1 > 0$.) Therefore, $\sqrt{n} \leq a_n < \sqrt{n+1}$ when $n \geq 4$, i.e. $\lfloor a_n^2 \rfloor = n$.

Problem 5. The quadrilateral $ABCD$ is inscribed in a circle. The lines AB and CD meet each other in the point E, while the diagonals AC and BD — in the point F. The circumcircles of the triangles AFD and BFC have a second common point, which is denoted by H. Prove that $\angle EHF = 90°$.

Solution. Let O be the circumcenter of $ABCD$. We shall prove that O is the second common point of the circumcircles of $\triangle AHB$ and $\triangle CHD$. (Since AB and CD are not parallel, then $O \neq H$.) After that we shall prove that the points E, H and O are colinear and $\angle OHF = 90°$.

We shall consider the possible positions of H.

Let G be the common point of AD and BC (these lines are not parallel because the circumcircles of $\triangle AFD$ and $\triangle BFC$ are not tangent). It is clear that H is in the interior of $\angle AGB$.

1) H is in $\triangle CGD$. Then, $\angle CHD = \angle CHF + \angle DHF = 180° - \angle CBF + 180° - \angle DAF = 360° - \widehat{CD} > 180°$ (from $\widehat{CD} < \widehat{AB}$ and $\widehat{AB} + \widehat{CD} < 360°$), which is impossible.

2) H is in $\triangle CFD$. Then, $\angle CHD = 360° - \angle CHF - \angle DHF = \angle CBF + \angle DAF = \widehat{CD} = \angle COD$ and $\angle AHB = \angle AHF + \angle BHF = \angle ADF + \angle BCF = \widehat{AB} = \angle AOB$.

3) H is in $\triangle ABF$. Analogously, $\angle CHD = \angle COD$ and $\angle AHB = \angle AOB$.

4) H is on AB. Again $\angle CHD = \angle COD$ and $180° = \angle AHB = \angle AOB$. Consequently, O is the midpoint of AB.

5) H is not in $\triangle ABG$. Then $\angle CHD = \angle COD$ and $\angle AHB = 360° - \angle AOB$.

Note that in the cases 2), 3) and 5) the points O and H are in one and the same semiplane with respect to the line AB and with respect to the line CD. Consequently, the points A, B, H and O are concyclic. We have the same for the points C, D, H and O.

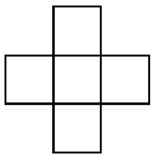
Now we shall prove that the line EH passes through the point O. Let this line meet the circumcircles of $\triangle AHB$ and $\triangle CHD$ at the points O_1 and O_2, respectively. Since these points lie on the ray $EH^{\rightarrow}$, it follows from the equalities $EH.EO_1 = EA.EB = EC.ED = EH.EO_2$ that $O_1 \equiv O_2 \equiv O$.

The points O and H are in one and the same semiplane with respect to CD and we can assume that the quadrilateral $COHD$ is convex. Let H be in $\triangle CFD$. Then, $\angle OHF = \angle FHC - \angle OHC = 180° - \angle FBC - \angle ODC = 180° - \frac{1}{2}.\angle COD - (90° - \frac{1}{2}.\angle COD) = 90°$. The other possibility for H is to be inside $\angle AGB$ and outside the non-convex quadrilateral $AFBG$.

Hence $\angle OHF = \angle OHC + \angle FHC = \angle ODC + \angle FBC = 90°$.

Problem 6. A square table of size 7×7 with the four corner squares deleted is given.

Figure 4.



a) What is the smallest number of squares which need to be colored black so that a 5-square entirely uncolored Greek cross (Figure 4) cannot be found on the table?

b) Prove that it is possible to writte integers in each square in a way that the sum of the integers in each Greek cross is negative while the sum of all integers in the square table is positive.

Solution. Denote the square in row i and column j by (i, j). Note that a cross is uniquely determined by its central cell. The cross with central cell (i, j) is denoted by C_{ij}, $2 \le i, j \le 6$. The number of all crosses is 25.

a) The squares $(1, i)$, $(i, 1)$, $(7, i)$, $(i, 7)$, $i = 2, \dots, 6$, are included in exactly one cross; the squares (2,2), (2,6), (6,2), (6,6) are included in exactly 3 crosses; the squares $(2, i)$, $(i, 2)$, $(6, i)$, $(i, 6)$, $i = 3, 4, 5$ — in exactly 4 crosses and finally, the squares $(i, j), 3 \le i, j \le 5$, are included in exactly 5 crosses. Since C_{22} contains a colored square, then at least one of (1,2), (2,1), (2,2), (2,3), (3,2) should be colored. Similarly, at least one of (2,7), (1,6), (2,6), (2,5), (3,6), at least one of (7,2), (6,1), (6,2), (5,2), (6,3) and at least one of (6,7), (7,6), (6,6), (6,5), (5,6) are colored. Any of them is contained in at most four crosses (the first two in each quintuple in 1 cross, the third one in 3 crosses and the remaining two — in 4 crosses). Denote by x the number of the colored squares among $(i, j), 3 \le i, j \le 5$. The number of crosses containing a colored square is not greater than $4.4 + 5x$, whence

$$16 + 5x \ge 25. \tag{1}$$

Thus, $x \ge 2$ and the number of the colored cells is at least 6.

Suppose that the number of the colored squares is 6. Then $x = 2$. Moreover, acording (1) there exists at most one Greek cross containing more than one colored square. If two squares with a common side or a common vertex are colored, then it is easy to check that there are two crosses with at least two colored squares each. Therefore, the squares (4,4), (3,3), (3,5), (5,3), (5,5) should be uncolored. With no loss of generality, let the two colored squares be (4,3) and (4,5). But now the crosses C_{34} and C_{54} do not contain colored squares.

To prove that the minimal number of colored squares is 7, color for example the squares (2,5), (3,2), (3,3), (4,6), (5,4), (6,2), (6,5).

b) Let us write -5 in each of the colored squares from a) and 1 in the remaining squares. Since every Greek cross contains a colored square, the sum of the numbers in its squares does not exceed $1+1+1+1-5 = -1 < 0$. The sum of all numbers in the table is $7.(-5)+(45-7).1 = 3 > 0$ and we are done.